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## Universal covering space of top spaces

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**Abstract** In this paper, it has been proved that the universal covering space of a top space (local top space) is a top space (local top space). In addition, it has been shown that the universal covering space of a top space with constant rank is an upper top space. However, there are some upper top spaces which are not an universal covering space. As follows, it has been proved that the fundamental group of a right top space is Abelian, and some properties of generalized left action of a top space on a manifold have been investigated.

**Keywords** Lie group · covering space · universal covering space · top space

**Mathematics Subject Classification (2010)** 22A15 · 22E60

### 1 Introduction

The notion of top space as a generalization of Lie groups was introduced by Molaei in 1998. Top spaces have been found in the process of working on constructing a geometric unified theory using Santilli's isotheory [8].

**Definition 1.1** *[9] The top space $T$ is a smooth manifold with an operation that called multiplication, such that:*

- $(xy)z = x(yz)$ *for all* $x, y, z \in T$;
- *For every* $x \in T$ *there exists an unique* $z \in T$ *such that* $xz = zx = x$ *(*$z$ *is denoted by* $e(x)$*);*
- *For every* $x \in T$ *there exists* $y \in T$ *such that* $xy = yx = e(x)$ *(*$y$ *is denoted by* $x^{-1}$*);*
- *The mapping* $m_1 : T \to T$ *defined by* $m_1(x) = x^{-1}$ *and the mapping* $m_2 : T \times T \to T$ *defined by* $m_2(x, y) = xy$ *are smooth maps;*

Top spaces with finite number of identities have been studied by several authors (Araujo, Molaei, Mehrabi, Oloomi, Tahmoresi, Farhangdoost, etc.). For example, it has been shown that a top space with finite number of

identities is a disjoint union of Lie groups [11]. Moreover, these Lie groups are homeomorphic, and the quotient space of these kinds of top spaces has been introduced [6].
The Lie algebras on these kinds of top spaces has been investigated in [10]. The definition of generalized action of top spaces on a manifold and the concept of the stabilizer of a top space have been introduced in [4]. It has been shown that these top spaces are $C^\infty$ principal fiber bundle and by this method the top spaces have been characterized [5].
A Rees matrix is an useful example of top spaces. Let the two smooth manifolds $I$ and $\Lambda$ be given. If $p : \Lambda \times I \to G$ is a mapping, then $I \times G \times \Lambda$ with the product $(i, a, \lambda)(j, b, \mu) = (i, ap(\lambda, j)b, \mu)$ is called a Rees matrix and denoted by $M(G, I, \Lambda, p)$. If $p$ is a smooth mapping then $M(G, I, \Lambda, p)$ is a top space. On the other hand, a top space with finite number of identities is homeomorphic with a subset of a Rees matrix [6].
The left and right translation by $g \in T$ are respectively the maps $l_g : T \to T$ and $r_g : T \to T$ defined by

$$l_g(p) = gp,$$

$$r_g(p) = pg,$$

for all $p \in T$.
Note that left and right translations in Lie groups are homeomorphism but in top spaces they do not have this property in general. Suppose that $T$ is a top space such that $Tt = T$, for some $t \in T$, where $Tt = \{gt : g \in T\}$. The following theorem implies that $r_t$ is a homeomorphism for all $t \in T$. In particular $r_{e(t)} = id$, for all $t \in T$ [14].

**Theorem 1.2** *[9] If $Tt \cap Tg \neq \emptyset$, then $Tt = Tg$, where $t, g \in T$. In particular, $Tt = Te(t)$.*

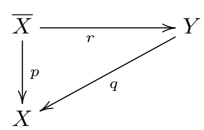
**Definition 1.3** *[1], [2] The top space $T$ called a right (left) top space if $Tt = T$ $(tT = T)$, for some $t \in T$.*

There are right (left) top spaces with infinite number of identities.
**Example 1.3.** [9] The $n$- dimensional torus $T^n = R^n/Z^n$ with the product $((a_1, a_2, ..., a_n)+Z^n, (b_1, b_2, ..., b_n)+Z^n) = (a_1+b_1, a_2+b_2, ..., a_{n-1}+b_{n-1}, a_n) + Z^n$ is a top space. $e((a_1, a_2, ..., a_n) + Z^n) = (0, 0, ..., 0, a_n) + Z^n$, thus $T^n$ has infinite number of identities. In addition $T^n a = T^n$ for all $a \in T^n$.
**Example 1.4.** Suppose that the $I$ in the definition of Rees matrices has just one member then $\Lambda \times G \times I \simeq \Lambda \times G$ is a right top space, which we call it a right Rees matrix and is denoted by $M(G, \Lambda, p)$. In addition, $e((\lambda, g)) = (\lambda, p(\lambda)^{-1})$ and consequently $card(e(\Lambda \times G)) = card(\Lambda)$.
The above mentioned contents have been proved for right top spaces and some aspects of Lie theorems are given in [2], [1]. In this paper some basic properties of the universal covering space of top spaces are investigated.
The organization of this paper is as follows:
In the next section, it has been proved that the universal covering space of a top space is a top space too. In addition, it has been shown that the

universal covering space of a right top space is a right top space.
In section 3, it has been proved that the fundamental group of a right top space is Abelian.
In section 4, it has been proved that the stabilizer of a top space is a top generalized normal subgroup, and the orbit map of a top spac has a constant rank. In the last section, it has been shown that the universal covering space of a local top space is a local top space and some important properties of a local top space can be lifted to its universal covering space.

