An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIV, 2018, f. 2

On matrices with the Pell, Pell-Lucas, $k-$Pell and $k-$Pell-Lucas quaternions

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Abstract In this paper, for any positive real number k, we introduce the $k-$Pell and $k-$Pell-Lucas quaternions and we consider certain matrices whose entries are Pell, Pell-Lucas, $k-$Pell and $k-$Pell-Lucas quaternions. We investigate the $g-$circulant, right circulant, left circulant and a special kind of a tridiagonal matrices whose entries are elements of these sequences. We present the determinant of these matrices and with the tridiagonal matrices we show that the determinant is equal to the nth term of the Pell, Pell-Lucas, $k-$Pell and $k-$Pell-Lucas quaternion sequences.

Keywords Pell numbers · Pell-Lucas numbers · Quaternions · Tridiagonal matrices · Circulant matrices

Mathematics Subject Classification (2010) 11B39 · 11C20 · 11B37 · 11R52

1 Introduction and background

Recently we find in the literature several research involving the study of sequences of quaternions defined by recurrence relations of second order. For example, the Fibonacci quaternions have been studied in several papers (see, for example, [15], [18], [20] and [21]). Also some generalizations of the Fibonacci quaternions have been presented in the literature (see, for example, [1], [11], [27] and [32]). Properties of the Fibonacci and Lucas quaternions can be found in [16] and [21] and different types of sequences of quaternions have been exploited by several authors (see, for example, [5], [11], [15], [26], [27] and [31]).

The Pell sequence $\{P_n\}_{n=0}^{\infty}$ is a sequence of nonnegative integers defined recursively by

$$P_{n+1} = 2P_n + P_{n-1}, \tag{1.1}$$

with the initial conditions $P_0 = 0$ and $P_1 = 1$.

Also the Pell-Lucas sequence $\{Q_n\}_{n=0}^{\infty}$ is a sequence of nonnegative integers defined by

$$Q_{n+1} = 2Q_n + Q_{n-1},$$

with the initial conditions $Q_0 = Q_1 = 2$. The Pell and Pell-Lucas sequences have also been studied by several authors (see, for example, [2], [8], [14] and [17]).

For any positive real number k, a generalization of these sequences are the $k-$Pell and $k-$Pell-Lucas sequences defined by

$$P_{k,0} = 0,\ P_{k,1} = 1,\ P_{k,n+1} = 2P_{k,n} + kP_{k,n-1},\ n \geq 1,$$

$$Q_{k,0} = Q_{k,1} = 2,\ Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1},\ n \geq 1,$$

respectively. For more details about these sequences see [4], [6] and [8].

The quaternion was formally introduced by W. R. Hamilton in 1843 and some background about this type of hypercomplex numbers can be found, for example, in [7] and [36]. A quaternion q is defined by

$$q = q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3,$$

where $q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{R}$ and $i_1,\ i_2$ and i_3 are complex operators such that $i_1^2 = i_2^2 = i_3^2 = -1,\ i_1 i_2 = -i_2 i_1 = i_3,\ i_2 i_3 = -i_3 i_2 = i_1,\ i_3 i_1 = -i_1 i_3 = i_2$ and $i_1 i_2 i_3 = -1$.

The algebra $\mathbb{H}$ of quaternions is a four-dimensional that is a noncommutative $\mathbb{R}-$algebra generated by four base elements $e_0 \cong 1,\ e_1 \cong i_1,\ e_2 \cong i_2$ and $e_3 \cong i_3$ that satisfy the following rules $e_l^2 = -1,\ l \in \{1, 2, 3\};\ e_m e_n = -e_n e_m = \beta_{mn} e_t,\ m \neq n,\ m, n \in \{1, 2, 3\}$, where β_{mn} and e_t are uniquely determined by e_m and e_n.

