An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIV, 2018, f. 2

The periods of the k-step Fibonacci and k-step Pell sequences in $D_{2n} \times \mathbb{Z}_{2^i}$ and $D_{2n} \times_\varphi \mathbb{Z}_{2^i}$

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Abstract The direct product $D_{2n} \times \mathbb{Z}_{2^i}$ and the semidirect product $D_{2n} \times_\varphi \mathbb{Z}_{2^i}$ for $n,\ i \geq 3$ are defined by the presentations

$$D_{2n} \times \mathbb{Z}_{2^i} = \left\langle x, y, z : x^2 = y^n = (xy)^2 = z^{2^i} = [x, z] = [y, z] = e \right\rangle$$

and

$$D_{2n} \times_\varphi \mathbb{Z}_{2^i} = \left\langle x, y, z : x^2 = y^n = (xy)^2 = z^{2^i} = e,\ z^{-1}xzx = e,\ z^{-1}yzy = e \right\rangle,$$

respectively. In this paper, we obtain the periods of the k-nacci sequences and the generalized order-k Pell sequences in the direct product $D_{2n} \times \mathbb{Z}_{2^i}$ and the semidirect product $D_{2n} \times_\varphi \mathbb{Z}_{2^i}$ for $n,\ i \geq 3$.

Keywords Period · sequence · direct product · semidirect product

Mathematics Subject Classification (2010) 20F05 · 20D60 · 11K31 · 11B39

1 Introduction and preliminaries

The recurrence sequences in groups was firstly studied by Wall [18] who calculated the periods of the Fibonacci sequences in cyclic groups. As a natural generalization of the problem, Wilcox [19] investigated the Fibonacci lenghts to abelian groups. In [3] the Fibonacci length of a 2-generator group is defined, thus extending the idea of forming a sequence of group elements based on a Fibonacci-like recurrence relation first introduced by Wall in [18]. The theory has been expanded to nilpotent groups, see for example [1,16]. Campbell *et al.* [2] examined the Fibonacci lengths of $D_{2n}^i,\ i > 1$ where D_{2n} is the dihedral group of order $2n$. Knox [14] proved that the periods of the k-nacci (k-step Fibonacci) sequences in dihedral groups were equal to $2k+2$. Lü and Wang [15] contributed to the study of the Wall number for the k-step

Fibonacci sequence. Falcon and Plaza [10] examined the periods k-Fibonacci sequences modulo m. Deveci and Karaduman [8] calculated the periods of the k-nacci sequences in the semidircet product $Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m}$. Deveci and Karaduman [6] expanded the concept to Pell sequences. Recently, many authors have studied some special linear recurrence sequences in groups; see for example, [5,7,12,17].

Let $f_n^{(k)}$ denote the nth number of the k-step Fibonacci sequence defined as

$$f_n^{(k)} = \sum_{j=1}^{k} f_{n-j}^{(k)} \text{ for } n > k \tag{1.1}$$

with boundary conditions $f_i^{(k)} = 0$ for $1 \leq i < k$ and $f_k^{(k)} = 1$. Reducing this sequence by a modulus m, we can get a repeating sequence, which we denote by

$$f(k,m) = \left(f_1^{(k,m)},\ f_2^{(k,m)}, \ldots,\ f_n^{(k,m)}, \ldots \right),$$

where $f_n^{(k,m)} = f_n^{(k)}$ $(mod\ m)$. We then have that $\left(f_1^{(k,m)}, f_2^{(k,m)},\ \ldots, f_k^{(k,m)} \right)$ $= (0, 0, \ldots 0, 1)$ and it has the same recurrence relation as in (1.1).
For more information see [15].

Theorem 1.1 *$f(k,m)$ is a periodic sequence [15].*

Let $h_k(m)$ denote the smallest period of $f(k,m)$, called the period of $f(k,m)$ or the Wall number of the k-step Fibonacci sequence modulo m. It is important to note that $h_k(2) = k+1$.
For more information see [15].

