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Tomul LXIV, 2018, f. 2

On different notions of higher curvature

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Abstract We compare different notions of higher scalar and Ricci curvature of Shen, of Wolfson and of the author. We show that k-positivity of curvature in the sense of Wolfson or Shen is equivalent to a system of polynomial inequalities in the higher curvatures considered by the author. The proof uses algebraic methods for positive quadratic forms and for symmetric polynomials. Moreover, we make this explicit in dimensions 3, 4 and 5.

Keywords Scalar curvature · Ricci curvature · higher curvature invariants · quadratic form · k-positivity

Mathematics Subject Classification (2010) 15A42 · 15A63 · 53A45 · 53A55 · 53B20

1 Summary of results

There exist several definitions of higher curvatures with different algebraic properties and geometric implications.

In [5], the author has shown that the positivity of the Ricci curvature Ric of an n-dimensional Riemannian manifold (M, g) is equivalent to the positivity of all higher scalar curvatures $scal_k$, $1 \leq k \leq n$, and that the positivity of the sectional curvature K is equivalent to the positivity of all higher Ricci curvatures Ric_k, $1 \leq k \leq n-1$. Positivity of the curvature operator was also considered but we do not use these results here. There are also similar statements for non-negative curvature. These results were proved by an algebraic lemma concerning positive definite quadratic forms.

We relate these algebraic notions to the higher Ricci curvature definitions of Z. Shen [7] and to the k-positive Ricci curvature of J. Wolfson [9] which have interesting geometrical applications.

2 Higher curvature notions

The author's results on the positivity relation between scalar, Ricci and sectional curvature in [5] can be summarized as follows:

$$\begin{aligned} Ric > 0 &\Leftrightarrow scal_k > 0 \quad \text{for all } k = 1, 2, .., n \\ K > 0 \quad &\Leftrightarrow Ric_k > 0 \quad \text{for all } k = 1, 2, .., n-1 \end{aligned}$$

On the left-hand side,

$$Ric : S(TM) \to \mathbb{R}$$

is the Ricci curvature of an n-dimensional Riemannian manifold (M, g) which is a quadratic function on the tangent unit sphere bundle, and

$$K : G_2(TM) \to \mathbb{R}$$

is the sectional curvature which is a function on the Grassmannian bundle of tangent 2-planes on M.

On the right-hand side, there are the higher scalar curvature functions

$$scal_k : M \to \mathbb{R},$$

which are defined as the elementary symmetric invariants

$$scal_k = \sigma_k(\lambda_1, \ldots, \lambda_n)$$

in the eigenvalues λ_i of the Ricci quadratic function. In particular, $scal_1 = scal$ is the standard scalar curvature.

The higher Ricci curvatures are homogeneous symmetric forms

$$Ric_k : S(TM) \to \mathbb{R}$$

of order $2k$ on the tangent unit sphere bundle, in particular $Ric_1 = Ric$ is the standard Ricci curvature. These are defined as the elementary symmetric invariants

$$Ric_k(v) = \sigma_k(\mu_1^v, \ldots, \mu_n^v)$$

in the eigenvalues μ_i^v of the (bi-)quadratic function

$$Q^v(w) := R(v, w, v, w)$$

with v fixed. As $Q^v(w)$ is only interesting on the orthogonal complement $v^\perp$, at least one of the eigenvalues μ_i^v vanishes and we have $Ric_n(v) = 0$.

The result above displays the information lost in the implications

$$K > 0 \quad \Rightarrow \quad Ric > 0 \quad \Rightarrow \quad scal > 0$$

which consists in the additional positivity of the higher scalar curvatures respectively higher Ricci curvatures.

In [5] these results were proved by an application of an algebraically invariant version of Sylvester's criterion on positive definite matrices and their elementary invariants:

Lemma 2.1 *(Positivity lemma, [5]) A real symmetric matrix is positive if and only if its elementary invariants are positive. In particular, a set of n real numbers λ_i $(1 \leq i \leq n)$ consists of positive numbers if and only if their elementary symmetric polynomials σ_i $(1 \leq i \leq n)$ are positive.*

Of course, only the 'if' part of this statement is interesting, whereas the 'only if' part is trivial. Moreover, in [5] the case of non-negative curvature is discussed and similar results were obtained. The main interest lies in the reduction of the number of dimensions from the left-hand to the right hand side.