Using the sequence of Pell quaternions and Pell-Lucas quaternions studied by Szynal-Liana and Włoch in [31], we continue the study of these quaternions using certain type of matrices whose entries are exactly elements of these quaternion sequences. We consider two types of matrices: circulant matrices ($g-$circulant, right and left circulant) and tridiagonal matrices. The $g-$circulant matrices have been one of the most important and active research field of applied mathematic and the study of this type of matrices is a long one (see, for example, [9], [13], [23], [30] and [37]) and more recently we can found several papers on certain types of circulant matrices (see, for example, [3], [19], [28], [29] and [33]). These matrices have many applications in several areas of applied mathematics, they are also incorporated in some mathematical software and they have been used in the study of geometric tools (see, for example, [9] and [25]).

In the literature there are some authors who study these type of matrices whose entries are elements of sequences of integers numbers defined recursively. For example, we have the work of Ipek in [19], the work of Shen, Cen and Hao in [28], the work of Solak in [29] and the work of Zhou and Jiang in [35], where the authors considered this type of matrices with the Fibonacci and Lucas numbers, the work of Bozkurt and Tam in [3] and the work of Gong, Jiang and Gao in [12], where the entries of these matrices are Jacobsthal and Jacobsthal-Lucas numbers and the work of Yazlik and

Taskara in [33] and in [34], where the $k-$Horadam mumbers are considered as entries.

For a natural number n we consider a $g-$*circulant* matrix as a square matrix of order n with the following form:

$$A_{g,n}=\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_{n-g+1} & a_{n-g+2} & \cdots & a_{n-g} \\ a_{n-2g+1} & a_{n-2g+2} & \cdots & a_{n-2g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g+1} & a_{g+2} & \cdots & a_g \end{pmatrix}, \tag{1.2}$$

where g is a nonnegative integer and each of the subscripts is understood to be reduced modulo n. The first row of $A_{g,n}$ is $(a_1, a_2, \ldots, a_n)$ and its $(j+1)$th row is obtained by giving its jth row a right circular shift by g positions.

Note that, $g = 1$ or $g = n + 1$ yields the standard *right circulant* matrix, or simply, *circulant* matrix. Thus a right circulant matrix is written as

$$RCirc(a_1, a_2, \ldots, a_n)=\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}. \tag{1.3}$$

If $g = n - 1$, we obtain the so called *left circulant* matrix, or *reverse circulant* matrix. In this case we write a left circulant matrix as

$$LCirc(a_1, a_2, \ldots, a_n)=\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_1 \\ \vdots & \vdots & & \vdots \\ a_n & a_1 & \cdots & a_{n-1} \end{pmatrix}. \tag{1.4}$$

In this paper we introduce the $k-$Pell and $k-$Pell-Lucas quaternions and we consider $g-$circulant, right and left circulant matrices whose entries are Pell, Pell-Lucas, $k-$Pell and $k-$Pell-Lucas quaternions instead of integer numbers. We present the determinant of these matrices and with the tridiagonal matrices we show that the determinant is equal to the nth term of the Pell, Pell-Lucas, $k-$Pell and $k-$Pell-Lucas quaternion sequences.

The Pell quaternions sequence $\{R_n\}_{n=0}^{\infty}$ was introduced in [31] and is defined by the recurrence relation

$$R_n=\sum_{l=0}^{3} P_{n+l}e_l, \tag{1.5}$$

where P_n is the nth Pell number. Note that the initial conditions of this sequence are given by

$$R_0 = \sum_{l=0}^{3} P_l e_l = P_0 e_0 + P_1 e_1 + P_2 e_2 + P_3 e_3 = e_1 + 2e_2 + 5e_3$$

and

$$R_1 = \sum_{l=0}^{3} P_{l+1} e_l = P_1 e_0 + P_2 e_1 + P_3 e_2 + P_4 e_3 = e_0 + 2e_1 + 5e_2 + 12e_3.$$

The Pell-Lucas quaternions sequence $\{S_n\}_{n=0}^{\infty}$ was introduced also in [31] and is defined by the recurrence relation

$$S_n = \sum_{l=0}^{3} Q_{n+l} e_l, \tag{1.6}$$

where Q_n is the nth Pell-Lucas number. Note that the initial conditions of this sequence are given by

$$S_0 = \sum_{l=0}^{3} Q_l e_l = Q_0 e_0 + Q_1 e_1 + Q_2 e_2 + Q_3 e_3 = 2e_0 + 2e_1 + 6e_2 + 14e_3$$

and

$$S_1 = \sum_{l=0}^{3} Q_{l+1} e_l = Q_1 e_0 + Q_2 e_1 + Q_3 e_2 + Q_4 e_3 = 2e_0 + 6e_1 + 14e_2 + 34e_3.$$