Definition 1.2 *A k-nacci sequence in a finite group is a sequence of group elements $x_0, x_1, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, \ldots, x_{j-1}$, each element is defined by*

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \leq n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \geq k. \end{cases}$$

We also require that the initial elements of the sequence, $x_0, \ldots, x_{j-1}$, generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group generated by $x_0, \ldots, x_{j-1}$ is denoted by $F_k(G; x_0, \ldots, x_{j-1})$ [14].
It is well-known that a sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence.

Theorem 1.3 *A k-nacci sequence in finite group is simply periodic [14].*

In [14], Knox had denoted the period of the sequence $F_k(G; x_0, \ldots, x_{j-1})$ by $P_k(G; x_0, \ldots, x_{j-1})$.
In [13], Kiliç and Taşci defined the k sequences of the generalized order-k Pell numbers as follows:
for $n > 0$ and $1 \leq$ i $\leq k$

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + \cdots + P_{n-k}^i, \tag{1.2}$$

with initial conditions

$$P_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

where P_n^i is the $n$thterm of the ith sequence. If $k = 2$, the generalized order-k Pell sequence, $\left\{P_n^k\right\}$, is reduced to the usual Pell sequence, $\{P_n\}$.
Reducing the generalized order-k Pell sequence by a modulus m, we can get a repeating sequence, denoted by

$$\left\{P^{k,m}\right\} = \left\{P_{1-k}^{k,m}, P_{2-k}^{k,m}, \ldots, P_0^{k,m}, P_1^{k,m}, P_2^{k,m}, \ldots, P_n^{k,m}, \ldots\right\},$$

where $P_n^{k,m} = P_n^k \pmod{m}$. It has the same recurrence relation as in (1.2). For more information see [6].

Theorem 1.4 $\left\{P^{k,m}\right\}$*is a periodic sequence [6].*

Let the notation $hP_k(m)$ denotes the smallest period of $\left\{P^{k,m}\right\}$, called the period of the generalized order-k Pell sequence modulo m. When $k = 2$, $hP_2(m)$ is the period of the Pell sequence modulo m [6].

Definition 1.5 *A generalized order-k Pell sequence in a finite group is a sequence of group elements $x_0, x_1, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, \ldots, x_{j-1}$, each element is defined by*

$$x_n = \begin{cases} x_0 x_1 \cdots (x_{n-1})^2 & \text{for } j \leq n < k, \\ x_{n-k} x_{n-k+1} \cdots (x_{n-1})^2 & \text{for } n \geq k. \end{cases}$$

It is required that the initial elements of the sequence, $x_0, \ldots, x_{j-1}$, generate the group, thus, forcing the generalized order-k Pell sequence to reflect the structure of the group. We denote the generalized order-k Pell sequence of a group G generated by $x_0, \ldots, x_{j-1}$ by $Q_k(G; x_0, \ldots, x_{j-1})$ [6].

Theorem 1.6 *A generalized order-k Pell sequence in a finite group is periodic [6].*

In [6], Deveci and Karaduman had denoted the period of the sequence $Q_k(G; x_0, \ldots, x_{j-1})$ by $PerQ_k(G; x_0, \ldots, x_{j-1})$.
From the definitions it is clear that the periods of both the k-nacci sequence and the generalized order-k Pell sequence in a group depend on the chosen generating set and the order in which the assignments of $x_0, x_1, x_2, \ldots, x_{j-1}$ are made.

2 Main results and proofs

We use the natural generating set for D_{2n}, as in [4], defined as satisfying $D_{2n} = \langle x, y : x^2 = y^n = (xy)^2 = e \rangle$. This is extended to direct product by using the following well known method of construction:
If $G_1 = \langle A : R_1 \rangle$ and $G_2 = \langle B : R_2 \rangle$, then $G_1 \times G_2 = \langle A, B : R_1,\ R_2,\ [A, B] \rangle$ where $[A, B] = \{[a, b] : a \in A,\ b \in B\}$, see [11].
The direct product $D_{2n} \times \mathbb{Z}_{2^i}$, $(n,\ i \geq 3)$ is defined by the presentation

$$D_{2n} \times \mathbb{Z}_{2^i} = \left\langle x, y, z : x^2 = y^n = (xy)^2 = z^{2^i} = [x, z] = [y, z] = e \right\rangle .$$