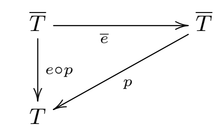
## 2 Universal covering space and upper top space

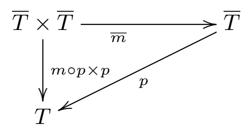
We recall the definition of universal covering space on a connected manifold. Let $X$ be a connected manifold with the base point $x_0$. A covering of $(X, x_0)$ is a triple consisting of a connected manifold $Y$ with a base point $y_0$ and a projection $q : Y \to X$ such that
(i) $q$ is a surjective local isomorphism;
(ii) $q(y_0) = x_0$;
(iii) for any $x \in X$ there exists a connected neighborhood $U$ of $x$ such that $q$ induces a homeomorphism of every connected component of $q^{-1}(U)$ onto $U$. The map $q$ is called the covering projection of $Y$ onto $X$. The covering $(\overline{X}, p, \overline{x_0})$ of $(X, x_0)$ is called a universal covering space if for any other covering $(Y, q, y_0)$ of $(X, x_0)$ there exists an unique differentiable map $r : \overline{X} \to Y$ such that $(\overline{X}, r, \overline{x_0})$ is a covering of $(Y, y_0)$ and the following diagram is commutative.




**Theorem 2.1** *Let $T$ be a connected top space. If $(\overline{T}, p, \overline{e})$ is the universal covering space of $(T, e)$, where $e \in e(T)$, then $(\overline{T}, p, \overline{e})$ is a top space.*

*Proof.* Since $(\overline{T}, p, \overline{e})$ is the universal covering space of $(T, e)$, the mapping $e \circ p$ has a unique lifting $\overline{e}$ such that $\overline{e}(\overline{e}) = \overline{e}$ and the following diagram commutes.
In addition, the mapping $\overline{m} : \overline{T} \times \overline{T} \to \overline{T}$ is an unique liftting of $m \circ p \times p$ such that the following diagram commutes. We prove that $\overline{m}(\bar{t}, \overline{e}(\bar{t})) = \bar{t}$. Since $T$ is a top space, $p \circ \overline{m}(\bar{t}, \overline{e}(\bar{t}))) = m \circ (p \times p)(\bar{t}, \overline{e}(\bar{t}))) = m(p(\bar{t}), p(\overline{e}(\bar{t}))) = p(\bar{t})$, for any $\bar{t} \in \overline{T}$.
Moreover, $p \circ id(\bar{t}) = p(\bar{t})$. Using the uniqueness of universal covering space,




$\overline{m}(\bar{t},\overline{e}(\bar{t}))=\bar{t}$.
Using the same method we can show that $\overline{m}(\overline{e}(\bar{t}),\bar{t})=\bar{t}$.
$p\circ(\overline{m}\circ(id_{\overline{T}}\times\overline{m}))=m\circ(p\times p)\circ(id_{\overline{T}}\times\overline{m})=m\circ(p\times p\circ\overline{m})=m\circ(p\times m\circ(p\times p))=m\circ(id\times m)\circ(p\times p\times p)$, and $p\circ(\overline{m}\circ(\overline{m}\times id_{\overline{T}})=m\circ(p\times p)\circ(\overline{m}\times id_{\overline{T}})=m\circ(p\circ\overline{m}\times p)=m\circ(m\circ(p\times p)\times p)=m\circ(id\times m)\circ(p\times p\times p)$.
It shows the multiplication on $\overline{T}$ is associative. Let $\bar{i}$ be the lifting of $i\circ p$, where $i$ is the inverse function on $T$. We have

$p\circ\overline{m}(\bar{t},\bar{i}(\bar{t}))=m\circ(p\times p)(\bar{t},\bar{i}(\bar{t}))=m(p(\bar{t}),i(p(\bar{t})))=e(p(\bar{t}))=p\circ\overline{e}(\bar{t})$.

By the above relationships $id$ and $\overline{m}(\bar{.},\bar{i}(\bar{.}))$ are the lifting of the same map, so they are equal. □

**Corollary.** If $(T,e)$ is a right top space and $(\overline{T},p,\overline{e})$ is its universal covering space then
$p\circ\overline{m}(\bar{t},\overline{e}(\bar{t}))=m\circ(p\times p)(\bar{t},\overline{e}(\overline{a}))=m(p(\bar{t}),p(\overline{e}(\overline{a})))=p(\bar{t})$, for any $\bar{t},\ \overline{a}\in\overline{T}$.
This implies
$\overline{T}$ is a right top space.

**Theorem 2.2** *Let $T$ be a connected top space such that $e:T\to T$ has a constant rank and $(\overline{T},p,\overline{e})$ be the universal covering space of $(T,e)$ then:*
*1) $\overline{e}:\overline{T}\to\overline{T}$ has a constant rank,*
*2)$\overline{e}^{-1}(\overline{e}(\bar{t}))$ is an embedded manifold for all $\bar{t}\in\overline{T}$.*

*Proof.* Using the equality $e\circ p=p\circ\overline{e}$ we have $de\circ dp=dp\circ d\overline{e}$. Since $p$ is a local homeomorphism, $dp$ is an isomorphism. Hence, $rank\ de=rank\ d\overline{e}$. (2) is an immediate consequence of (1). □

**Theorem 2.3** *let $T$ be a top space such that the rank of $e$ is constant. Then $\overline{e}^{-1}(\overline{e}(\bar{t}))$ is the universal covering space of $e^{-1}(e(t))$.*