It is easy to show that the sequences $\{R_n\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$ are more two examples of sequences defined by a recurrence relation of order two. From the recurrence relations (1.5), using the recurrence relations (1.1) and some properties of summation formulas we obtain that

$$\begin{aligned} R_{n+1} &= \sum_{l=0}^{3} P_{n+l+1} e_l \\ &= \sum_{l=0}^{3} (2P_{n+l} + P_{n+l-1}) e_l \\ &= 2(\sum_{l=0}^{3} P_{n+l} e_l) + (\sum_{l=0}^{3} P_{n+l-1} e_l), \end{aligned}$$

and so,

$$R_{n+1} = 2R_n + R_{n-1},\ n \geq 1.$$

In a similar way we obtain

$$S_{n+1} = 2S_n + S_{n-1},\ n \geq 1.$$

Next we introduce a generalization of the previous sequences. For any positive real number k, we define the $k-$Pell quaternions sequence $\{R_{k,n}\}_{n=0}^{\infty}$ by the recurrence relation

$$R_{k,n} = \sum_{l=0}^{3} P_{k,n+l} e_l, \tag{1.7}$$

where $P_{k,n}$ is the nth $k-$Pell number. Note that the initial conditions of this sequence are given by

$$\begin{aligned} R_{k,0} &= \sum_{l=0}^{3} P_{k,l} e_l \\ &= P_{k,0} e_0 + P_{k,1} e_1 + P_{k,2} e_2 + P_{k,3} e_3 \\ &= e_1 + 2e_2 + (4+k) e_3 \end{aligned}$$

and

$$\begin{aligned} R_{k,1} &= \sum_{l=0}^{3} P_{k,l+1} e_l \\ &= P_{k,1} e_0 + P_{k,2} e_1 + P_{k,3} e_2 + P_{k,4} e_3 \\ &= e_0 + 2e_1 + (4+k) e_2 + (8+4k) e_3. \end{aligned}$$

Now we define the $k-$Pell-Lucas quaternions sequence $\{S_{k,n}\}_{n=0}^{\infty}$ by the recurrence relation

$$S_{k,n} = \sum_{l=0}^{3} Q_{k,n+l} e_l, \tag{1.8}$$

where $Q_{k,n}$ is the nth $k-$Pell-Lucas number. Note that the initial conditions of this sequence are given by

$$\begin{aligned} S_{k,0} &= \sum_{l=0}^{3} Q_{k,l} e_l \\ &= Q_{k,0} e_0 + Q_{k,1} e_1 + Q_{k,2} e_2 + Q_{k,3} e_3 \\ &= 2e_0 + 2e_1 + (4+2k) e_2 + (8+6k) 4 e_3 \end{aligned}$$

and

$$\begin{aligned} S_{k,1} &= \sum_{l=0}^{3} Q_{k,l+1} e_l \\ &= Q_{k,1} e_0 + Q_{k,2} e_1 + Q_{k,3} e_2 + Q_{k,4} e_3 \\ &= 2e_0 + (4+2k) e_1 + (8+6k) e_2 + (16+16k+2k^2) e_3. \end{aligned}$$

Note that for $k = 1$ the sequence $\{R_{k,n}\}_{n=0}^{\infty}$ defined in (1.7) reduces to $\{R_n\}_{n=0}^{\infty}$ defined in (1.5) and the sequence $\{S_{k,n}\}_{n=0}^{\infty}$ defined in (1.8) reduces to $\{S_n\}_{n=0}^{\infty}$ defined in (1.6).

In the next two sections we consider the circulant matrices whose entries are the Pell quaternions and their generalizations and the Pell-Lucas quaternions and their generalizations, respectively, and in both cases we present the determinant of these matrices. In what concerning the tridiagonal matrices, in Section 4 we show that the determinant is equal to the nth term of the Pell, Pell-Lucas, $k-$Pell and $k-$Pell-Lucas quaternion sequences and we end this paper with some conclusions and plans for further investigation.