The usual notation $G_1 \times_\varphi G_2$ is used for the semidirect product of the group G_1 by G_2, where $\varphi : G_2 \to \mathrm{Aut}\,(G_1)$ is a homomorphism such that $b\varphi = \varphi_b$ and $\varphi_b : G_1 \to G_1$ is an element $\mathrm{Aut}\,(G_1)$.
The semidirect product $D_{2n} \times_\varphi \mathbb{Z}_{2^i}$, $(n,\ i \geq 3)$ is defined by the presentation

$$D_{2n} \times_\varphi \mathbb{Z}_{2^i} = \left\langle x, y, z : x^2 = y^n = (xy)^2 = z^{2^i} = e,\ z^{-1}xzx = e,\ z^{-1}yzy = e \right\rangle .$$

Where if $\mathbb{Z}_{2^i} = \langle z \rangle$, then $\varphi : \mathbb{Z}_{2^i} \to \mathrm{Aut}\,(D_{2n})$ is a homomorphism such that $z\varphi = \varphi_z$; $\varphi_z : D_{2n} \to D_{2n}$ is defined by $x\varphi_z = x$ and $y\varphi_z = y^{-1}$.
For more information see [9].

Theorem 2.1 *The periods of the k-nacci sequences in the direct product* $D_{2n} \times \mathbb{Z}_{2^i}$, $(n,\ i \geq 3)$ *are as follows:*

i. $P_2(D_{2n} \times \mathbb{Z}_{2^i}; x, y, z) = \mathrm{lcm}\left[h_2\left(2^i\right), h_2\left(n\right)\right]$*, the least common multiple of the* $h_2\left(2^i\right)$ *and the* $h_2\left(n\right)$*.*

ii. *If* $k \geq 3$*, then* $P_k(D_{2n} \times \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^i,\, n\right](k+1)}{2}$.

Proof. We first note that in the group defined

$$\left\langle x, y, z : x^2 = y^n = (xy)^2 = z^{2^i} = [x, z] = [y, z] = e \right\rangle ,$$

$xz = zx$ and $yz = zy$.

i. If $k = 2$, we have the sequence

$$x_0 = x,\ x_1 = y,\ x_2 = z,\ x_3 = yz,\ x_4 = z^2y,\ x_5 = z^3y^2,$$
$$x_6 = z^5y^3,\ x_7 = z^8y^5,\ x_8 = z^{13}y^8,\ x_9 = z^{21}y^{13},\ x_{10} = z^{34}y^{21}, \ldots .$$

From this sequence we obtain a subsequence as follow:

$$a_1 = y,\ a_2 = z,\ a_3 = yz,\ a_4 = z^2y,\ a_5 = z^3y^2, a_6 = z^5y^3,$$
$$a_7 = z^8y^5,\ a_8 = z^{13}y^8,\ a_9 = z^{21}y^{13},\ a_{10} = z^{34}y^{21}, \ldots .$$

In fact it is easy to see that the 2-nacci sequence conforms to the following pattern:

$$a_{t+1} = z^{f_{t+1}^{(2)}} y^{f_t^{(2)}},$$
$$a_{t+2} = z^{f_{t+2}^{(2)}} y^{f_{t+1}^{(2)}}.$$

We need the smallest t, satisfying $a_{t+1} = y$ and $a_{t+2} = z$. Letting

$$\mathrm{lcm}\left[h_2\left(2^i\right), h_2\left(n\right)\right] = \alpha,$$

then we have $2^i \big| f^{(2)}_{\alpha+1}$, $n \big| f^{(2)}_{\alpha+1}$, $f^{(2)}_{\alpha} \equiv 1 \bmod 2^i$, $f^{(2)}_{\alpha} \equiv 1 \bmod n$, $f^{(2)}_{\alpha+2} \equiv 1 \bmod 2^i$ and $f^{(2)}_{\alpha+2} \equiv 1 \bmod n$ (where by $2^i \big| f^{(2)}_{\alpha+1}$ we mean that 2^i divides $f^{(2)}_{\alpha+1}$). If we choose $t = \alpha$, then we obtain $a_{\alpha+1} = y$ and $a_{\alpha+2} = z$. So we get $P_2(D_{2n} \times \mathbb{Z}_{2^i}; x, y, z) = \mathrm{lcm}\left[h_2\left(2^i\right), h_2\left(n\right)\right]$.