In order to give an impression of these higher invariants, we cite from [5] the explicit expressions for $k = 2$ and 3 for the higher scalar curvatures

$$\begin{aligned} scal_2 &= \tfrac{1}{2}((R_i^i)^2 - R_i^j R_j^i) \\ scal_3 &= \tfrac{1}{6}((R_i^i)^3 - 3R_i^i R_j^k R_k^j + 2R_i^j R_j^k R_k^i) \end{aligned}$$

and for the higher Ricci curvatures:

$$\begin{aligned} Ric_2(v) &= \tfrac{1}{2}((R^i_{\alpha i\beta} v^\alpha v^\beta)^2 - R^j_{\alpha i\beta} v^\alpha v^\beta R^i_{\gamma j\delta} v^\gamma v^\delta) \\ Ric_3(v) &= \tfrac{1}{6}((R^i_{\alpha i\beta} v^\alpha v^\beta)^3 - 3R^i_{\alpha i\beta} v^\alpha v^\beta R^k_{\gamma j\delta} v^\gamma v^\delta R^j_{\mu k\nu} v^\mu v^\nu \,. \\ &\qquad +2R^j_{\alpha i\beta} v^\alpha v^\beta R^k_{\gamma j\delta} v^\gamma v^\delta R^i_{\mu k\nu} v^\mu v^\nu) \end{aligned}$$

This definition of Ric_k is different from the notion of higher positive Ricci curvature given by Z. Shen in [7]:

$$Ric^{(k)}_x \geq c :\Leftrightarrow \sum_{i=1}^{k+1} R_{klkl} e^i_k v_l e^i_k v_l \geq c$$

where the right-hand side has to hold for all $(k+1)$-dimensional subspaces $V^{k+1} \subset T_xM$ and for all unit vectors $v \in V^{k+1}$, and where e^i, $i = 1, 2, \ldots, k+1$ denotes an (arbitrary) orthonormal basis (ONB) of V^{k+1}. In order to have a real function, we *define kth Shen-Ricci curvature* $Ric^{(k)}(v)$ to be the minimum of the left hand side with respect to all V^{k+1} that contain v:

$$Ric^{(k)}(v) := \min\{\sum_{i=1}^{k+1} R_{klkl} e^i_k v_l e^i_k v_l | v \in V^{k+1} \subset T_xM, e^i \text{ ONB of } V^{k+1}\}$$

Note that the minimum is realized by some V^{k+1} as the corresponding Grassmann manifold is compact. This gives a continuous function

$$Ric^{(k)} : S(TM) \to \mathbb{R}.$$

There are interesting geometric applications, for example, kth positive Ricci curvature in the sense of Shen gives restriction on the embedding of totally geodesic submanifolds which sometimes can be used to derive

vanishing results for characteristic numbers: see remark 2.4, p. 267 in the work of C. Böhm and B. Wilking on manifolds with positive curvature operator [1] and section 6 in A. Dessai's work on obstructions to positive curvature [4].

Moreover, J. Wolfson defines in [9] k-positive Ricci curvature by the positivity of the sum of the first k eigenvalues (where the eigenvalues are increasingly ordered)

$$\lambda_1 + \lambda_2 + \ldots + \lambda_k$$

of the Ricci quadratic function Ric (in an arbitrary point x). As Wolfson shows, this notion of positivity is strongly related to topological invariants and surgery theory. See also the recent work of D. Crowley and D. J. Wraith [2] for applications of k-positive Ricci curvature to highly connected manifolds which are based on the work of S. Hoelzel on surgery stable curvature conditions [3].

However, note that our definition of higher scalar and Ricci curvature is non-linear in the eigenvalues and thus does not satisfy the positive cone criterion of S. Hoelzel for surgery stability in [3].

In the next section, we consider the minimum of traces of restrictions of quadratic forms to subspaces and its connection to the sum of its first k eigenvalues. This gives us a reformulation of Shen's k-th Ricci curvature $Ric_x^{(k)}$ and Wolfson's sum of first k eigenvalues of the Ricci form.

3 Restrictions of quadratic forms and eigenvalues

By an *n-dimensional quadratic form* A, we understand an n-dimensional euclidean vector space V together with a real symmetric bilinear form A on V. Choosing an orthonormal basis of V, A is represented by a symmetric n-matrix. By Sylvester's law, we can find an orthonormal basis which diagonalizes A. As usual, we call the real diagonal elements

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

the eigenvalues of A and the elementary symmetric functions

$$\sigma_k(\lambda_1, \lambda_2, \ldots \lambda_n)$$

for $k = 1, \ldots, n$ the *elementary invariants* of A. In particular, $\sigma_1 = tr(A)$ is the trace and $\sigma_n = det(A)$ is the determinant of A. It is well-known how to compute the elementary invariants in terms of traces of powers of A by the Newton formula (e.g. see [5]).