2 Circulant matrices with the Pell quaternions and their generalizations

In this section, we consider right circulant matrices $A_{k,n}$ and A_n as in (1.3), whose entries are elements of the sequences $\{R_{k,n}\}_{n=0}^{\infty}$ and $\{R_n\}_{n=0}^{\infty}$, respectively. We give a determinant formula for these matrices following the idea of Gong, Jiang and Gao in [12]. First we state that $det(A_{k,1}) = R_{k,1}$, $det(A_1) = R_1$ and for $n \geq 2$ we have

Theorem 2.1 *For $n \geq 2$, let $A_{k,n} = RCirc(R_{k,1}, R_{k,2}, \dots, R_{k,n})$ and $A_n = RCirc(R_1, R_2, \dots, R_n)$ be right circulant matrices. Then we have*

1. $det(A_{k,n}) = (R_{k,1} - R_{k,n+1})^{n-2}(R_{k,1}\beta_{k,n} - R_{k,0}\alpha_{k,n})$,
2. $det(A_n) = (R_1 - R_{n+1})^{n-2}(R_1\beta_n - R_0\alpha_n)$,

where

$$\alpha_{k,n} = \sum_{l=1}^{n-1} \left(\frac{kR_{k,n} - kR_{k,0}}{R_{k,1} - R_{k,n+1}} \right)^{n-(l+1)} R_{k,l+1}, \tag{2.1}$$

$$\beta_{k,n} = (R_{k,1} - 2R_{k,n}) + \sum_{l=1}^{n-2} \left(\frac{kR_{k,n} - kR_{k,0}}{R_{k,1} - R_{k,n+1}} \right)^{n-(l+1)} kR_{k,l}, \tag{2.2}$$

$$\alpha_n = \sum_{l=1}^{n-1} \left(\frac{R_n - R_0}{R_1 - R_{n+1}} \right)^{n-(l+1)} R_{l+1} \tag{2.3}$$

and

$$\beta_n = (R_1 - 2R_n) + \sum_{l=1}^{n-2} \left(\frac{R_n - R_0}{R_1 - R_{n+1}} \right)^{n-(l+1)} R_l. \tag{2.4}$$

Proof. 1. In the case $n \geq 2$ and k any positive real number, we consider the following square matrices of order n used in the theory of circulant matrices and given by

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -k & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & -k \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & -k & \cdots & 0 & 0 & 0 \end{pmatrix}$$

and

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{kR_{k,n}-kR_{k,0}}{R_{k,1}-R_{k,n+1}}\right)^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \left(\frac{kR_{k,n}-kR_{k,0}}{R_{k,1}-R_{k,n+1}}\right)^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{kR_{k,n}-kR_{k,0}}{R_{k,1}-R_{k,n+1}} & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that

$$det(\Gamma) = det\Pi = (-1)^{\frac{(n-1)(n-2)}{2}}. \quad (2.5)$$

Considering the product $\Gamma A_{k,n}\Pi$ of matrices we obtain the following matrix

$$\begin{pmatrix} R_{k,1} & \alpha_{k,n} & R_{k,n-1} & \cdots & R_{k,3} & R_{k,2} \\ kR_{k,0} & \beta_{k,n} & kR_{k,n-2} & \cdots & kR_{k,2} & kR_{k,1} \\ 0 & 0 & R_{k,1}-R_{k,n+1} & & & \\ 0 & 0 & k(R_{k,0}-R_{k,n}) & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & k(R_{k,0}-R_{k,n}) & R_{k,1}-R_{k,n+1} \end{pmatrix},$$

where $\alpha_{k,n}$ and $\beta_{k,n}$ are given by (2.1) and (2.2), respectively. Now calculating the determinant of the matrix $\Gamma A_{k,n}\Pi$ we obtain

$$det(\Gamma A_{k,n}\Pi) = (R_{k,1}-R_{k,n+1})^{n-2} det \begin{pmatrix} R_{k,1} & \alpha_{k,n} \\ R_{k,0} & \beta_{k,n} \end{pmatrix}.$$