*Proof.* First, we show that $\overline{e}^{-1}(\overline{e}(\bar{t}))$ is a cover of $e^{-1}(e(t))$. Set $p|_{\overline{e}^{-1}(\overline{e}(\bar{t}))}=p'$. Suppose that $\overline{e}(\overline{s})=\overline{e}(\bar{t})$. Since $e(t)=p'(\overline{e}(\bar{t}))=p'(\overline{e}(\overline{s}))=e(p'(\overline{s}))$, $p'(\overline{s})\in e^{-1}(e(t))$, so, $p'$ is well defined.
On the other hand, for $s\in e^{-1}(e(t))$ there is the neighborhood $U$ of $s$ in $T$ such that $p^{-1}(U)=\cup_i U_i$ where $U_i$'s are neighborhoods of $p^{-1}(s)$. Let

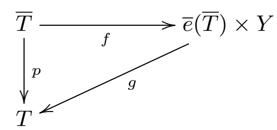
$U' = U \cap e^{-1}(e(t))$, so $p^{-1}(U') = \cup_i (U_i \cap \overline{e}^{-1}(\overline{e}(\overline{t}))$. $p : U_i \to U$ is a homeomorphism for all $i$. Also $e^{-1}(e(t))$ and $\overline{e}^{-1}(\overline{e}(\overline{t}))$ are embedded manifolds. Thus, $p' : U_i \cap \overline{e}^{-1}(\overline{e}(\overline{t})) \to U \cap e^{-1}(e(t))$ is a homeomorphism. Therefore, $\overline{e}^{-1}(\overline{e}(\overline{t}))$ is a cover of $e^{-1}(e(t))$.

Now, we prove that $(\overline{e}^{-1}(\overline{e}(\overline{t})), p', \overline{e})$ is the universal covering space of $(e^{-1}(e(t)), e)$. Let $(Y, q, y_0)$ be a cover of $(e^{-1}(e(t)), e)$. Since $e^{-1}(e(t))$ is isomorphic with $e^{-1}(e(t'))$ for all $t, t' \in T$, $(Y, q, y_0)$ is a cover of $e^{-1}(e(t'))$, for all $e(t) \in e(T)$. Consider the set $\overline{e}(\overline{T}) \times Y$. We prove that $(\overline{e}(\overline{T}) \times Y, g, y_0)$ is a cover of $(T, e)$ where $g(\overline{e}(\overline{t}), y) = q(y)$ for all $y \in Y$.

It is clear that $g$ is surjective. We show that $g$ is a local homeomorphism. Let $t \in T$ be an arbitrary member. Since $T = \cup e^{-1}(e(t))$, $t \in e^{-1}(e(t'))$ for some $t' \in T$ and $(Y, y_0)$ is a cover of $e^{-1}(e(t'))$, then there is the neighborhood $U$ of $t$ in $e^{-1}(e(t'))$ such that $q^{-1}(U) = \cup U_i$, where $U_i$'s are disjoint open sets in $Y$. Hence $g^{-1}(U) = \cup(\overline{e}(\overline{T}) \times U_i)$.

Since $(\overline{T}, \overline{e})$ is the universal covering of $(T, e)$, there is an unique differentiable map $f : \overline{T} \to \overline{e}(\overline{T}) \times Y$ such that the following diagram commutes.

Set $f' = f|\overline{e}^{-1}(\overline{e}(\overline{t}))$. Hence, $g \circ f' = p$. It is clear that $f'$ is unique.




□

**Remark 2.1** If $T$ is a right top space, $T \cong e(T) \oplus eT$, where $e(t)T = e^{-1}(e(t))$ for some $t \in T$ [1]. Hence, $\overline{e}\overline{T}$ is the universal covering space of $eT$.

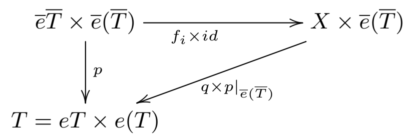
We recall that if $T$ is a top space, then $e^{-1}(e(t)) = e(t)Te(t)$ for all $t \in T$.

**Theorem 2.4** *If $T$ is a right top space and $(\overline{T}, p, \overline{e})$ is the universal covering space of $(T, e)$, for some $e \in e(T)$, then $\overline{e}(\overline{T})$ is the universal covering space of $e(T)$.*

*Proof.* Let $p' = p|_{\overline{e}(\overline{T})}$. It is clear that $p'$ is surjective. First, we prove that $(\overline{e}(\overline{T}), p', \overline{e})$ is a covering space of $(e(T), e)$. Since $T$ is a right top space, $e(T)$ is an imbedded submanifold of $T$. Let $e(t) \in e(T)$ be arbitrary. Let $U' = U \cap e(T)$ where $U$ is the neighborhood of $e(t)$ in $T$ which is given by the covering definition. ${p'}^{-1}(U') = p^{-1}(U) \cap \overline{e}(\overline{T})$. Since the components of $p^{-1}(U)$ are homeomorphic with $U$, then the components of ${p'}^{-1}(U')$ are homeomorphic with $U'$. Therefore, the proof is complete.