If U is a subspace of V, the restriction of A to U gives a quadratic form A_U. Now we define the *k-subspace trace* of A as

$$tr^{(k)}(A) := \min\{tr(A_U) | dim(U) = k\}.$$

The next lemma answers the question how small $tr(A_U)$ can be.

Proposition 3.1 *Let A be n-dimensional quadratic form with eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$

Then the k-subspace trace, which is the minimum of $tr(A_U)$ for all k-dimensional subspaces U of V, is given by the sum of the first k eigenvalues:

$$tr^{(k)}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_k.$$

Proof. Let P be the orthogonal projection operator for U, i.e. P is a symmetric matrix with $P^2 = P$ and $tr(P) = dim(U) = k$. Then the restriction of A_U is, up to a trivial summand of dimension $n-k$, given by $P^T AP$. Hence $tr(A_U) = tr(P^T AP) = tr(AP)$ and by choosing an orthonormal basis which diagonalizes A we obtain $tr(AP) = \sum_{i=1}^n p_{ii}\lambda_i$. As $p_{ii} = \sum_j p_{ij}^2 \geq 0$ and $\sum_{i=1}^n p_{ii} = k$, $tr(AP)$ is in the convex hull of the sums of k eigenvalues λ_i and takes its minimum if $p_{ii} = 1$ for $i = 1, \ldots k$ and $p_{ii} = 0$ for $i > k$. This is exactly the case if P is the orthogonal projection to the subspace U spanned by the eigenvectors of the first k eigenvalues. □

A quadratic form is called *k-positive* if $tr^{(k)}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_k > 0$. As an example, $tr^{(k)}(Ric) > 0$ is just the condition of Wolfson's k-positive Ricci curvature.

Note that $\sigma_k(A)$ is a polynomial function of A and hence is also differentiable with respect to A. In contrast, the k-subspace trace $tr^{(k)}(A)$ depends continuously on A, but is in general not differentiable with respect to A. For example, the restriction of $tr^{(1)}(A)$ to the linear subspace of diagonal matrices A is given by the minimum of the diagonal elements.

Moreover, Shen's higher Ricci curvature

$$Ric^{(k)}(v) = \min\{\sum_{i=1}^{k+1} R_{klkl} e_k^i v_l e_k^i v_l | v \in V^{k+1} \subset T_xM, e^i \text{ ONB of } V^{k+1}\}$$

is given by

$$Ric^{(k)}(v) = tr^{(k)}(Q^v|_{v^\perp})$$

where $Q^v(w) = R(v,w,v,w)$ denotes the biquadratic form associated to sectional curvature, which is quadratic in v and in w, and $Q^v|_{v^\perp}$ denotes its restriction to $v^\perp$. Note that $Q^v(v) = 0$, hence the minimum over all V^{k+1} which contain v equals the minimum over all V^k in $v^\perp$. This is just the definition of $tr^{(k)}$ applied to $Q^v|_{v^\perp}$.

The proof of the following lemma is straightforward just by inserting the definitions:

Lemma 3.2 *An n-dimensional quadratic form A is positive definite if and only if $tr^{(1)}(A) > 0$, and A is non-negative if and only $tr^{(1)}(A) \geq 0$. Moreover, A is non-negative with degeneracy k (number of eigenvalues which are zero) if and only if*

$$tr^{(1)}(A) = tr^{(2)}(A) = \ldots = tr^{(k)}(A) = 0$$

and $tr^{(k+1)}(A) > 0$. *It holds* $\lambda_i = tr^{(i)}(A) - tr^{(i-1)}(A)$ *and*

$$\lambda_1 = tr^{(1)}(A) \leq \frac{1}{2}tr^{(2)}(A) \leq \frac{1}{3}tr^{(3)}(A) \leq \ldots \leq \frac{1}{n}tr^{(n)}(A) = \frac{1}{n}tr(A).$$

Moreover, if $tr^{(i)}(A) > 0$ *then it follows also* $\lambda_k > 0$ *and* $tr^{(k)}(A) > 0$ *for all* $k \geq i$.