Using the property of the determinant of a product of matrices and the identity (2.5), we conclude that

$$det(A_{k,n}) = det(\Gamma A_{k,n}\Pi)$$

and the result follows.
2. For $k = 1$ in 1. of this theorem we obtain the result required. □

Let $B_{k,n} = LCirc(R_{k,1}, R_{k,2}, \ldots, R_{k,n})$ and $B_n = LCirc(R_1, R_2, \ldots, R_n)$ be left circulant matrices as in (1.4), whose entries are elements of the sequences $\{R_{k,n}\}_{n=0}^{\infty}$ and $\{R_n\}_{n=0}^{\infty}$, respectively.

Next we give a determinant formula for these matrices following once more the idea used by Gong, Jiang and Gao in [12]. Lemma 5 in [22] will help us to obtain the determinant of these matrices. The authors in [22] define the following matrix

$$\Delta := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

that is an orthogonal cyclic shift matrix (and a left circulant matrix) of order n. They stated that

$$LCirc(a_1, a_2, \ldots, a_n) = \Delta RCirc(a_1, a_2, \ldots, a_n). \tag{2.6}$$

Using the fact that $det(\Delta) = (-1)^{\frac{(n-1)(n-2)}{2}}$, calculating the determinant in both sides of the identity (2.6) and according the results obtained in Theorem 2.1, the following result is easily proved.

Theorem 2.2 *For $n \geq 2$, let $B_{k,n} = LCirc(R_{k,1}, R_{k,2}, \ldots, R_{k,n})$ and $B_n = LCirc(R_1, R_2, \ldots, R_n)$ be left circulant matrices. Then we have*

1. $det(B_{k,n}) = (-1)^{\frac{(n-1)(n-2)}{2}} (R_{k,1} - R_{k,n+1})^{n-2} (R_{k,1}\beta_{k,n} - R_{k,0}\alpha_{k,n})$,
2. $det(B_n) = (-1)^{\frac{(n-1)(n-2)}{2}} (R_1 - R_{n+1})^{n-2} (R_1\beta_n - R_0\alpha_n)$,

where $\alpha_{k,n}$ and $\beta_{k,n}$ are given by (2.1) and (2.2), respectively and α_n and β_n are given by (2.3) and (2.4), respectively.

Now let $C_{k,n}$ and C_n be $g-$circulant matrices as in (1.2), which entries are elements of the sequences $\{R_{k,n}\}_{n=0}^{\infty}$ and $\{R_n\}_{n=0}^{\infty}$, respectively. Let $\mathbb{Q}_g$ be a $g-$circulant matrix with the first row $e^* = [1, 0, \ldots, 0]$. In order to obtain the determinant formula of the matrices $C_{k,n}$ and C_n we use the following result of [22]:

Lemma 2.3 (Lemma 7 of [22]) *$A_{g,n}$ is a $g-$circulant matrix with the first row $[a_1, a_2, \ldots, a_n]$ if and only if $A_{g,n} = \mathbb{Q}_g RCirc(a_1, a_2, \ldots, a_n)$.*

From this lemma and Theorem 2.1, we deduce the following results.

Theorem 2.4 *For $n \geq 2$, let $C_{k,n}$ and C_n be $g-$circulant matrices whose entries are elements of the sequences $\{R_{k,n}\}_{n=0}^{\infty}$ and $\{R_n\}_{n=0}^{\infty}$, respectively. Then*

1. $det(C_{k,n}) = det(\mathbb{Q}_g)[(R_{k,1} - R_{k,n+1})^{n-2}(R_{k,1}\beta_{k,n} - R_{k,0}\alpha_{k,n})]$,
2. $det(C_n) = det(\mathbb{Q}_g)[(R_1 - R_{n+1})^{n-2}(R_1\beta_n - R_0\alpha_n)]$,

where $\alpha_{k,n}$ and $\beta_{k,n}$ are given by (2.1) and (2.2), respectively and α_n and β_n are given by (2.3) and (2.4), respectively.