Now, we prove that $(\overline{e}(\overline{T}), p', \overline{e})$ is the universal covering space of $(e(T), e)$. If $\overline{e}(\overline{T})$ is not the universal covering of $e(T)$, then there are two homomorphisms $f_1, f_2 : \overline{e}(\overline{T}) \to X$ where $(X, q)$ is a cover of $e(T)$ such that $q \circ f_i = p'$

for $i = 1, 2$. On the other hand, since $\overline{T}$ is a right top space $\overline{T} = \overline{e}\overline{T} \oplus \overline{e}(\overline{T})$ is the universal covering space of $T = eT \oplus e(T)$. Consider the cover $X \times \overline{e}(\overline{T})$ of $T$. The following diagram commutes.
i.e, $(q \times p|_{\overline{e}(\overline{T})}) \circ (f_i \times id)(\overline{e}\overline{t}, \overline{e}_1) = (q \times p|_{\overline{e}(\overline{T})})(f_i(\overline{e}\overline{t}), \overline{e}_1) =$




$(q \circ f_i(\overline{e}(\overline{t})), p(\overline{e}_1)) = p(\overline{e}\overline{t}, \overline{e}_1)$.
This is a contradiction to the fact that $\overline{T}$ is the universal covering space of $T$. □

2.1 Upper top space and universal covering space

If $(\overline{e^{-1}(e(t))}, p_t, \overline{e(t)})$ is the universal covering space of $(e^{-1}(e(t)), e(t))$, then $(\overline{e^{-1}(e(t))}, p_t, \overline{e(t)})$ is a Lie group, with the multiplication $\overline{m_t}(\overline{t_1}, \overline{t_2})$ with $\overline{t_1}, \overline{t_2} \in \overline{e^{-1}(e(t))}$ such that $p_t \circ \overline{m_t}(\overline{t_1}, \overline{t_2}) = m_t \circ p_t(\overline{t_1}, \overline{t_2})$ where $m_t$ is the restriction of $m$ on $e^{-1}(e(t)) \times e^{-1}(e(t))$. Let $\overline{T}$ be the disjoint union of $\overline{e^{-1}(e(t))}$. Then one can define the product $\overline{m}$ on $\overline{T} \times \overline{T}$ such that $p_{st} \circ \overline{m}(\overline{s}, \overline{t}) = m(p_s(\overline{s}), p_t(\overline{t}))$ and $\overline{m}(\overline{e(s)}, \overline{e(t)}) = \overline{e(st)}$. It is proved in [12] that $(\overline{T}, \overline{m})$ is a top space. The mapping $p : \overline{T} \to T$ defined by $p(\overline{t}) = p_t(\overline{t})$ is a homomorphism of top spaces. The pair $(\overline{T}, p)$ is called the upper top space of $T$.

**Corollary 2.5** *If $(\overline{T}, \overline{e})$ is the universal covering space of $(T, e)$ where rank $e$ is constant, then using theorem 2.3, $(\overline{T}, \overline{e})$ is an upper top space.*

**Example 2.6.** This example shows that upper top space and universal covering space of a top space can be different.
Let $T = S^1 \times S^1 \times S^1$ with the following multiplication $m : T \times T \to T$ such that $m((e^{2\pi it}, e^{2\pi it'}, e^{2\pi it''}), (e^{2\pi is}, e^{2\pi is'}, e^{2\pi is''})) =$
$(e^{2\pi it}, e^{2\pi it'} p(e^{2\pi it''}, e^{2\pi is}) e^{2\pi is'}, e^{2\pi is''}) = (e^{2\pi it}, e^{2\pi i(t'+t''+s+s')}, e^{2\pi is''})$.
Where $p : S^1 \times S^1 \to S^1$ and $p(e^{2\pi it}, e^{2\pi it'}) = e^{2\pi i(t+t')}$. Its identity is $e(e^{2\pi it}, e^{2\pi it'}, e^{2\pi it''}) = (e^{2\pi it}, e^{-2\pi i(t+t'')}, e^{2\pi it''})$. *rank* $e = 2$ and that universal covering space is $R^3$ but its upper top space is $S^1 \times R \times S^1$.

## 3 Fundamental group

The fundamental group is a mathematical group associated with any topological space that provides a way to determine when two paths, starting and

ending at a fixed point, can be continuously deformed into each other. It records information about the basic shape of topological space. Fundamental group of Lie groups are studied in [7].
In this section we show that the fundamental group of a right top space is Abelian. We recall that if $(Y, q, y_0)$ is a covering space of $(X, x_0)$. A homeomorphism $\varphi : Y \to Y$ called a deck transformation if $q \circ \varphi = q$. Let $(\overline{X}, \overline{x_0})$ be the universal covering space of $(X, x_0)$. Then any loop $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$ can be lifted to the unique curve $\overline{\gamma} : [0,1] \to \overline{X}$ such that
(i)$\overline{\gamma}(0) = \overline{x_0}$;
(ii) $p \circ \overline{\gamma} = \gamma$.
The end point $\overline{\gamma}(1)$ of $\overline{\gamma}$ is in $p^{-1}(x_0)$. This map induces a bijection of $\pi_1(X, x_0)$ onto $p^{-1}(x_0)$. On the other hand, for any $x \in p^{-1}(x_0)$ there exists an unique deck transformation of $\overline{X}$ to $X$ which maps $\overline{x_0}$ into $x$. In this way, we construct a map from the fundamental group $\pi_1(X, x_0)$ onto the group of the deck transformations of $\overline{X}$. This map is a group isomorphism. Therefore, $\pi_1(X, x_0)$ acts on $\overline{X}$, and $X$ is the quotient of $\overline{X}$ with respect to this action.

