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Uniqueness results related to certain q-shift difference polynomials

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Abstract In this paper, we investigate the uniqueness problems of certain q-shift difference polynomials sharing a value or a small function or a polynomial with finite weights. The results of this paper improve and extend the results due to Wang, Xu and Zhan [Adv. Differ. Equ., 249 (2014), 1-16].

Keywords Uniqueness · Small function · Shift · Difference polynomial

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1 Introduction, definitions and main results

In this paper, by meromorphic function we shall always mean meromorphic function in the complex plane. We adopt the standard notations of Nevanlinna Theory (See [4,8,17]). We use $S(r, f)$ to denote any quantity satisfying $S(r, f) = o\{T(r, f)\}$ for all r outside a possible exceptional set E of finite logarithmic measure. A meromorphic function $\alpha(z)$ is called a small function with respect to f, provided that $T(r, \alpha(z)) = S(r, f)$. We define $\rho(f)$, the order of f and $\Theta(a, f)$ as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}, \quad \Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where a is a value in the extended complex plane. Moreover, if for some $a \in \mathbb{C} \cup \{\infty\}$, the zeros of $f(z) - a$ and $g(z) - a$ coincide in location and multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicities) and if they coincide in location only then we say that $f(z)$

and $g(z)$ share a IM (ignoring multiplicities). The difference operators for a meromorphic function f are defined as

$$\Delta_c f(z) = f(z+c) - f(z), (c \neq 0),$$

$$\Delta_q f(z) = f(qz) - f(z), (q \neq 0, 1).$$

We now recall the following definitions.

Definition 1.1 ([7]). *For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f| = 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f| \leq m)$ $(N(r, a; f| \geq m))$ the counting function of those a-points of f whose multiplicities are not greater (less) than m where each a-point is counted according to its multiplicity. $\overline{N}(r, a; f| \leq m)$ $(\overline{N}(r, a; f| \geq m))$ are defined analogously, where in counting the a-points of f we ignore the multiplicities.*

Definition 1.2 ([6]). *Let a be any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then*

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ \ldots \ + \overline{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.3 ([5,6]). *Let l be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, $E_l(a, f)$ denotes the set of all a-points of f with multiplicity m where each a-point is counted m times if $m \leq l$ and $l+1$ times if $m > l$. We say that f and g share the value a with weight l if $E_l(a, f) = E_l(a, g)$. Clearly f, g share a CM and IM when $l = \infty$ and $l = 0$ respectively.*

Recently the topic of difference equation and difference products have attracted many mathematicians from various parts of the world. Many papers have been published on the uniqueness of differences and difference polynomials of meromorphic functions sharing values or certain functions ([2,3, 10,12–14,18]).

In 2007, Laine and Yang [9] investigated the value distribution of difference product of entire functions and obtained the following result.

Theorem A. *Let f be a transcendental entire function of finite order and η be a nonzero complex constant. Then for $n \geq 2$, $f^n(z)f(z+\eta)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Example 1.1 ([9]). *Let $f(z) = 1 + e^z$. Then $f(z)f(z+\pi i) - 1 = -e^{-2z}$ has no zeros. This example shows that Theorem A does not hold for $n = 1$.*

In 2010, Qi, Yang, Liu [11] proved the following uniqueness result which corresponds to Theorem A.

Theorem B. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and η be a nonzero complex constant and let $n \geq 6$ be an integer. If $f^n(z)f(z+\eta)$ and $g^n(z)g(z+\eta)$ share 1 CM, then either $f(z)g(z) = t_1$ or $f(z) = t_2 g(z)$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.*

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a nonzero polynomial where a_0, $a_1, \ldots, a_n (\neq 0)$ are complex constants and n an integer. Let $\Gamma_0 = m_1 + m_2$ and $\Gamma_1 = m_1 + 2m_2$ where m_1 is the number of simple zeros of $P(z)$ and m_2 is the number of multiple zeros of $P(z)$. In 2011, Luo and Lin [10] studied the uniqueness of difference polynomials of entire functions and obtained the following result.