By definition, $tr^{(n)}(A) = tr(A) = \sigma_1(A)$. The relation between $\sigma_2(A)$ and the k-subspace traces of A is more complicated:

Proposition 3.3 *Let* A *be an* n*-dimensional quadratic form with eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$

Then it holds

$$\sigma_2(A) = \lambda_1\lambda_2 + tr^{(2)}(A)\lambda_3 + tr^{(3)}(A)\lambda_4 + \ldots + tr^{(n-1)}(A)\lambda_n.$$

If $n \geq 4$, *then* $tr^{(2)}(A) > 0$ *implies* $\sigma_2(A) > 0$. *This is not the case for* $n \leq 3$. *Moreover, for* $n > 4$ *there are* A *with* $tr^{(2)}(A) < 0$ *but* $\sigma_2(A) > 0$.

Proof. The first equation is easily checked by inserting the definition of $tr^{(k)}(A)$. Now, the condition $\lambda_1 + \lambda_2 = tr^{(2)}(A) > 0$ is equivalent to $-\lambda_2 < \lambda_1 \leq \lambda_2$. Products of the form $\lambda_1\lambda_k$ are critical as they can be negative, whereas products $\lambda_i\lambda_j$ are positive for all $2 \leq i < j$. But the sum of the products $\lambda_1\lambda_k + \lambda_2\lambda_k$ is positive for $k \geq 3$. The sum of the remaining critical product $\lambda_1\lambda_2$ with $\lambda_2\lambda_4$ is also positive. This proves $\sigma_2(A) > 0$ for $n \geq 4$. Counter examples for $n \leq 3$ are given by $(\lambda_1, \lambda_2) = (-0.5, 1)$ and $(\lambda_1, \lambda_2, \lambda_3) = (-0.6, 1, 1)$. For the last statement, consider $(\lambda_1, \lambda_2, \ldots, \lambda_n) = (-1, 0.5, 1, 1, \ldots, 1)$. □

It seems to be difficult to generalize this method to find a relation between $\sigma_k(A)$ and $tr^{(k)}(A)$ for $k > 2$. Instead we will use methods from the theory of symmetric polynomials in the next section in order to give equivalent conditions for k-positivity.

4 *k*-positivity of quadratic forms

Now we will use methods from the theory of symmetric polynomials in order to show how to measure k-positivity of a quadratic form by its elementary invariants.

The idea is to consider not only the special k-fold sum $\lambda_1 + \lambda_2 + \ldots + \lambda_k$ of the lowest eigenvalues but all such sums of k eigenvalues

$$\Lambda_I := \lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_k}$$

with I denoting the multi-index given by $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. The number of these multi-indices is $N := \binom{n}{k}$ and we can form the elementary symmetric polynomials of the new variables Λ_I:

$$a_j := \sigma_j(\Lambda_I | \text{ all multi-indices } I)$$

for $j = 1, 2, \ldots N$. Thus we have

$$\sum_{j=0}^{N} a_j t^j = \prod_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} (t + \lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_k})$$

As the a_j are symmetric in the λ_i by definition, the main theorem on symmetric polynomials ([8], chapter 7) yields that they can be uniquely expressed as universal polynomials $A_j^{n,k}(\sigma_1, \ldots, \sigma_n)$ in the elementary symmetric polynomials σ_i of the variables λ_i.

Proposition 4.1 *A quadratic form A is k-positive if and only if its elementary invariants $\sigma_i(A)$ satisfy $A_j^{n,k}(\sigma_1, \ldots, \sigma_n) > 0$ for all $j = 1, 2, \ldots, N := \binom{n}{k}$.*

Proof. k-positivity is is defined as $\Lambda_{12..k} > 0$ where the λ_i are the eigenvalues of A. As $\Lambda_I \geq \Lambda_{12..k}$ for all I, this is equivalent to the positivity of all Λ_I. By the positivity lemma, this is equivalent to $A_j^{n,k}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ for all $j = 1, \ldots, N$. □

Note that these universal polynomials also express the j-th Chern class $c_j(\Lambda^k E)$ of the k-th exterior power of an n-dimensional complex vector bundle E by its Chern classes $c_i(E)$:

$$c_j(\Lambda^k E) = A_j^{n,k}(c_1(E), c_2(E), \ldots, c_n(E)).$$

This follows from the splitting principle, from the fact

$$\Lambda^k(L_1 \oplus L_2 \oplus \ldots \oplus L_n) \cong \bigoplus_I L_{i_1} \otimes L_{i_2} \otimes \cdots \otimes L_{i_k}$$

and because of $c_1(L \otimes L') = c_1(L) + c_1(L')$ for line bundles (e.g., see chapter 14 in [6]).