In the case of $n = 1$, we immediately obtain that $det(B_{k,1}) = det(A_{k,1}) = det(C_{k,1}) = R_{k,1}$ and $det(B_1) = det(A_1) = det(C_1) = R_1$.

3 Circulant matrices with the Pell-Lucas quaternions and their generalizations

In this section we get similar results to those stated in the previous section. The proofs are similar and for this reason we decide do not include them. Therefore let $D_{k,n}$ and D_n be right circulant matrices whose entries are elements of the sequences $\{S_{k,n}\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$, respectively. We give a determinant formula for these matrices in the following result:

Theorem 3.1 *For $n \geq 2$, let $D_{k,n} = RCirc(S_{k,1}, S_{k,2}, \ldots, S_{k,n})$ and $D_n = RCirc(S_1, S_2, \ldots, S_n)$ be right circulant matrices. Then we have*

1. $det(D_{k,n}) = (S_{k,1} - S_{k,n+1})^{n-2}(S_{k,1}\delta_{k,n} - S_{k,0}\mu_{k,n})$,
2. $det(D_n) = (S_1 - S_{n+1})^{n-2}(S_1\delta_n - S_0\mu_n)$,

where

$$\mu_{k,n} = \sum_{l=1}^{n-1} \left(\frac{kS_{k,n} - kS_{k,0}}{S_{k,1} - S_{k,n+1}} \right)^{n-(l+1)} S_{k,l+1}, \tag{3.1}$$

$$\delta_{k,n} = (S_{k,1} - 2S_{k,n}) + \sum_{l=1}^{n-2} \left(\frac{kS_{k,n} - kS_{k,0}}{S_{k,1} - S_{k,n+1}} \right)^{n-(l+1)} kS_{k,l}, \tag{3.2}$$

$$\mu_n = \sum_{l=1}^{n-1} \left(\frac{S_n - S_0}{S_1 - S_{n+1}} \right)^{n-(l+1)} S_{l+1} \tag{3.3}$$

and

$$\delta_n = (S_1 - 2S_n) + \sum_{l=1}^{n-2} \left(\frac{S_n - S_0}{S_1 - S_{n+1}} \right)^{n-(l+1)} S_l. \tag{3.4}$$

About the left circulant matrices, let $E_{k,n} = LCirc(S_{k,1}, S_{k,2}, \ldots, S_{k,n})$ and $E_n = LCirc(S_1, S_2, \ldots, S_n)$ be left circulant matrices as in (1.4), whose entries are elements of the sequences $\{S_{k,n}\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$, respectively.

As before, Lemma 5 in [22] and the results obtained in Theorem 3.1 will help us in the following result where we state the determinant formula of these matrices.

Theorem 3.2 *For $n \geq 2$, let $E_{k,n} = LCirc(S_{k,1}, S_{k,2}, \ldots, S_{k,n})$ and $E_n = LCirc(S_1, S_2, \ldots, S_n)$ be left circulant matrices. Then we have*

1. $det(E_{k,n}) = (-1)^{\frac{(n-1)(n-2)}{2}}(S_{k,1} - S_{k,n+1})^{n-2}(S_{k,1}\delta_{k,n} - S_{k,0}\mu_{k,n})$,

2. $det(E_n) = (-1)^{\frac{(n-1)(n-2)}{2}}(S_1 - S_{n+1})^{n-2}(S_1\delta_n - S_0\mu_n)$,

where $\mu_{k,n}$ and $\delta_{k,n}$ are given by (3.1) and (3.2), respectively and μ_n and δ_n are given by (3.3) and (3.4), respectively.