Theorem C. *Let f and g be two transcendental entire functions of finite order, c be a nonzero complex constant, and let $n > 2\Gamma_1 + 1$ be an integer. If $P(f)f(z+c)$ and $P(g)g(z+c)$ share 1 CM, then one of the following results holds:*

(i) $f \equiv tg$ *for a constant* t *such that* $t^d = 1$ *where* $d = GCD\{\lambda_0 + 1, \lambda_1 + 1, \ldots,$
$\lambda_n + 1\}$ *and* $\lambda_i = \begin{cases} i, & a_i \neq 0 \\ n, & a_i = 0, \end{cases} \quad i = 0, 1, 2, \ldots, n;$

(ii) f *and* g *satisfy the algebraic equation* $R(f,g) = 0$, *where* $R(\omega_1, \omega_2) = P(\omega_1)$
$\omega_1(z+c) - P(\omega_2)\omega_2(z+c);$

(iii) $f(z) = e^{\alpha(z)}$, $g(z) = e^{\beta(z)}$, *where* $\alpha(z)$ *and* $\beta(z)$ *are two polynomials,* b *is a constant satisfying* $\alpha + \beta \equiv b$, *and* $a_n^2 e^{(n+1)b} = 1$.

The q-difference variant of the same topic was also of special interest to many authors. In this direction, Zhang and Korhonen [18] proved the following results:

Theorem D. *Let f and g be two transcendental entire functions with zero order. Suppose that q is a non-zero constant and n is an integer satisfying $n \geq 4$. If $f^n(z)f(qz)$ and $g^n(z)g(qz)$ share 1 CM, then $f \equiv tg$ for $t^{n+1} = 1$.*

Theorem E. *Let f and g be two transcendental entire functions with zero order. Suppose that q is a non-zero constant and n is an integer satisfying $n \geq 6$. If $f^n(z)(f(z)-1)f(qz)$ and $g^n(z)(g(z)-1)g(qz)$ share 1 CM, then $f \equiv g$.*

In 2014, Cao, Liu and Xu [2] studied the q-shift difference polynomials and obtained the following result:

Theorem F. *Let f and g be two transcendental entire functions with zero order. Suppose that q is a non-zero constant and n is an integer satisfying $n \geq 2k+m+6$. If $[f^n(z)(f^m(z)-a)f(qz+c)]^{(k)}$ and $[g^n(z)(g^m(z)-a)g(qz+c)]^{(k)}$ share 1 CM, then $f \equiv tg$ where $t^{n+1} = t^m = 1$.*

In the same year, Wang, Xu and Zhan [14] proved the following theorems:

Theorem G. *Let f, g be two transcendental entire functions with zero order, n, d, $s_j (j = 1, 2, ..., d) \in \mathbb{N}_+$, c_j, $q_j \in \mathbb{C}\backslash\{0\}$ $(j = 1, 2, ..., d)$ be distinct contants, $F(z) = P(f)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}$ and $G(z) = P(g)\prod_{j=1}^{d} g(q_j z + c_j)^{s_j}$. Suppose that $n > max\{2(\Gamma_1 + 2d) - \lambda, \lambda\}$. If $F(z)$ and $G(z)$ share 1 CM, then one of the following cases holds:*

(I) $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD\{\lambda_0 + \lambda, \lambda_1 + \lambda, \ldots, \lambda_n + \lambda\}$ and

$$\lambda_i = \begin{cases} i+1, & a_i \neq 0, \\ n+1, & a_i = 0, \end{cases} \quad i = 0, 1, 2, \ldots, n;$$

(II) f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(\omega_1, \omega_2) = P(\omega_1)\prod_{j=1}^{d} \omega_1(q_j z + c_j)^{s_j} - P(\omega_2)\prod_{j=1}^{d} \omega_2(q_j z + c_j)^{s_j},$$

where $\lambda = s_1 + s_2 + \ldots + s_d$.