Now we give first some trivial cases of these universal polynomials:

Lemma 4.2 *In any dimension n, it holds $A_j^{n,1} = \sigma_j$ for all j,*

$$A_1^{n,k} = \tbinom{n-1}{k-1}\sigma_1$$

for all k, $A_1^{n,n} = \sigma_1$ and $A_j^{n,k} = 0$ for all $j > N := \binom{n}{k}$.

Proof. The first statement is trivial as for $k = 1$, the Λ_I are just given by the λ_i. The second statement follows as $\sum_I \lambda_I$ consists of a $k\binom{n}{k}$-fold sum of the λ_j which is symmetric and thus has to be a multiple of σ_1. As σ_1 has n summands, the factor is given by $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$. The third statement is trivial again. The fourth statement follows from the fact that the number of variables Λ_I is N. □

The next two cases are more complicated:

Lemma 4.3 *In any dimension n, it holds*

$$A_2^{n,2} = \binom{n-1}{2}\sigma_1^2 + (n-2)\sigma_2.$$

Proof. There are $\binom{n}{2}$ variables $\Lambda_{ij} = \lambda_i + \lambda_j$ with $i < j$. Hence $a_2 = \sum(\lambda_i + \lambda_j)(\lambda_k + \lambda_l)$ where the sum is over all lexicographically ordered pairs $(ij) < (kl)$ (with $i < j$ and $k < l$). As a symmetric polynomial of degree 2, it can be expressed as a linear combination of σ_1^2 and σ_2.

In order to find the coefficients of this decomposition, we have just to look at the term λ_1^2 (which correspond to $\sigma_1^2 - 2\sigma_2$) and at the term $\lambda_1\lambda_2$ (which correspond to σ_2). Now λ_1^2 occurs in a_2 by multiplying the ordered pairs $(1, j)$ and $(1, l)$ (with $1 < j < l$) and there are exactly $\binom{n-1}{2}$ of such pairs. Moreover $\lambda_1\lambda_2$ occurs in a_2 by multiplying the ordered pairs $(1, j)$ and $(2, l)$ (with $1 < j$ and $2 < l$) or by multiplying the ordered pairs $(1, 2)$ and $(1, i)$ (with $2 < i$). Hence the coefficient is given by $(n-1)(n-2)+n-2 = n(n-2)$. Thus we have

$$a_2 = \binom{n-1}{2}(\sigma_1^2 - 2\sigma_2) + n(n-2)\sigma_2$$

which proves the statement. □

Lemma 4.4 *In any dimension n, it holds*

$$A_l^{n,n-1} = \sum_{j=0}^{l}(-1)^{l-j}\binom{n-l+j}{j}\sigma_1^j\sigma_{l-j}$$

$$= (-1)^l\sigma_l + (-1)^{l-1}\binom{n-l+1}{1}\sigma_1\sigma_{l-1} + \ldots + \binom{n-2}{l-2}\sigma_1^{l-2}\sigma_2 + \binom{n}{l-1}\sigma_1^l.$$

Proof. Here, we denote the Λ_I as complementary variables $\bar{\lambda}_i := \sigma_1 - \lambda_i$. Moreover, the elementary symmetric functions $a_j^{(n-1)}$ in the $\bar{\lambda}_i$ are denoted by $\bar{\sigma}_k$. Then we have

$$\sum_{k=0}^{n} t^k\bar{\sigma}_{n-k} = \prod_{i=1}^{n}(t + \bar{\lambda}_i) = \prod_{i=1}^{n}((t + \sigma_1) - \lambda_i) =$$

$$= \sum_{k=0}^{n} (t+\sigma_1)^k (-1)^{n-k} \sigma_{n-k} = \sum_{i,j\geq 0}^{i+j\leq k} (-1)^{n-i-j} \binom{i+j}{j} \sigma_1^j \sigma_{n-i-j} t^i.$$

This proves that $\bar{\sigma}_l = \sum_{j=0}^{l} (-1)^{l-j} \binom{n-l+j}{j} \sigma_1^j \sigma_{l-j}$. Note that the last two terms ($j = l$ and $j = l-1$) give

$$\binom{n}{l} \sigma_1^l \sigma_0 - \binom{n-1}{l-1} \sigma_1^{l-1} \sigma_1 = \binom{n}{l-1} \sigma_1^l$$

which proves the statement. □

Corollary 4.5 *For $n = 3$ it holds*

$$\begin{aligned} A_1^{3,2} &= 2\sigma_1 & A_1^{3,3} &= \sigma_1 \\ A_2^{3,2} &= \sigma_2 + \sigma_1^2 \\ A_3^{3,2} &= -\sigma_3 + \sigma_2\sigma_1 \end{aligned}$$

Already the next dimension $n = 4$ needs a more elaborate computation.