Now let $G_{k,n}$ and G_n be $g-$circulant matrices as in (1.2), which entries are elements of the sequences $\{S_{k,n}\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$, respectively. In order to obtain the determinant formula of these matrices we use Lemma 2.3 (Lemma 7 of [22]) and the results of Theorem 3.1:

Theorem 3.3 *For $n \geq 2$, let $G_{k,n}$ and G_n be $g-$circulant matrices whose entries are elements of the sequences $\{S_{k,n}\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$, respectively. Then*

1. $det(G_{k,n}) = det(\mathbb{Q}_g)[(S_{k,1} - S_{k,n+1})^{n-2}(S_{k,1}\delta_{k,n} - S_{k,0}\mu_{k,n})]$,
2. $det(G_n) = det(\mathbb{Q}_g)[(S_1 - S_{n+1})^{n-2}(S_1\delta_n - S_0\mu_n)]$,

where $\mu_{k,n}$ and $\delta_{k,n}$ are given by (3.1) and (3.2), respectively and μ_n and δ_n are given by (3.3) and (3.4), respectively.

Finally for $n = 1$, we have that $det(E_{k,1}) = det(D_{k,1}) = det(G_{k,1}) = S_{k,1}$ and $det(E_1) = det(D_1) = det(G_1) = S_1$.

4 Tridiagonal matrices with the Pell, Pell-Lucas quaternions and their generalizations

Following [10], we know that the determinant of a special kind of tridiagonal matrices is related to a special nth element of a sequence. If we consider the $(n \times n)$ tridiagonal matrices N_n, defined as:

$$\begin{pmatrix} a & b & & & & \\ c & d & e & & & \\ & c & d & e & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & d & e \\ & & & & c & d \end{pmatrix},$$

computing the sequence of determinants, we obtain:

$$\begin{aligned} |N_1| &= a \\ |N_2| &= d\,|N_1| - bc \\ |N_3| &= d\,|N_2| - ce\,|N_1| \\ |N_4| &= d\,|N_3| - ce\,|N_2| \\ &\vdots \\ |N_{n+1}| &= d\,|N_n| - ce\,|N_{n-1}|, \end{aligned}$$

and therefore we can obtain the following result related with the sequence $\{R_{k,n}\}_{n=0}^{\infty}$:

Proposition 4.1 *The* $(n \times n)$ *tridiagonal matrices*

$$M_{k,n} = \begin{pmatrix} R_{k,2} & R_{k,1} & & & & \\ -k & 2 & 1 & & & \\ & -k & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -k & 2 & 1 \\ & & & & -k & 2 \end{pmatrix}$$

satisfy

$$|M_{k,n}| = R_{k,n+1}$$

that is, the nth element of the sequence $\{R_{k,n}\}_{n=0}^{\infty}$ *may be obtained through the computation of the the determinant of the* $((n-1) \times (n-1))$ *tridiagonal matrix* $M_{k,n-1}$.

Proof. If we consider $a = R_{k,2},\ b = R_{k,1},\ c = -k,\ d = 2$ and $e = 1$, it is straightforward to see that the sequence of determinants becomes:

$$\begin{aligned} |N_1| &= |M_{k,1}| = R_{k,2}(x) \\ |N_2| &= |M_{k,2}| = 2R_{k,2} + kR_{k,1} = R_{k,3} \\ |N_3| &= |M_{k,3}| = 2R_{k,3} + kR_{k,2} = R_{k,4} \\ &\vdots \\ |N_{n-1}| &= |M_{k,n-1}| = 2R_{k,n-1} + kR_{k,n-2} = R_{k,n}, \\ &\vdots \end{aligned}$$

as required. □

About the sequence $\{S_{k,n}\}_{n=0}^{\infty}$ we get a similar result using the following matrix

$$T_{k,n} = \begin{pmatrix} S_{k,2} & S_{k,1} & & & & \\ -k & 2 & 1 & & & \\ & -k & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -k & 2 & 1 \\ & & & & -k & 2 \end{pmatrix}$$

and the result:

Proposition 4.2 *The $(n \times n)$ tridiagonal matrices $T_{k,n}$ satisfy*

$$|T_{k,n}| = S_{k,n+1}$$

that is, the nth element of the sequence $\{S_{k,n}\}_{n=0}^{\infty}$ may be obtained through the computation of the the determinant of the $((n-1)\times(n-1))$ tridiagonal matrix $T_{k,n-1}$.