Theorem H. *Under the assumptions of Theorem G, If*

$$E_l(1, P(f)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}) = E_l(1, P(g)\prod_{j=1}^{d} g(q_j z + c_j)^{s_j})$$

and l, n, d are integers satisfying one of the following conditions:

(I) $l \geq 3$, $n > max\{2\Gamma_1 + 4d - \lambda, \lambda\}$;
(II) $l = 2$, $n > max\{2\Gamma_1 + \Gamma_0 + 5d - \lambda - d\chi, \lambda\}$;
(III) $l = 1$, $n > max\{2\Gamma_1 + 2\Gamma_0 + 6d - \lambda - 2d\chi, \lambda\}$;
(IV) $l = 0$, $n > max\{2\Gamma_1 + 3\Gamma_0 + 7d - \lambda - 3d\chi, \lambda\}$,

then the conclusions of Theorem G hold, where $\chi = min\{\Theta(0, f), \Theta(0, g)\}$.

Regarding Theorems G and H it is natural to ask the following questions.

Question 1.1. *What happens if one replaces the difference polynomial $P(f)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}$ by $(P(f)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j})^{(k)}$ where $k(\geq 0)$ is any integer?*

Question 1.2. *Is it possible in any way to reduce the lower bound of n in Theorems G and H ?*

Question 1.3. *What can be said if sharing value* 1 *is replaced by sharing a nonzero polynomial or a small function having finitely many zeros ?*

The results of this paper reflects the answers to the above questions. We prove the following three theorems which not only extend Theorems G and H at the same time improve them by reducing the lower bound of n. The followings are the main results of the paper.

Henceforth, we assume that

$$F_1(z) = P(f)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}, \quad G_1(z) = P(g)\prod_{j=1}^{d} g(q_j z + c_j)^{s_j}, \quad (1.1)$$

where $c_j \in \mathbb{C}$, q_j's are nonzero complex constants and $j = 1, 2, \ldots, d$.

Theorem 1.1. *Let f and g be two transcendental entire functions of zero order such that f and g share 0 CM. Suppose that $F = F_1^{(k)}$ and $G = G_1^{(k)}$, where F_1 and G_1 are given as in (1.1). If $E_l(1; F) = E_l(1; G)$ and l, n, $d(> 0)$, $s_j(> 0)(j = 1, 2, \ldots, d)$ are integers satisfying one of the following conditions:*

(I) $l \geq 2$, $n > 2\Gamma_1 + 2km_2 + 2(k + 2)d - \lambda$;
(II) $l = 1$, $n > \frac{1}{2}(4\Gamma_1 + \Gamma_0 + 5km_2 + (5k + 9)d - 2\lambda)$;
(III) $l = 0$, $n > 2\Gamma_1 + 3\Gamma_0 + 5km_2 + (5k + 7)d - \lambda$.

Then one of the following cases holds:

(i) $f \equiv tg$ *for a constant t such that $t^\kappa = 1$, where $\kappa = GCD\{\lambda_0 + \lambda, \lambda_1 + \lambda, \ldots, \lambda_n + \lambda\}$ and*

$$\lambda_i = \begin{cases} i, & a_i \neq 0, \\ n, & a_i = 0, \end{cases} \quad i = 0, 1, 2, \ldots, n;$$

(ii) *f and g satisfy the algebraic equation $R(f, g) = 0$, where*

$$R(w_1, w_2) = P(w_1)\prod_{j=1}^{d} w_1(q_j z + c_j)^{s_j} - P(w_2)\prod_{j=1}^{d} w_2(q_j z + c_j)^{s_j};$$

Theorem 1.2. *Let f and g be two transcendental entire functions of zero order such that f and g share 0 CM and $P_0(z)$ be a polynomial. Suppose that $F = F_1^{(k)}$ and $G = G_1^{(k)}$, where F_1 and G_1 be defined as in (1.1). If F and G share $P_0(z)$ with weight l and l, n, $d(> 0)$, $s_j(> 0)(j = 1, 2, \ldots, d)$ are integers satisfying one of I, II or III of Theorem 1.1, then one of the conclusions of Theorem 1.1 holds.*

Theorem 1.3. *Let f and g be two transcendental entire functions of zero order such that f and g share 0 CM and $\alpha(z)(\neq \infty)$ be a small function with respect to both f and g having finitely many zeros. Suppose that $F = F_1^{(k)}$ and $G = G_1^{(k)}$, where F_1 and G_1 are defined as in (1.1). If F and G share $\alpha(z)$ with weight l and l, n, $d(>0)$, $s_j(>0)(j = 1, 2, \ldots, d)$ are integers satisfying one of I, II or III of Theorem 1.1, then one of the conclusions of Theorem 1.1 holds.*

Now we will state some lemmas which will be needed in proving the theorems.