Theorem 4.6 *For $n = 4$ it holds*

$$\begin{aligned} A_1^{4,2} &= 3\sigma_1 & A_1^{4,3} &= 3\sigma_1 \\ A_2^{4,2} &= 2\sigma_2 + 3\sigma_1^2 & A_2^{4,3} &= \sigma_2 + 3\sigma_1^2 \\ A_3^{4,2} &= 4\sigma_2\sigma_1 + \sigma_1^3 & A_3^{4,3} &= -\sigma_3 + 2\sigma_2\sigma_1 + \sigma_1^3 \\ A_4^{4,2} &= -4\sigma_4 + \sigma_3\sigma_1 + \sigma_2^2 + 2\sigma_2\sigma_1^2 & A_4^{4,3} &= \sigma_4 - \sigma_3\sigma_1 + \sigma_2\sigma_1^2 \\ A_5^{4,2} &= -3\sigma_4\sigma_1 + \sigma_3\sigma_1^2 + \sigma_2^2\sigma_1 \\ A_6^{4,2} &= -\sigma_4\sigma_1^2 - \sigma_3^2 + \sigma_3\sigma_2\sigma_1 \end{aligned}$$

and $A_1^{4,4} = \sigma_1$.

Proof. We have already computed $A_1^{4,2}$, $A_2^{4,2}$, $A_j^{4,3}$ (for $j = 1, 2, 3, 4$) and $A_1^{4,4}$. Of course it is possible to compute the remaining terms by applying symbolic computational software such as Maple, Mathematica or Sage to expand the product

$$\begin{aligned} &(t+\lambda_1+\lambda_2)(t+\lambda_1+\lambda_3)(t+\lambda_1+\lambda_4)(t+\lambda_2+\lambda_3)(t+\lambda_2+\lambda_4)(t+\lambda_3+\lambda_4) \\ &\quad = t^6 + A_1^{4,2} t^5 + A_2^{4,2} t^4 + A_3^{4,2} t^3 + A_4^{4,2} t^2 + A_5^{4,2} t^1 + A_6^{4,2} \end{aligned}$$

and to find the polynomial expressions of the coefficients $A_j^{4,2}$ in terms of the elementary symmetric functions. Instead we describe here the combinatorial computation by hand, avoiding the brute force expansion of the product which involves $3^6 = 729$ terms.

In order to expand the remaining terms $A_j^{4,2}$ for $j = 3, 4, 5$ and 6 in elementary symmetric polynomials, we utilize the following standard method of combinatorial analysis. Let $I := (i_1, i_2, \ldots, i_r)$ be a partition of d, i.e.

$i_1 \geq i_2 \geq \ldots \geq i_r > 0$ and $i_1 + i_2 + \ldots + i_r = d$. Denote the symmetric polynomial which is the symmetrization of $\lambda^I := \lambda_1^{i_1}\lambda_2^{i_2}\cdots\lambda_r^{i_r}$ by

$$m^I := \lambda_1^{i_1}\lambda_2^{i_2}\cdots\lambda_r^{i_r} + \text{sym.}$$

In particular, we have

$$m^{(11..1)} = \sigma_r$$

with r-times index 1 in $m^{(11..1)}$. The m^I form the so-called *monomial basis* of the ring of all symmetric polynomials and λ^I is called the *leading term* of m^I. Hence $m^{(1)}$ is the only basis vector of degree 1, $m^{(2)}$ and $m^{(11)}$ form the basis in degree 2, $m^{(3)}$, $m^{(21)}$ and $m^{(111)}$ form the basis in degree 3, and so on. Then the $A_j^{n,k}$ and also products of elementary symmetric polynomials can be expanded in the m^I and this enables us to expand the $A_j^{n,k}$ in elementary symmetric polynomials.

Moreover, a formula can be checked by applying the *augmentation homomorphism* to both sides of an equation. This gives a necessary (not sufficient, in general) condition that an equation is true. Augmentation associates an integer $\epsilon(p)$ to every polynomial p by inserting 1 for every variable. In particular, $\epsilon(\sigma_j) = \binom{n}{j}$, $\epsilon(A_j^{n,k}) = k^j\binom{N}{j}$, and $\epsilon(m^I) = \frac{n!}{l_1!l_2!\cdots l_s!}$ is the symmetry factor of the partition I, i.e. $(l_1, l_2, \ldots, l_s)$ give the size of groups of equal indices (exponents) i_j in I. Note that we count here also the frequency of the exponent 0 in a monomial. For example, $\epsilon(m^{(k)}) = n$ (one time exponent k and $n-1$ times exponent 0), $\epsilon(m^{(k,k)}) = \binom{n}{2}$ (two times exponent k and $n-2$ times exponent 0) if $k > 0$ and $\epsilon(m^{(k,l)}) = n(n-1)$ if $k > l > 0$.