ii. If $k = 3$, then we have the sequence

$$\begin{aligned}
&x_0 = x,\ x_1 = y,\ x_2 = z,\ x_3 = xyz,\ x_4 = z^2 x,\\
&x_5 = z^4 y^{-1},\ x_6 = z^7 y^{-2},\ x_7 = z^{13} y^3 x,\ x_8 = z^{24} x,\\
&x_9 = z^{44} y,\ x_{10} = z^{81} y^4, \ldots, x_{16} = z^{3136} x,\\
&x_{17} = z^{5768} y,\ x_{18} = z^{10609} y^8, \ldots, x_{24} = z^{410744} x,\\
&x_{25} = z^{755476} y,\ x_{26} = z^{1389537} y^{12}, \ldots, x_{32} = z^{53798080} x,\\
&x_{33} = z^{98950096} y,\ x_{34} = z^{181997601} y^{18}, \ldots .
\end{aligned}$$

Using the above, the 3-nacci sequence becomes:

$$x_{\alpha 8} = z^{\frac{\alpha}{t} 4u_\alpha} x,\ x_{\alpha 8+1} = z^{\frac{\alpha}{t} 4u_{\alpha+1}} y,\ x_{\alpha 8+2} = z^{\frac{\alpha}{t} 4u_{\alpha+2}+1} y^{4\alpha}, \ldots,$$

Where t is odd and $\alpha \in \mathbb{N}$ such that $\alpha = 2^\sigma.t$ and $u_\alpha,\ u_{\alpha+1},\ u_{\alpha+2} \in \mathbb{N}$ such that $\mathrm{lcm}\left[u_\alpha,\ u_{\alpha+1},\ u_{\alpha+2}\right] = 1$. We need the smallest α, satisfying $x_{\alpha 8} = x$, $x_{\alpha 8+1} = y$ and $x_{\alpha 8+2} = z$. If we choose $\alpha = \frac{\mathrm{lcm}[2^i,\, n]}{4}$, then we obtain $x_{2\mathrm{lcm}[2^i,\, n]} = x$, $x_{2\mathrm{lcm}[2^i,\, n]+1} = y$ and $x_{2\mathrm{lcm}[2^i,\, n]+2} = z$. So we get $P_k(D_{2n} \times \mathbb{Z}_{2^i}; x, y, z) = 2\mathrm{lcm}\left[2^{\mathrm{i}},\ n\right]$.

If $k \geq 4$, we have the sequence

$$\begin{aligned}
&x_0 = x,\ x_1 = y,\ x_2 = z,\ x_3 = xyz,\\
&x_4 = z^2, \ldots,\ x_k = z^{2^{k-3}},\ x_{k+1} = z^{2^{k-2}} x,\\
&x_{k+2} = z^{2^{k-1}} y^{-1}, x_{k+3} = z^{2^k - 1} y^{-2},\ x_{k+4} = z^{2^{k+1}-3} y^3 x,\\
&x_{k+5} = z^{2^{k+2}-2.4},\ x_{k+6} = z^{2^{k+3}-2^2.5}, \ldots,\\
&x_{2k+1} = z^{2^{2k-2}-2^{k-3}.k},\ x_{2k+2} = z^{2^{2k-1}-2^{k-2}.(k+1)} x,\\
&x_{2k+3} = z^{2^{2k}-2^{k-1}.(k+2)} y, x_{2k+4} = z^{2^{2k+1}-2^k.(k+3)+1} y^4, \ldots .
\end{aligned}$$