Let F and G be two nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$. We define

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

2 Lemmas

Lemma 2.1 ([17, Theorem 1.12., page 27]). *Let f be a nonconstant meromorphic function and $P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0$ where $a_n(\neq 0)$, a_{n-1},..., a_0 are complex constants. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.2 ([19]). *Let f be a nonconstant meromorphic function and p, k be two positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \qquad (2.1)$$

$$N_p(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \qquad (2.2)$$

Lemma 2.3 ([6]). *Let f and g be two nonconstant meromorphic functions. If $E_2(1; f) = E_2(1; g)$ then one of the following three cases holds:*

(i) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$;

(ii) $f = g$;

(iii) $fg = 1$,

where $T(r) = max\{T(r, f), T(r, g)\}$ *and* $S(r) = o\{T(r)\}$.

Lemma 2.4 ([1]). *Let F and G be two transcendental meromorphic functions. If $E_1(1; F) = E_1(1; G)$ and $H \not\equiv 0$, then*

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2}\overline{N}(r, 0; F)$$
$$+ \frac{1}{2}\overline{N}(r, \infty; F) + S(r, F) + S(r, G).$$

Lemma 2.5 ([1]). *Let F and G be two nonconstant meromorphic functions sharing 1 IM and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) \leq\ & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) \\ & +2\overline{N}(r,0;F) + \overline{N}(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\ & +S(r,F) + S(r,G). \end{aligned}$$

Lemma 2.6 ([15]). *Let f be a transcendental meromorphic function of zero order, and $q(\neq 0)$, η be complex constants. Then*

$$\begin{aligned} T(r,f(qz+\eta)) &= T(r,f(z)) + S(r,f); \\ N(r,\infty,f(qz+\eta)) &\leq N(r,\infty;f(z)) + S(r,f); \\ N(r,0;f(qz+\eta)) &\leq N(r,0;f(z)) + S(r,f); \\ \overline{N}(r,\infty,f(qz+\eta)) &\leq \overline{N}(r,\infty;f(z)) + S(r,f); \\ \overline{N}(r,0;f(qz+\eta)) &\leq \overline{N}(r,0;f(z)) + S(r,f). \end{aligned}$$

Lemma 2.7 ([14]). *Let f be a transcendental entire function. Suppose that F_1 is defined as in (1.1), and $\lambda = s_1 + s_2 + \cdots + s_d$. Then*

$$T(r,F_1(z)) = (n+\lambda)T(r,f) + S(r,f).$$

Lemma 2.8. *Let f and g be two entire functions, n, k be two positive integers, $q_j(\neq 0)$ and c_j are complex constants and let $F = F_1^{(k)}$, $G = G_1^{(k)}$. If there exist two nonzero constants a and b such that $\overline{N}(r,a;F) = \overline{N}(r,0;G)$ and $\overline{N}(r,b;G) = \overline{N}(r,0;F)$, then $n \leq 2\Gamma_0 + 2km_2 + 2(k+1)d - \lambda$.*

Proof. By the second main theorem of Nevanlinna, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,a;F) + S(r,F) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F). \end{aligned} \tag{2.3}$$