For $n = 4$, the combinatorics of products of the Λ_{ij} can be visualized by the following matrix which allows an easy computation of the coefficients of $A_j^{4,2}$ in the monomial basis by a quick counting of all combinations of rows which produce the corresponding leading term.

	λ_1	λ_2	λ_3	λ_4
Λ_{12} :	1	1		
Λ_{13} :	1		1	
Λ_{14} :	1			1
Λ_{23} :		1	1	
Λ_{24} :		1		1
Λ_{34} :			1	1

We apply this method to $A_3^{4,2}$ and this gives

$$A_3^{4,2} = m^{(3)} + 7m^{(2,1)} + 18m^{(1,1,1)},$$

checking augmentation of both sides gives the value 160:

$$2^3\binom{6}{3} = \frac{4!}{1!3!} + 7\frac{4!}{1!1!2!} + 18\frac{4!}{3!1!}.$$

Moreover, we compute the decomposition of products of elementary symmetric polynomials into the monomial basis as follows. In order to find the coefficient of m^I for a product of σ_j, we have to count all possibilities how the leading term λ^I can be produced by the factors σ_j. E.g. $\lambda^{(21)}$ can be produced from $\sigma_1^3 = \sigma_1\sigma_1\sigma_1$ by the combinations $\lambda_2\lambda_1\lambda_1$, $\lambda_1\lambda_2\lambda_1$ or $\lambda_1\lambda_1\lambda_2$, which shows that the corresponding coefficient is given by 3. More generally, we have to know how the product of m^I and m^J decomposes in the monomial basis. The same argument as above gives

$$m^I \cdot m^J = \sum_K c_K^{I,J} m^K$$

where the coefficient $c_K^{I,J}$ is given by the number of decompositions of the partition K into non-negative multi-indices I' and J' (i.e., $k_l = i'_l + j'_l$ for all l) such that I' and J' are given by permutation of the partitions I and J, respectively (with 0 inserted at suitable places, if necessary). For example, it is straightforward to see that

$$\begin{aligned} m^{(1)}m^{(1)} &= m^{(2)} + 2m^{(11)} \\ m^{(2)}m^{(1)} &= m^{(3)} + m^{(21)} \\ m^{(11)}m^{(1)} &= m^{(21)} + 3m^{(111)} \end{aligned}$$

This method gives the following matrix

$$\begin{aligned} \sigma_1^3 &= m^{(3)} + 3m^{(21)} + 6m^{(111)} \\ \sigma_2\sigma_1 &= m^{(21)} + 3m^{(111)} \\ \sigma_3 &= m^{(111)} \end{aligned}$$

and we easily obtain from this the stated expansion of $A_3^{4,2}$ in elementary symmetric polynomials. See the first four sections of chapter 7 in [8] for more properties of this decomposition (in particular, the coefficients form a triangle matrix with 1 as diagonal elements).

As next we compute $A_4^{4,2}$ which gives

$$A_4^{4,2} = 2m^{(3,1)} + 5m^{(2,2)} + 13m^{(2,1,1)} + 30m^{(1,1,1,1)}$$

with augmentation of both sides equal to 240:

$$2^4\binom{6}{4} = 2\frac{4!}{1!1!2!} + 5\frac{4!}{2!2!} + 13\frac{4!}{1!2!1!} + 30\frac{4!}{4!}.$$

Because of

$$\begin{aligned} \sigma_1^4 &= m^{(4)} + 4m^{(31)} + 6m^{(22)} + 12m^{(211)} + 24m^{(1111)} \\ \sigma_2\sigma_1^2 &= m^{(31)} + 2m^{(22)} + 5m^{(211)} + 12m^{(1111)} \\ \sigma_2^2 &= m^{(22)} + 2m^{(211)} + 6m^{(1111)} \\ \sigma_3\sigma_1 &= m^{(211)} + 4m^{(1111)} \\ \sigma_4 &= m^{(1111)} \end{aligned}$$

we obtain the stated expansion of $A_4^{4,2}$.