**Theorem 3.1** *Let $T$ be a right top space then, $\pi_1(T, e) \cong p^{-1}\{e\}$ where $p$ is the covering map.*

*Proof.* Let $D$ be the collection of the deck transformations on $\overline{T}$ and $h : \pi_1(T, e) \to D$ be a map defined by $h(\gamma) = \overline{\gamma}(1)$ where $\overline{\gamma}$ is a loop on $\overline{T}$. $h$ is an isomorphism [7].
We define $g : p^{-1}\{e\} \to D$ by $\bar{t} \mapsto \overline{g}(\bar{t}) : \overline{T} \to \overline{T}$, where $\overline{g}(\bar{t})$ is the deck transformation associated with $\bar{t}$. It is enough to prove that $g$ is an isomorphism. $p : \overline{T} \to T$ is a covering map and $p \circ g(\bar{t}) = p$. On the other hand, $p \circ l_{\bar{t}} = p$. $g(\bar{t}) = l_{\bar{t}}$. Since $T$ is a right top space, $g(\overline{t_1 t_2})(\overline{e}) = l_{\overline{t_1 t_2}}(\overline{e}) = \overline{t_1 t_2} = l_{\overline{t_1}}(l_{\overline{t_2}}(\overline{e})) = g(\overline{t_1})g(\overline{t_2})(\overline{e})$. Therefore, $g$ is an isomorphism.
Define $f : p^{-1}\{e\} \to \pi_1(T, e)$ by $f(\bar{t}) = h^{-1} \circ g(\bar{t})$. It is clear that $f$ is an isomorphism and the proof is complete. □

**Theorem 3.2** *If $T$ is a right top space then $p^{-1}\{e\}$ is a Lie group.*

*Proof.* It is sufficient to show that $\overline{e}$ is unique identity element for $p^{-1}\{e\}$, where $p(\overline{e}) = e$. Suppose that $\overline{e_1} \in p^{-1}(e) \cap \overline{e}(\overline{T})$. Since $\overline{T}$ is a covering of $T$, $p \circ \overline{m}(\overline{e_1}, \bar{t}) = m \circ (p \times p)(\overline{e_1}, \bar{t}) = m(e, p(\bar{t}))$ and $p \circ \overline{m}(\overline{e}, \bar{t}) = m(e, p(\bar{t}))$. Hence, $\overline{m}(\overline{e_1}, .)$ and $\overline{m}(\overline{e}, .)$ are lifting of $m(e, p(.))$ and consequently are equal. Now we have $\overline{m}(\overline{e}, \overline{e_1}) = \overline{m}(\overline{e_1}, \overline{e_1}) = \overline{e_1}$ and $\overline{e_1} = \overline{e}$. Therefore, $p^{-1}\{e\}$ is a group. Since $p^{-1}\{e\}$ is closed so it is a Lie group. □

**Corollary 3.3** *If $T$ is a connected right top space then $p|\overline{e}(\overline{T}) : \overline{e}(\overline{T}) \to e(T)$ is a homeomorphism.*

**Corollary 3.4** *$p^{-1}\{e\} \subseteq \overline{e}\overline{T}$ which $\overline{e}\overline{T}$ is a cover of $eT$. Because $p^{-1}\{e\} = kerp|_{\overline{e}\overline{T}}$, it is a normal subgroup of $\overline{e}\overline{T}$. Hence, $p^{-1}\{e\}$ is a central subgroup of $\overline{e}\overline{T}$. Therefore, the fundumental group $\pi_1(T, e)$ is Abelian.*

## 4 Generalized left action of a top space

**Definition 4.1** *The top generalized subgroup $N$ of the top space $T$ is called a top generalized normal subgroup of $T$ if for every $a \in T$, $N_a$ is a normal subgroup of $e^{-1}(e(a))$, where $N_a := N \cap e^{-1}(e(a))$.*

*Example 4.1* Let $\varphi : T \to H$ be a morphism between the right top space $T$ and the Lie group $H$. Then, $ker\varphi$ is a normal subgroup of $T$.

**Definition 4.2** *A generalized left action of a top space $T$ on the manifold $M$ is the smooth map $\varphi : T \times M \to M$ such that:*
*1) for every $t_1$ and $t_2 \in T, \varphi(t_1, \varphi(t_2, m)) = \varphi(t_1 t_2, m)$;*
*2) for every $m \in M$ and $t \in T, \varphi(e(t), m) = m$.*

**Definition 4.3** *Let $\varphi$ be an action of the top space $T$ on the manifold $M$. Then $H(m) = \{t : \varphi(t, m) = m\}$ is called the stabilizer of $\varphi$.*

**Theorem 4.4** *Let $T$ be a top space which acts on the manifold $M$, then $H(m)$ is a top generalized normal subgroup of the top space $T$ for any $m \in T$.*

*Proof.* Since $e^{-1}(e(a))$ is a Lie group, $\varphi|e^{-1}(e(a)) : e^{-1}(e(a)) \times M \to M$ is an action of a Lie group on $M$ and consequently, $H(m) \cap e^{-1}(e(a))$ is the stabilizer of $\varphi|e^{-1}(e(a))$. Hence, it is a normal subtop space. □

**Remark 4.1** The differentiable map $\rho(m) : T \to M$ given by $\rho(m)(t) = tm$ is the orbit map of $m$. Its image is called the $T$-orbit of $m \in M$.