From (2.1) we see that

$$T(r,F_1) \leq T(r,F) - \overline{N}(r,0;F) + N_{k+1}(r,0;F_1) + S(r,f).$$

Using Lemma 2.7, (2.2) and (2.3) we obtain from above

$$\begin{aligned}(n+\lambda)T(r,f) &\leq \overline{N}(r,0;G) + N_{k+1}(r,0;F_1) + S(r,f)\\ &\leq N_{k+1}(r,0;G_1) + N_{k+1}(r,0;F_1) + S(r,f) + S(r,g)\\ &\leq N_{k+1}(r,0;P(f)\textstyle\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}) + N_{k+1}(r,0;\\ &\quad P(g)\textstyle\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}) + S(r,f) + S(r,g)\\ &\leq N_{k+1}(r,0;P(f)) + N_{k+1}(r,0;\textstyle\prod_{j=1}^{d} f(q_jz\\ &\quad +c_j)^{s_j}) + N_{k+1}(r,0;P(g)) + N_{k+1}(r,0;\textstyle\prod_{j=1}^{d} g(q_jz\\ &\quad +c_j)^{s_j}) + S(r,f) + S(r,g)\\ &\leq [m_1+(k+1)m_2]T(r,f) + (k+1)\overline{N}(r,0;\textstyle\prod_{j=1}^{d} f(q_jz\\ &\quad +c_j)^{s_j}) + [m_1+(k+1)m_2]T(r,g) + (k+1)\overline{N}(r,0;\\ &\quad \textstyle\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}) + S(r,f) + S(r,g)\\ &\leq [\Gamma_0+km_2+(k+1)d](T(r,f)+T(r,g))+S(r,f)+S(r,g).\end{aligned} \tag{2.4}$$

Similarly,

$$\begin{aligned}(n+\lambda)T(r,g) \leq [\Gamma_0 + km_2 + (k+1)d](T(r,f) + T(r,g))\\ +S(r,f) + S(r,g).\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5) we get

$$[n+\lambda-2\Gamma_0-2km_2-2(k+1)d](T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

which gives $n \leq 2\Gamma_0 + 2km_2 + 2(k+1)d - \lambda$. □

Lemma 2.9. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of zero orders, q_j, $c_j \in \mathbb{C}\backslash\{0\}$ $(j = 1, 2, ..., d)$ be distinct constants, n, d, $s_j (j = 1, 2, ..., d) \in \mathbb{N}_+$, and $P_0(z)$ be a polynomial. Then*

$$(P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})^{(k)}(P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j})^{(k)} \not\equiv (P_0(z))^2.$$

Proof. In the contrary, we may assume that

$$(P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})^{(k)}(P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j})^{(k)} = (P_0(z))^2. \tag{2.6}$$

From (2.6), it is clear that $P(f)$ and $P(g)$ have at most finitely many zeros as f, g are transcendental. Suppose $P(u)$ has two zeros, say u_1, u_2 $(u_1 \neq u_2)$.

Then $P(f) = a_n(f-u_1)^{n_1}(f-u_2)^{n_2}$ where n_1 and n_2 are positive integers and $n_1+n_2=n$. Therefore $f-u_1$ and $f-u_2$ have atmost finitely many zeros which is a contradiction by the second main theorem.

Next we consider $P(u)$ has only one zero. Then we can write $P(f) = a_n(f-a)^n$, where a is a complex constant. Hence we obtain

$$f(z) = e^{\alpha(z)} + a, \quad g(z) = e^{\gamma(z)} + a, \tag{2.7}$$

where $\alpha(z)$ and $\gamma(z)$ are nonzero polynomials. But f and g are functions of order zero, hence $\alpha(z)$ and $\gamma(z)$ must be constants, which contradicts the fact that f and g are transcendental. This proves the lemma. □

Lemma 2.10. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of zero orders, q_j, $c_j \in \mathbb{C}\backslash\{0\}$, $(j=1,2,...,d)$ be distinct constants, n, d, $s_j(j=1,2,...,d) \in \mathbb{N}_+$, and $\alpha(z)$ be a small function of f and g with finitely many zeros. Then*

$$(P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})^{(k)}(P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j})^{(k)} \not\equiv \alpha(z).$$

Proof. In the contrary, we may assume that

$$(P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})^{(k)}(P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j})^{(k)} = \alpha(z). \tag{2.8}$$

Then proceeding in a similar manner as in Lemma 2.9 we see that $P(f) = a_n(f-a)^n$, where a is a complex constant. If $k=0$, then (2.8) becomes

$$a_n(f-a)^n\prod_{j=1}^{d} f(q_jz+c_j)^{s_j} = \frac{\alpha(z)}{a_n(g-a)^n\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}}.$$

Using Lemma 2.6 and noting that $\alpha(z)$ has finitely many zeros, we obtain from above

$$\begin{aligned}\overline{N}(r,a;f)+\overline{N}(r,0;f(q_jz+c_j)) &\le \overline{N}(r,a;f)+\overline{N}(r,0;f)+S(r,f)\\ &\le N(r.0;\alpha(z)) = S(r,f).\end{aligned}$$