Using the above, the k-nacci sequence becomes:

$$\begin{aligned}
&x_{\alpha(2k+2)-k+3} = z^{\frac{\alpha}{t} 4\tau_\alpha},\ x_{\alpha(2k+2)-k+4} = z^{\frac{\alpha}{t} 4\tau_{\alpha+1}}, \ldots,\\
&x_{\alpha(2k+2)-1} = z^{\frac{\alpha}{t} 4\tau_{\alpha+k-4}}, x_{\alpha(2k+2)} = z^{\frac{\alpha}{t} 4\varepsilon_\alpha} x,\\
&x_{\alpha(2k+2)+1} = z^{\frac{\alpha}{t} 4\varepsilon_{\alpha+1}} y,\ x_{\alpha(2k+2)+2} = z^{\frac{\alpha}{t} 4\varepsilon_{\alpha+2}+1} y^{4\alpha}, \ldots,
\end{aligned}$$

where t is a positive odd integer and α is a positive integer such that $\alpha = 2^\sigma.t$,

$$\tau_\alpha,\ \tau_{\alpha+1}, \ldots,\ \tau_{\alpha+k-4},\ \varepsilon_\alpha,\ \varepsilon_{\alpha+1},\ \varepsilon_{\alpha+2} \in \mathbb{N}$$

and

$$\mathrm{lcm}\left[\tau_\alpha,\ \tau_{\alpha+1}, \ldots,\ \tau_{\alpha+k-4},\ \varepsilon_\alpha,\ \varepsilon_{\alpha+1},\ \varepsilon_{\alpha+2}\right] = 1.$$

We need the smallest α, satisfying $x_{\alpha(2k+2)-l}=e,\ x_{\alpha(2k+2)}=x,\ x_{\alpha(2k+2)+1} = y$ and $x_{\alpha(2k+2)+2} = z$, where $k-3 \leq l \leq 1$. If we choose $\alpha = \frac{\mathrm{lcm}[2^i,\, n]}{4}$, then we obtain

$$x_{\frac{\mathrm{lcm}[2^i,\, n](k+1)}{2}-l} = e,\ x_{\frac{\mathrm{lcm}[2^i,\, n](k+1)}{2}} = x,$$
$$x_{\frac{\mathrm{lcm}[2^i,\, n](k+1)}{2}+1} = y \text{ and } x_{\frac{\mathrm{lcm}[2^i,\, n](k+1)}{2}+2} = z.$$

So we get $P_k(D_{2n} \times \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^i,\, n\right](k+1)}{2}$. □

Theorem 2.2 *The periods of the k-nacci sequences in the semidirect product $D_{2n} \times_\varphi \mathbb{Z}_{2^i}$, $(n,\ i \geq 3)$ are as follows:*

i. $P_{2,3}(D_{2n} \times_\varphi \mathbb{Z}_{2^i}; x, y, z) = h_{2,3}\left(2^i\right)$

ii. $P_{4,5}(D_{2n} \times_\varphi \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^i,\, n\right](k+1)}{2}$.

iii. *Let $k \geq 6$.*

i'. *If there is no $\omega \in [3,\ k-3]$ such that ω is an odd factor of n, then $P_k(D_{2n} \times_\varphi \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^i,\, n\right](k+1)}{2}$.*

ii'. *Let λ be the biggest odd factor of n in $[3,\ k-3]$, then two cases occur:*

1. *If $\lambda 3^j \notin [3,\ k-3]$ for $j \in \mathbb{N}$, then*

$$P_k(D_{2n} \times_\varphi \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^i,\ n\right](k+1)\,\lambda}{2}.$$

2. *If μ is the biggest odd number which is in $[3,\ k-3]$ and $\mu = \lambda 3^j$ for $j \in \mathbb{N}$, then*

$$P_k(D_{2n} \times_\varphi \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^i,\ n\right](k+1)\,\mu}{2}.$$