**Theorem 4.5** *Let $T$ be a top space. Then $e^{-1}(e(t))$ and $e^{-1}(e(s))$ are homeomorphic, for $t, s \in T$.*

*Proof.* The map $\varphi_{e(t)e(s)} : e^{-1}(e(t)) \to e^{-1}(e(s))$, defined by, $\varphi_{e(t)e(s)}(t') = e(s)t'e(s)$, is a homeomorphism. □

**Theorem 4.6** *Let $T$ be a right top space. Then for any $m \in M$, the orbit map $\rho(m) : T \to M$ has a constant rank.*

*Proof.* For any $t_1, t_2 \in T$ we have $(\tau_{t_1} \circ \rho(m))(t_2) = \tau_{t_1}(mt_2) = m(t_2 t_1) = \rho(m)(t_2 t_1) = (\rho(m) \circ r(t_1))(t_2)$. Therefore, $\tau(t_1) \circ \rho(m) = \rho(m) \circ r(t_1)$ for any $t_1 \in T$. If we calculate the derivative of this map at the identity in $T$, then

$$d_m(\tau(t)) \circ d_e(\rho(m)) = d_{et}(\rho(m)) \circ d_e(r(t))$$

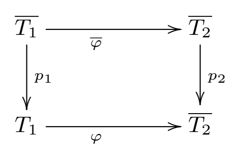
for any $t \in T$ and $e \in e(T)$. $\tau(t)$ and $r(t)$ are homeomorphisms. This implies $rankT_e(\rho(m)) = rankT_{et}(\rho(m))$. Since $T$ is a right top space, $T = e(T) \oplus tT$ for all $t \in T$. Thus, $d_s\rho(m) : e(T)_{e(s)} \oplus (tT)_{ts} \to M_{\varphi(s,m)}$ defined by $d_s\rho(m) = d_{e(s)}\rho(m) \oplus d_{ts}\rho(m)$. Since $\rho(m)$ on $e(T)$ is constant, then $d_{ts}\rho(m) = 0$. On the other hand, $e^{-1}(e(t))$ and $e^{-1}(e(s))$ are homeomorphic with function $f(t) = e(s)te(s)$ so, $\rho(m)(f(t)) = \varphi(f(t), m) = \varphi(e(s)te(s), m) = \varphi(e(s)t, m) = \varphi(e(s), \varphi(t, m)) = \varphi(t, m) = \rho(m)(t)$. Hence, $rankd_{e(s)}\rho(m)$ is constant. Therefore, $rankd_s\rho(m)$ is constant. □

**Theorem 4.7** *[7] If $d\varphi$ is a linear isomorphism on Lie groups, then $\varphi$ is a covering projection.*

**Theorem 4.8** *Let $\varphi : T_1 \to T_2$ be a top space homomorphism of connected right top spaces $T_1$ and $T_2$. Then*
*i)If $\varphi$ is a covering projection, then $d(\varphi) : T_{e_1} \to T_{e_2}$, where $e_1 \in e(T_1)$ and $e_2 \in e(T_2)$, are linear isomorphisms.*
*ii)If $d(\varphi) : T_{e_1} \to T_{e_2}$, where $e_1 \in e(T_1)$ and $e_2 \in e(T_2)$, are linear isomorphisms then $\varphi|e^{-1}(e(t_1))$ is a covering projection.*

*Proof.* i) If $\varphi$ is a covering projection then it is obvious that $d_e(\varphi)$ is an isomorphism.
ii) Since $\varphi$ is a homomorphism, $\varphi(e(T_1)) \subseteq e(T_2)$ and $\varphi(e_1T_1) \subseteq \varphi(e_1)T_2$. Since $T_1$ and $T_2$ are right top spaces, there are isomorphisms $f : T_1 \to e(T_1) \oplus e_1T_1$ and $g : T_2 \to e(T_2) \oplus \varphi(e_1)T_2$. For example, $f$ is defined by $f(t) = (e(t), e_1t)$. In addition, we define $\varphi_1 : e(T_1) \to e(T_2)$ and $\varphi_2 : e_1T_1 \to \varphi(e_1)T_2$ respectively by, $\varphi_1(e) = e(\varphi(e))$ and $\varphi_2(e_1t_1) = \varphi(e_1)\varphi(t_1)$. Now we have $g \circ \varphi = (\varphi_1, \varphi_2) \circ f$. Hence, $\varphi(e(t_1), e_1t_1) = (\varphi_1(e(t_1)), \varphi_2(e_1t_1))$, where $\varphi_1 : e(T_1) \to e(T_2)$ and $\varphi_2 : e_1T_1 \to e_2T_2$. using the assumption of the theorem $d\varphi_2$ is a linear isomorphism and consequently $\varphi_2$ is a covering projection. □

**Lemma 4.9** *Let $\varphi : T_1 \to T_2$ be a top space homomorphism from a simply connected right top space $T_1$ into a connected right top space $T_2$. Let $(\overline{T_1}, p_1)$ and $(\overline{T_2}, p_2)$ be the universal covering of $T_1$ and $T_2$ respectively. Then there exists an unique homomorphism $\overline{\varphi} : \overline{T_1} \to \overline{T_2}$ such that the following diagram commutes.*