Using Nevanlinna's three small function theorem we obtain

$$\begin{aligned}T(r,f) &\le \overline{N}(r,0;f)+\overline{N}(r,a;f)+\overline{N}(r,\infty;f)\\ &\le S(r,f),\end{aligned}$$

a contradiction. If $k \ge 1$, then arguing in a similar manner as in Lemma 2.9 we arrive at a contradiction. This proves the lemma. □

3 Proof of theorems

Proof of Theorem 1.2. Let $F_0(z) = \frac{F(z)}{P_0(z)}$ and $G_0(z) = \frac{G(z)}{P_0(z)}$. Since F and G share $P_0(z)$ with weight l, F_0 and G_0 share 1 with weight l. Now, by Lemmas 2.2 and 2.7, we get

$$\begin{aligned} N_2(r,0;F_0) &\le N_2(r,0;F_1^{(k)}) + O\{\log r\} \\ &\le T(r,F_1^{(k)}) - T(r,F_1) + N_{k+2}(r,0;F_1) + O\{\log r\} \\ &= T(r,F) - (n+\lambda)T(r,f) + N_{k+2}(r,0;F_1) + O\{\log r\}. \end{aligned}$$

From this we get

$$\begin{aligned} (n+\lambda)T(r,f) &\le T(r,F) + N_{k+2}(r,0;F_1) - N_2(r,0;F_0) + O\{\log r\} \\ &\le T(r,F_0) - N_2(r,0;F_0) + N_{k+2}(r,0;F_1) + O\{\log r\}. \end{aligned} \tag{3.1}$$

Also by Lemma 2.2 we have

$$\begin{aligned} N_2(r,0;F_0) &\le N_2(r,0;F_1^{(k)}) + O\{\log r\} \\ &\le N_{k+2}(r,0;F_1) + O\{\log r\}. \end{aligned} \tag{3.2}$$

Case 1. *Let $l \ge 2$. Suppose (i) of Lemma 2.3 holds. Then,*

$$T(r,F_0) \le N_2(r,0;F_0) + N_2(r,0;G_0) + S(r).$$

Then, from (3.1) and (3.2), we have

$$\begin{aligned} (n+\lambda)T(r,f) &\le N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + O\{\log r\} \\ &\le N_{k+2}(r,0;P(f)) + N_{k+2}(r,0;\textstyle\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}) \\ &\quad + N_{k+2}(r,0;P(g)) + N_{k+2}(r,0;\textstyle\prod_{j=1}^{d} g(q_jz+c_j)^{s_j} \\ &\quad + O\{\log r\} \\ &\le (m_1+(k+2)m_2)T(r,f) + (k+2)\overline{N}(r,0;\textstyle\prod_{j=1}^{d} f(q_jz \\ &\quad +c_j)^{s_j}) + (m_1+(k+2)m_2)T(r,g) + (k+2)\overline{N}(r,0; \\ &\quad \textstyle\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}) + O\{\log r\} \\ &\le (\Gamma_1 + km_2 + (k+2)d)(T(r,f)+T(r,g)) + O\{\log r\}. \end{aligned} \tag{3.3}$$

Similarly,

$$(n+\lambda)T(r,g) \le (\Gamma_1 + km_2 + (k+2)d)(T(r,f)+T(r,g)) + O\{\log r\}. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$(n+\lambda-2\Gamma_1-2km_2-2(k+2)d)(T(r,f)+T(r,g)) \le O\{\log r\},$$

which contradicts our assumption that $n > 2\Gamma_1 + 2km_2 + 2(k+2)d - \lambda$. *We thus have either* $F_0G_0 = 1$ *or* $F_0 = G_0$.

If $F_0G_0 = 1$, *then*

$$(P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})^{(k)}(P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j})^{(k)} = (P_0(z))^2,$$

a contradiction by Lemma 2.9.