Proof. We first note that in the group defined

$$D_{2n} \times_\varphi \mathbb{Z}_{2^i} = \left\langle x, y, z : x^2 = y^n = (xy)^2 = z^{2^i} = e,\ z^{-1}xzx = e,\ z^{-1}yzy = e \right\rangle,$$

$xy = y^{-1}x,\ \ xz = zx$ and $yz = zy^{-1}$.

i. If $k = 2$, we have the sequence

$$x_0 = x,\ x_1 = y,\ x_2 = z,\ x_3 = yz,\ x_4 = z^2y^{-1},$$
$$x_5 = z^3y^{-2},\ x_6 = z^5y^{-1},\ x_7 = z^8y,\ x_8 = z^{13},\ x_9 = yz^{21}, \ldots.$$

The 2-nacci sequence can be said to form layers of length 6. Using the above, the 2-nacci sequence becomes:

$$x_{6\alpha+1} = z^{f^{(2)}_{6\alpha+1}}y,\ x_{6\alpha+2} = z^{f^{(2)}_{6\alpha+2}}, \ldots.$$

We need the smallest α, satisfying $x_{6\alpha+1} = y$ and $x_{6\alpha+2} = z$. If we choose $\alpha = 2^{i-2}$, we obtain that

$$x_{6.2^{i-2}+1} = z^{f^{(2)}_{6.2^{i-2}+1}} y,\ x_{6.2^{i-2}+2} = z^{f^{(2)}_{6.2^{i-2}+2}} \ldots .$$

Since $h_2\left(2^i\right) = 2^{i-1}.h_2(2) = 2^{i-1}.3 = 6.2^{i-2}$, we have $2^i \big| f^{(2)}_{6.2^{i-2}+1}$ and $f^{(2)}_{6.2^{i-2}+2} \equiv 1 \mod 2^i$. Thus, $x_{6.2^{i-2}+1} = y$ and $x_{6.2^{i-2}+2} = z$ that is $P_2(D_{2n} \times_\varphi \mathbb{Z}_{2^i}; x, y, z) = h_2\left(2^i\right) = 2^{i-1}.3$.

If $k = 3$, we have the sequence

$$\begin{array}{l} x_0 = x,\ x_1 = y,\ x_2 = z,\ x_3 = xyz,\ x_4 = z^2xy^{-2}, \\ x_5 = z^4y^{-1},\ x_6 = z^7y^{-2},\ x_7 = yxz^{13},\ x_8 = z^{24}x, \\ x_9 = z^{44}y,\ x_{10} = z^{81},\ x_{11} = xyz^{149}, \ldots . \end{array}$$

The 3-nacci sequence can be said to form layers of length 8. Using the above, the 3-nacci sequence becomes:

$$x_{8\alpha} = z^{f^{(3)}_{8\alpha+1}} x,\ x_{8\alpha+1} = z^{f^{(3)}_{8\alpha+2}} y,\ x_{8\alpha+2} = z^{f^{(3)}_{8\alpha+3}}, \ldots .$$

We need the smallest α, satisfying $x_{8\alpha} = x,\ x_{8\alpha+1} = y$ and $x_{8\alpha+2} = z$. If we choose $\alpha = 2^{i-2}$, we obtain that

$$x_{8.2^{i-2}} = z^{f^{(3)}_{8.2^{i-2}+1}} x,\ x_{8.2^{i-2}+1} = z^{f^{(3)}_{8.2^{i-2}+2}} y,\ x_{8.2^{i-2}+2} = z^{f^{(3)}_{8.2^{i-2}+3}} \ldots .$$

Since

$$h_3\left(2^i\right) = 2^{i-1}.h_3(2) = 2^{i-1}.4 = 8.2^{i-2},$$

we have $2^i \big| f^{(3)}_{8.2^{i-2}+1},\ 2^i \big| f^{(3)}_{8.2^{i-2}+2}$ and $f^{(3)}_{8.2^{i-2}+3} \equiv 1 \bmod 2^i$. Thus, $x_{8.2^{i-2}} = x,\ x_{8.2^{i-2}+1} = y$ and $x_{8.2^{i-2}+2} = z$ that is $P_3(D_{2n} \times_\varphi \mathbb{Z}_{2^i}; x, y, z) = h_3\left(2^i\right) = 2^{i-1}.4$.