*Proof.* Let $T_1$ and $T_2$ be connected right top spaces and $\varphi : T_1 \to T_2$ be a homomorphism. Assume that $T_1$ is simply connected. Thus, there is a unique lifting $\overline{\varphi} : \overline{T_1} \to \overline{T_2}$ for $\varphi \circ p_1$ such that $\overline{\varphi}(\overline{e_1}) = \overline{e_2}$, where $e_1 \in e(T_1)$ and $\overline{e_2} \in e(\overline{T_2})$. Since
$p_2 \circ \overline{m_2} \circ (\overline{\varphi} \times \overline{\varphi}) = m_2 \circ (p_2 \times p_2) \circ (\overline{\varphi} \times \overline{\varphi}) = m_2 \circ ((p_2 \circ \overline{\varphi}) \times \overline{\varphi})) = m_2 \circ (\varphi \times \varphi)$
$= \varphi \circ m_1 = p_2 \circ \overline{\varphi} \circ m_1$.
the maps $\overline{m} \circ (\overline{\varphi} \times \overline{\varphi})$ and $\overline{\varphi} \circ m$ are the lifts of the same map and they are equal at $(e_1, e_1)$. Consequently, they are identical. Therefore, $\overline{\varphi}$ is a top space homomorphism such that $p_2 \circ \overline{\varphi} = \varphi$. □

## 5 Covering of local top spaces

**Definition 5.1** *The smooth manifold $H$ called a local top space if there exists*

- *a set $e(H) \subset H$, the identity elements,*
- *a smooth product map $\mu : U \to H$ defined on an open subset $(e(H) \times H) \cup (H \times e(H)) \subset U \subset (H \times H)$,*
- *a smooth inversion map $i : V \to H$ defined on an open subset $e(H) \subset V \subset H$ such that $V \times i(V) \subset U$, and $i(V) \times V \subset U$,*

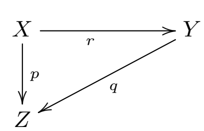
*all satisfying the following properties:*
*(i) Identity: For each $x \in H$ there is a unique element $e(x)$ such that $\mu(e(x), x) = x = \mu(x, e(x))$,*
*(ii) Inverse: $\mu(i(x), x) = \mu(x, i(x)) = e(x)$ for all $x \in V$,*
*(iii) Associativity: If $(x, y), (y, z), (\mu(x, y), z)$ and $(x, \mu(y, z))$ all belongs to $U$, then $\mu(\mu(x, y), z) = \mu(x, \mu(y, z))$,*
*(iv) $\mu(e(x), e(y)) = e(\mu(x, y))$, for each $x, y \in H$,*
*(v) $e : H \to H$ is a smooth map.*

One can use the symbol $(H, \mu, U, i, V)$ for a local top space $H$ with the functions $\mu$, $i$, $U$ and $V$ as the above definition [3].

**Remark 5.1** [3] 1) Using (i), (iii) and (iv) in the above definition, we have

$$\mu(x, \mu(e(x), e(x))) = \mu(\mu(x, e(x)), e(x)) = \mu(x, e(x)) = x.$$

Hence uniqueness of the identity element implies that $\mu(e(x), e(x)) = e(x)$ and consequently $e(e(x)) = e(x)$, for every $x \in H$.
2) $\mu(\mu(\mu(e(x), e(y)), e(x)), e(x)) = \mu(\mu(e(x), e(y)), \mu(e(x), e(x))) = \mu(\mu(e(x), e(y)), e(x))$.
Using (i) and (iv) in definition 2.1 implies that $\mu(\mu(e(x), e(y)), e(x)) = e(x)$.
3) $e(i(x)) = e(\mu(i(x), \mu(x, i(x)))) = \mu(e(i(x)), e(x)) = e(\mu(i(x)), x)) = e(e(x)) = e(x)$. Hence $e(i(x)) = e(x)$.




**Definition 5.2** *[3] Let $(H, \mu, U, i, V)$ and $(\widetilde{H}, \widetilde{\mu}, \widetilde{U}, \widetilde{i}, \widetilde{V})$ be local top spaces. A smooth map $\varphi : H \to \widetilde{H}$ is called a local top space homomorphism if*

- *$\varphi \times \varphi(U) \subset \widetilde{U}$, $\varphi(V) \subset \widetilde{V}$, $\varphi(e(t)) = \widetilde{e}(\varphi(t))$,*
- *$\varphi(\mu(g, h)) = \widetilde{\mu}(\varphi(g), \varphi(h))$ for $(g, h) \in U$,*
- *$\varphi(i(g)) = \widetilde{i}(\varphi(g))$ for $g \in V$.*

*A local top space homomorphism is called a local top space homeomorphism if it is one-to-one and onto with a smooth inverse.*

**Definition 5.3** *[3] The local top space $H$ is globalizable if there exists a homeomorphism $\varphi : H \to N$ which mapping $H$ onto a neighborhood $e(T) \subset N \subset T$ of the normal top space $T$.*

**Definition 5.4** *[3] The local top space $H$ is called globally inversional if the inversion map $i$ is defined everywhere, so that $V = H$.*

**Definition 5.5** *[3] A local top space $H$ is associative to order $n$ if, for every $3 \leq m \leq n$, and every ordered $m$-tuple of group elements $(x_1, ..., x_m) \in L^m$, all corresponding well defined $m$-fold products are equal. A local top space is called globally associative if it is associative to every order $n \geq 3$.*

**Lemma 5.6** *If $H$ is a local top space then $e^{-1}(e(t))$ is a local Lie group, for every $t \in H$.*

**Theorem 5.7** *[13] The local Lie group $L$ is globalizable if and only if it is globally associative.*

**Theorem 5.8** *The local top space $(H, \mu_H, U, i_H, V)$, that $e^{-1}(e(t))$ is homeomorphic with $p^{-1}(p(t'))$, for all $t, t' \in H$, is globalizable if and only if it is globally associative.*