Thus $F_0 = G_0$. *Then*

$$(P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})^{(k)} = (P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j})^{(k)}.$$

Integrating once we obtain

$$(P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})^{(k-1)} = (P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j})^{(k-1)} + c_{k-1},$$

where c_{k-1} *is a constant.*

If $c_{k-1} \neq 0$, *from Lemma 2.8, we have* $n \leq 2\Gamma_0 + 2(k-1)m_2 + 2kd - \lambda$, *which is a contradiction. Hence we must have* $c_{k-1} = 0$. *Repeating this process* k *times, we deduce that*

$$P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j} = P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}.$$

Then the conclusion of the theorem follows from the proof of Case 2 of Theorem 1.3 [14].

Case 2. *Let* $l = 1$ *and* $H \not\equiv 0$, *Using Lemma 2.4 and (3.1), we get*

$$\begin{aligned}(n+\lambda)T(r,f) &\leq T(r,F_0) - N_2(r,0;F_0) + N_{k+2}(r,0;F_1) + O\{\log r\}\\ &\leq N_2(r,0;G_0) + \frac{1}{2}\overline{N}(r,0;F_0) + N_{k+2}(r,0;F_1) + O\{\log r\}\\ &\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + \frac{1}{2}N_{k+1}(r,0;F_1) + O\{\log r\}\\ &\leq [m_1 + (k+2)m_2 + (k+2)d](T(r,f) + T(r,g))\\ &\quad + \frac{1}{2}[m_1 + (k+1)m_2 + (k+1)d]T(r,f) + O\{\log r\}.\end{aligned}$$

Similarly,

$$(n+\lambda)T(r,g) \le (m_1+(k+2)m_2+(k+2)d)(T(r,f)+T(r,g)) + \frac{1}{2}[m_1+(k+1)m_2+(k+1)d]T(r,g)+O\{\log r\}.$$

Combining the above two inequalities, we obtain

$$\left\{n+\lambda-2m_1-2(k+2)m_2-2(k+2)d-\tfrac{1}{2}[m_1+(k+1)m_2+(k+1)d]\right\} (T(r,f)+T(r,g)) \le O\{\log r\}$$
$$\text{i.e., } [2n+2\lambda-4\Gamma_1-\Gamma_0-5km_2-(5k+9)d]\,(T(r,f)+T(r,g)) \le O\{\log r\},$$

which contradicts with the fact that $n > \frac{1}{2}(4\Gamma_1+\Gamma_0+5km_2+(5k+9)d-2\lambda)$.

We now assume $H \equiv 0$. *Then*

$$\frac{F_0''}{F_0'} - \frac{2F_0'}{F_0-1} = \frac{G_0''}{G_0'} - \frac{2G_0'}{G_0-1}.$$

Integrating both sides twice, we get

$$\frac{1}{F_0-1} = \frac{A}{G_0-1}+B, \tag{3.5}$$

where $A(\neq 0)$ *and* B *are constants.*

We now consider the following subcases.

Subcase 2.1. *Let* $B \neq 0$ *and* $A=B$. *Then we have from (3.5)*

$$\frac{1}{F_0-1} = \frac{BG_0}{G_0-1}.$$

If $B=-1$, *then we obtain* $F_0G_0=1$, *which can be dealt with as in case 1. If* $B\neq -1$, *we have* $\frac{1}{F_0} = \frac{BG_0}{(1+B)G_0-1}$ *and so* $\overline{N}(r,\frac{1}{B+1};G_0) = \overline{N}(r,0;F_0)$. *From the second main theorem of Nevanlinna we get,*

$$\begin{aligned} T(r,G_0) &\le \overline{N}(r,0;G_0)+\overline{N}(r,\frac{1}{B+1};G_0)+S(r,G_0) \\ &\le \overline{N}(r,0;F_0)+\overline{N}(r,0;G_0)+S(r,G_0). \end{aligned}$$

From this we obtain using (2.1) and (2.2) that

$$\begin{aligned} T(r,G) &\le \overline{N}(r,0;F)+\overline{N}(r,0;G)+O\{\log r\} \\ &\le N_{k+1}(r,0;F_1)+T(r,G)+N_{k+1}(r,0;G_1)-(n+\lambda)T(r,g) \\ &\quad +O\{\log r\} \end{aligned}$$

i.e., $(n+\lambda)T(r,g) \leq N_{k+1}(r,0;F_1)+N_{k+1}(r,0;G_1)+O\{\log r\}$
$\leq [m_1+(k+1)m_2+(k+1)d](T(r,f)+T(r,g))+O\{\log r\}.$