ii. The proof is similar to the proof of Theorem 2.1.ii. and is omitted.

iii. If $k \geq 6$, we have the sequence

$$\begin{array}{l} x_0 = x,\ x_1 = y,\ x_2 = z,\ x_3 = xyz, \\ x_4 = z^2y^{-2},\ x_5 = z^4y^{-4},\ x_6 = z^8y^{-8}, \ldots, \\ x_k = z^{2^{2^{k-3}}} y^{-2^{2^{k-3}}},\ x_{k+1} = xy^{-2^{2^{k-2}}} z^{2^{2^{k-2}}},\ x_{k+2} = z^{2^{2^{k-1}}} y^{-1}, \\ x_{k+3} = z^{2^k-1}y^{-2},\ x_{k+4} = xz^{2^{k+1}-3}y, \\ x_{k+5} = z^{2^{k+2}-2.4}y^4,\ x_{k+6} = z^{2^{k+3}-2^2.5}y^{12}, \ldots, \\ x_{2k+1} = z^{2^{2k-2}-2^{k-3}.k} y^{4\vartheta_1},\ x_{2k+2} = z^{2^{2k-1}-2^{k-2}.(k+1)} xy^{4\vartheta_2}, \\ x_{2k+3} = z^{2^{2k}-2^{k-1}.(k+2)} y,\ x_{2k+4} = z^{2^{2k+1}-2^k.(k+3)+1} z, \ldots, \end{array}$$

where $\vartheta_1,\ \vartheta_2 \in \mathbb{N}$. Using the above, the $k-nacci$ sequence becomes:

$$\begin{aligned}
x_{\alpha(2k+2)-k+3} &= z^{\frac{\alpha}{t}4\beta_{\alpha}}y^{4\alpha},\\
x_{\alpha(2k+2)-k+4} &= z^{\alpha 4\beta_{\alpha+1}}y^{8\alpha^2+4\alpha},\\
x_{\alpha(2k+2)-k+5} &= z^{\alpha 4\beta_{\alpha+2}}y^{4v_{\alpha}},\ldots,\\
x_{\alpha(2k+2)-1} &= z^{\alpha 4\beta_{\alpha+k-4}}y^{4v_{\alpha+k-6}},\\
x_{\alpha(2k+2)} &= z^{\alpha 4\beta_{\alpha+k-3}}xy^{4v_{\alpha+k-5}},\\
x_{\alpha(2k+2)+1} &= z^{\alpha 4\beta_{\alpha+k-2}}y,\\
x_{\alpha(2k+2)+2} &= z^{\alpha 4\beta_{\alpha+k-1}}z,\ldots,
\end{aligned}$$

where t is a positive odd integer and α is a positive integer such that $\alpha= 2^{\sigma}.t$,

$$\beta_{\alpha},\ \beta_{\alpha+1},\ldots,\beta_{\alpha+k-1},\ v_{\alpha},\ v_{\alpha+1},\ldots,v_{\alpha+k-5} \in \mathbb{N},$$

$$\mathrm{lcm}\left[\beta_{\alpha},\ \beta_{\alpha+1},\ldots,\beta_{\alpha+k-1}\right] = 1$$

and

$$\mathrm{lcm}\left[v_{\alpha},\ v_{\alpha+1},\ldots,v_{\alpha+k-5}\right] = 1.$$

We need the smallest α, satisfying $x_{\alpha(2k+2)-l} = e,\ x_{\alpha(2k+2)} = x,\ x_{\alpha(2k+2)+1} = y$ and $x_{\alpha(2k+2)+2} = z$, where $k-3 \le l \le 1$.

i'. The proof is similar to the proof of Theorem 2.2.ii. and is omitted.

ii'. Let λ be the biggest odd factor of n is $[3,\ k-3]$, then two cases occur:

1. If $\lambda 3^j \notin [3,\ k-3]$ for $j \in \mathbb{N}$, then we obtain

$$x_{\frac{\mathrm{lcm}[2^{i},\ n](k+1)\lambda}{2}-l} = e,\ x_{\frac{\mathrm{lcm}[2^{i},\ n](k+1)\lambda}{2}} = x,$$
$$x_{\frac{\mathrm{lcm}[2^{i},\ n](k+1)\lambda}{2}+1} = y \text{ and } x_{\frac{\mathrm{lcm}[2^{i},\ n](k+1)\lambda}{2}+2} = z$$

for $\alpha = \frac{\mathrm{lcm}[2^{i},\ n]\lambda}{4}$. So we get $P_k(D_{2n} \times \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^{i},\ n\right](k+1)\lambda}{2}$.

2. If μ is the biggest odd number which is in $[3,\ k-3]$ and $\mu = \lambda.3^j$ for $j \in \mathbb{N}$, then we obtain

$$x_{\frac{\mathrm{lcm}[2^{i},\ n](k+1)\mu}{2}-l} = e,\ x_{\frac{\mathrm{lcm}[2^{i},\ n](k+1)\mu}{2}} = x,$$
$$x_{\frac{\mathrm{lcm}[2^{i},\ n](k+1)\mu}{2}+1} = y \text{ and } x_{\frac{\mathrm{lcm}[2^{i},\ n](k+1)\mu}{2}+2} = z$$

for $\alpha = \frac{\mathrm{lcm}[2^{i},\ n]\mu}{4}$. So we get $P_k(D_{2n} \times \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^{i},\ n\right](k+1)\mu}{2}$. $\square$

Theorem 2.3 *The periods of the generalized order-k Pell sequences in the direct product $D_{2n} \times \mathbb{Z}_{2^i}$, $(n,\ i \ge 3)$ are as follows:*

i. $PerQ_2(D_{2n} \times \mathbb{Z}_{2^i}; x, y, z) = \mathrm{lcm}\left[hP_2\left(2^{i}\right), hP_2\left(n\right)\right]$.

ii. *If $k \ge 3$, then* $PerQ_k(D_{2n} \times \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^{i},\ n\right]hP_k(2)}{2}$.

Theorem 2.4 *The periods of the generalized order-k Pell sequences in the semidirect product $D_{2n} \times_{\varphi} \mathbb{Z}_{2^i}$, $(n,\ i \geq 3)$ are as follows:*

i. $PerQ_{2,3}(D_{2n} \times_{\varphi} \mathbb{Z}_{2^i}; x, y, z) = hP_{2,3}\left(2^i\right)$

ii. $PerQ_{4,5}(D_{2n} \times_{\varphi} \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^{\mathrm{i}},\ n\right] hP_k(2)}{2}$.

iii. *Let $k \geq 6$.*

i'. *If there is no $\omega \in [3,\ k-3]$ such that ω is an odd factor of n, then*

$$PerQ_k(D_{2n} \times_{\varphi} \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^{\mathrm{i}},\ n\right] hP_k\left(2\right)}{2}.$$

ii'. *Let λ be the biggest odd factor of n in $[3,\ k-3]$, then two cases occur:*

1. *If $\lambda 3^j \notin [3,\ k-3]$ for $j \in \mathbb{N}$, then $PerQ_k(D_{2n} \times_{\varphi} \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^{\mathrm{i}},\ n\right] hP_k(2)\lambda}{2}$.*

2. *If μ is the biggest odd number which is in $[3,\ k-3]$ and $\mu = \lambda 3^j$ for $j \in \mathbb{N}$, then*

$$PerQ_k(D_{2n} \times_{\varphi} \mathbb{Z}_{2^i}; x, y, z) = \frac{\mathrm{lcm}\left[2^{\mathrm{i}},\ n\right] hP_k\left(2\right)\mu}{2}.$$

The proofs of the Theorem 2.3 and Theorem 2.4 are similar to the proofs of Theorem 2.1 and Theorem 2.2, respectively and are omitted.

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Received: 25.I.2017 / Accepted: 21.II.2017

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