*Proof.* If $H$ is globalizable, then it is globally associative since it is homeomorphic with an associative local top space. Conversely, if it is globally associative then $e^{-1}(e(t))$ is associative and consequently it is globalizable. Since it is globalizable, there is a neighborhood, $N$, of the identity element of a Lie group $G$ and a map $\varphi_0 : e^{-1}(e_i) \to N$. Now since $e^{-1}(e(t))$ is homeomorphic with $e^{-1}(e(t'))$, one can define a covering map from $H$ to $N$ by $\varphi(h) = \varphi_0(\varphi_{e(t)e(t')}(h))$, where $e(h) = e(t')$. By the above lemma, $e^{-1}(e(t))$ is a local Lie group. In addition, $\varphi|(e^{-1}(e(t)))$ is local homeomorphism, since $\varphi_{e(t)e(t')}$ and $\varphi_0$ are local homeomorphisms.
$\varphi \times \varphi(u_1, u_2) = (\varphi_0(\varphi_{ji}(u_1)), \varphi_0(\varphi_{ki}(u_2))) =$
$(\varphi_0(\mu_H(\mu_H(e_i, u_1), e_i)), \varphi_0(\mu_H(\mu_H(e_i, u_2), e_i))) \subseteq U_N$.
Let $e(t) = e_j$.
$\varphi_0(\varphi_{ji}(e(t))) = \varphi_0(\mu_H(\mu_H(e_i, e(t)), e_i)) = \mu(\mu(\varphi_0(e_i), \varphi_0(e(t))), \varphi_0(e_i)) =$
$\mu(\mu(e(\varphi_0(e_i)), e(\varphi_0(t)), e(\varphi_0(e_i))) =$
$e(\varphi_0(\mu_H(\mu_H((e_i, t), e_i)))) = e(\varphi(t))$.
If $e(h) = e_k$, $e(s) = e_l$ and $\mu(e_k, e_l) = e_j$ then
$\varphi(\mu(h, s)) = \varphi_0(\varphi_{ji}(\mu_H(h, s))) = \varphi_0(\mu_H(\varphi_{ji}(h), \varphi_{ji}(s))) =$
$\mu(\varphi(h), \varphi(s))$.
In addition, we have $\varphi(i_H(v)) = i(\varphi(v))$, which completes the proof. □

**Theorem 5.9** *Let $(H, e)$ be a local top space and $(\overline{H}, \overline{e})$ be its universal covering space. Then $\overline{H}$ is a local top space.*

**Lemma 5.10** *If the local top space $H$ is globally associative, invertional, connected, then $\overline{H}$ is globally associative, invertional, connected too.*

**Theorem 5.11** *Suppose that $H$ is a local top space such that $e^{-1}(e(t))$ is homeomorphic with $p^{-1}(p(t'))$, for all $t, t' \in H$. If $H$ is globalizable then $\overline{H}$ is globalizable.*

*Proof.* It is an immediate consequence of 5.9 and 5.11. □

## References


1. Ebrahimi, N. – *Left invariant vector fields of a class of top spaces*, Applied Sciences, **14** (2012), 37-44.
2. Ebrahimi, N. – *Exponential map of a class of top spaces*, Journal of Mahani Mathematical Research Center, **1** (2012), 47-54.
3. Ebrahimi, N. – *Local Lie groups and local top spaces*, Journal of the Egyptian Mathematical Society, **24** (2016), no. 1, 20-24.
4. Farhangdoost, M.R. – *Action of generalized Lie groups on manifolds*, Acta Mathematica Universitatis Comenianae (N.S.), **80** (2011), no. 2, 221-227.
5. Farhangdoost, M.R. – *Fiber bundles and Lie algebras of top spaces*, Bulletin of the Iranian Mathematical Society, **39** (2013), no. 4, 589-598.
6. Farhangdoost, M.R.; Molaei, M.R. – *Characterization of top spaces by diffeomorphic Lie groups*, Differential Geometry Dynamical Systems, **11** (2009), 130-134.
7. Miličić, D. – *Lectures on Lie groups*, University Of Chicago Press (January 15, 1983), 153 pp.
8. Molaei, M.R. – *Generalized groups*, International conference on Algebra, România, Buletinul Institutului Politehnic din Iaşi, Secţ. I. Mat. Mec. Teor. Fiz. **45**(49) (1999)/(2001), no. 3-4, 21-24.
9. Molaei, M.R. – *Top spaces*, Journal of Interdisciplinary Mathematics, **7** (2004), no. 2, 173-181.
10. Molaei, M.R.; Farhangdoost, M.R. – *Lie algebras of a class of top spaces*, Balkan Journal of Geometry and its Applications, **14** (2009), no. 1, 46-51.
11. Molaei, M.R.; Khadekar, G.S.; Farhangdoost, M.R. – *On top spaces*, Balkan Journal of Geometry and its Applications, **11** (2006), no. 2, 101-106.
12. Molaei, M.R.; Farhangdoost, M.R. – *Upper top spaces*, Applied Sciences, **8** (2006), no. 1, 128-131.
13. Olver, P.J. – *Non-associative local lie groups*, J. Lie Theory **6** (1996), no. 1, 23-51.
14. Warner, F.W. – *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, New York-Berlin (1983), 272 pp.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