From this we get

$$[n+\lambda-2m_1-2(k+1)m_2-2(k+1)d]T(r) \leq O\{\log r\},$$

where $T(r)=\max\{T(r,f),T(r,g)\}$. *Therefore we have* $n \leq 2\Gamma_0+2km_2+2(k+1)d-\lambda$, *a contradiction.*

Subcase 2.2. *Let* $B \neq 0$ *and* $A \neq B$, *then from (3.5) we have* $F_0 = \frac{(B+1)G_0-(B-A+1)}{BG_0+(A-B)}$ *and* $\overline{N}(r,\frac{B-A+1}{B+1};G_0)=\overline{N}(r,0;F_0)$. *Proceeding similarly as in subcase 1 we arrive at a contradiction.*

Subcase 2.3. *Let* $B=0$. *Then from (3.5) we see that* $F_0=\frac{G_0+A-1}{A}$ *and* $G_0=AF_0-(A-1)$. *If* $A\neq 1$, *then we have* $\overline{N}(r,\frac{A-1}{A};F_0)=\overline{N}(r,0;G_0)$ *and* $\overline{N}(r,1-A;G_0)=\overline{N}(r,0;F_0)$. *Then by Lemma 2.8 we have* $n\leq 2\Gamma_0+2km_2+2(k+1)d-\lambda$, *a contradiction. Thus* $A=1$ *and then* $F_0=G_0$, *and the conclusion follows from the proof of Case 1.*

Case 3. *Let* $l=0$ *and* $H\not\equiv 0$. *Then from Lemma 2.5, (3.1) and (3.2), we get*

$$\begin{aligned}(n+\lambda)T(r,f) &\leq T(r,F_0)-N_2(r,0;F_0)+N_{k+2}(r,0;F_1)+O\{\log r\}\\ &\leq N_2(r,0;G_0)+2\overline{N}(r,0;F_0)+\overline{N}(r,0;G_0)\\ &\quad +N_{k+2}(r,0;F_1)+O\{\log r\}\\ &\leq N_2(r,0;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+N_{k+2}(r,0;F_1)\\ &\quad +O\{\log r\}\\ &\leq N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+2N_{k+1}(r,0;F_1)\\ &\quad +N_{k+1}(r,0;G_1)+O\{\log r\}\\ &\leq (m_1+(k+2)(m_2+d))(T(r,f)+T(r,g))\\ &\quad +2(m_1+(k+1)(m_2+d))\\ &\quad T(r,f)+(m_1+(k+1)(m_2+d))T(r,g)+O\{\log r\}.\end{aligned} \tag{3.6}$$

Similarly,

$$\begin{aligned}(n+\lambda)T(r,g) &\leq (m_1+(k+2)(m_2+d))(T(r,f)+T(r,g))\\ &\quad +2(m_1+(k+1)(m_2+d))\\ &\quad T(r,g)+(m_1+(k+1)(m_2+d))T(r,f)+O\{\log r\}.\end{aligned} \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$[n+\lambda-5m_1-7m_2-5km_2-(5k+7)d](T(r,f)+T(r,g)) \leq O\{\log r\},$$

which contradicts with the assumption that $n \geq 2\Gamma_1 + 3\Gamma_0 + 5km_2 + (5k + 7)d - \lambda$. *Therefore we must have* $H \equiv 0$ *and the conclusion of the theorem follows from the proof of Case 2.*

□

Proof of Theorem 1.1. This theorem can be proved in a similar way as in the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let $F_0(z) = \frac{F(z)}{\alpha(z)}$ and $G_0(z) = \frac{G(z)}{\alpha(z)}$. Since F and G share $\alpha(z)$ with weight l, F_0 and G_0 share 1 with weight l except the zeros and poles of $\alpha(z)$. Then using Lemma 2.10 and proceeding in a like manner as in Theorem 1.2 we obtain the conclusions of the theorem. □

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