Taking the particular case of $k = 1$, the matrices $M_{k,n}$ of Proposition 4.1 and $T_{k,n}$ of Proposition 4.2 give the following tridiagonal matrices M_n and T_n that correspond to $\{R_n\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$, respectively.

$$M_n = \begin{pmatrix} R_2 & R_1 & & & & \\ -1 & 2 & 1 & & & \\ & -1 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & 1 \\ & & & & -1 & 2 \end{pmatrix}$$

and

$$T_n = \begin{pmatrix} S_2 & S_1 & & & & \\ -1 & 2 & 1 & & & \\ & -1 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & 1 \\ & & & & -1 & 2 \end{pmatrix}.$$

The corresponding results involving these matrices are stated in the following proposition:

Proposition 4.3 *The $(n \times n)$ tridiagonal matrices M_n and T_n satisfy*

1. $|M_n| = R_{n+1}$,
2. $|T_n| = S_{n+1}$,

that is, the nth element of the sequences $\{R_n\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$ may be obtained through the computation of the determinant of the $((n-1)\times(n-1))$ tridiagonal matrices M_n and T_n.

Following the ideas of [24], we can relate the nth order Pell, Pell-Lucas, $k-$Pell and $k-$Pell-Lucas quaternions using the computation of others tridiagonal matrices. In [24], the following result is presented:

Theorem 4.4 *Let $\{x_n\}_n$ be any second order linear sequence, defined recursively as:*

$$x_{n+1} = Ax_n + Bx_{n-1}, \quad n \geq 1,$$

with $x_0 = C,\ x_1 = D$. *Then, for all* $n \geq 0$*:*

$$x_n = \begin{vmatrix} C & D & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & B & 0 & \cdots & 0 & 0 \\ 0 & -1 & A & B & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A & B \\ 0 & 0 & 0 & 0 & \cdots & -1 & A \end{vmatrix}_{(n+1)\times(n+1)}.$$

In the case of the $k-$Pell and $k-$Pell-Lucas quaternion sequences, we have $A = 2$, $B = k$, where C and D are the initial conditions of the sequences, and then, a direct application of Theorem 4.4 leads to the following proposition (for the Pell and Pell-Lucas case consider $k = 1$):

Proposition 4.5 *For* $n \geq 0$*, we have*

1. $$R_{k,n} = \begin{vmatrix} R_{k,0} & R_{k,1} & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & k & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & k \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}_{(n+1)\times(n+1)};$$

2. $$S_{k,n} = \begin{vmatrix} S_{k,0} & S_{k,1} & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & k & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & k \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}_{(n+1)\times(n+1)};$$

3. $$R_n = \begin{vmatrix} R_0 & R_1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}_{(n+1)\times(n+1)};$$

4. $$S_n = \begin{vmatrix} S_0 & S_1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}_{(n+1)\times(n+1)}.$$

5 Conclusions

In this paper we introduced the $k-$Pell quaternions and the $k-$Pell-Lucas quaternions for any positive real number k. We have considered certain types of $g-$circulant, right and left circulant matrices whose entries are Pell, Pell-Lucas, $k-$Pell and $k-$Pell-Lucas quaternions. In this work we have considered the determinant formula of these matrices following the idea present in [12].
Types of tridiagonal matrices whose entries are Pell, Pell-Lucas, $k-$Pell and $k-$Pell-Lucas quaternion were considered and we presented a way to obtain the nth term of the Pell, Pell-Lucas, $k-$Pell and $k-$Pell-Lucas quaternion sequences using these matrices.
In the future, we intend to discuss the invertibility of these circulant type matrices associated with these type of sequences, following, for example, the studies of Shen in [28] in the case of Fibonacci and Lucas numbers, Yazlik in [34] with Generalized $k-$Horadam numbers and Bozkurt in [3] with Jacobsthal and Jacobsthal-Lucas numbers, among others.

Acknowledgements The research was financed by Portuguese Funds through FCT – Fundação para a Ciência e a Tecnologia, within the Project UID/MAT/00013/2013. Also, this work was financially supported by National Funds through FCT – Fundação para a Ciência e a Tecnologia under the project UID/CED/00194/2013.

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Received: 11.VII.2017 / Accepted: 13.IX.2017

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