For $A_5^{4,2}$ we obtain the following decomposition:

$$A_5^{4,2} = m^{(3,2)} + 3m^{(3,1,1)} + 7m^{(2,2,1)} + 15m^{(2,1,1,1)}$$

with augmentation of both sides equal to 192:

$$2^5\binom{6}{5} = \frac{4!}{1!1!2!} + 3\frac{4!}{1!2!1!} + 7\frac{4!}{2!1!1!} + 15\frac{4!}{1!3!}.$$

Now we use the table (computed for the case of 5 variables, we list here the coefficients of the decomposition as a matrix)

	$m^{(5)}$	$m^{(41)}$	$m^{(32)}$	$m^{(311)}$	$m^{(221)}$	$m^{(2111)}$	$m^{(11111)}$
σ_1^5 :	1	5	10	20	30	60	120
$\sigma_2\sigma_1^3$:		1	3	6	12	27	60
$\sigma_2^2\sigma_1$:			1	2	5	12	30
$\sigma_3\sigma_1^2$:				1	2	6	20
$\sigma_3\sigma_2$:					1	3	10
$\sigma_4\sigma_1$:						1	5
σ_5 :							1

In our case, $m^{(5)}$ and $m^{(41)}$ do not occur as no variable appears more than 3 times in $A_5^{4,2}$. Moreover, $m^{(11111)} = \sigma_5 = 0$ as there are only four variables. After a straightforward computation this yields the stated expansion of $A_5^{4,2}$.

For $A_6^{4,2}$ we get the following decomposition by a tedious computation

$$A_6^{4,2} = m^{(3,2,1)} + 2m^{(3,1,1,1)} + 2m^{(2,2,2)} + 4m^{(2,2,1,1)}$$

with augmentation of both sides equal to 64:

$$2^6 = \frac{4!}{1!1!1!1!} + 2\frac{4!}{1!3!} + 2\frac{4!}{3!1!} + 4\frac{4!}{2!2!}.$$

Now we use the following matrix computed for the case of 6 variables, where the columns denote the coefficients of the decomposition of products of elementary symmetric polynomials into the monomial basis with order $m^{(6)}$, $m^{(51)}$, $m^{(42)}$, $m^{(411)}$, $m^{(33)}$, $m^{(321)}$, $m^{(3111)}$, $m^{(222)}$, $m^{(2211)}$, $m^{(21111)}$ and $m^{(111111)}$:

σ_1^6 :	1	6	15	30	20	60	120	90	180	360	720
$\sigma_2\sigma_1^4$:		1	4	9	6	22	48	36	78	168	360
$\sigma_2^2\sigma_1^2$:			1	2	2	8	18	15	34	78	180
$\sigma_3\sigma_1^3$:				1	0	3	10	6	18	48	120
σ_2^3 :					1	3	6	6	15	36	90
$\sigma_3\sigma_2\sigma_1$:						1	3	3	8	22	60
$\sigma_4\sigma_1^2$:							1	0	2	9	30
σ_3^2 :								1	2	6	20
$\sigma_4\sigma_2$:									1	4	15
$\sigma_5\sigma_1$:										1	6
σ_6 :											1

In our case, the basis polynomials from $m^{(6)}$ to $m^{(33)}$ do not occur as no variable appears more than 3 times and also terms like $\lambda_1^3\lambda_2^3$ cannot occur. After a straightforward computation this yields the stated expansion of $A_6^{4,2}$. □

5 Application to Wolfson's *k*-positivity

Now we apply the universal polynomials to Wolfson's k-positive Ricci curvature.

Theorem 5.1 *Wolfson's k-positive Ricci curvature is equivalent to the nonlinear system of N inequalities given by the $A_j^{n,k}$ with variables σ_i replaced by the higher scalar curvatures $scal_i$:*

$$A_j^{n,k}(scal_1, scal_2, \ldots, scal_n) > 0$$

for all $j = 1, 2, \ldots, N$ with $N := \binom{n}{k}$.

Proof. Wolfson's k-positive Ricci curvature is equivalent to $\Lambda_{12..k} > 0$ where the λ_i are the eigenvalues of the Ricci curvature form Ric (in an arbitrary point x which we fix at the moment). As $\Lambda_I \geq \Lambda_{12..k}$ for all I, this is equivalent to the positivity of all Λ_I. By the positivity lemma, this is equivalent to $A_j^{n,k}(\sigma_1, \ldots, \sigma_n)$ for all $j = 1, \ldots, N$. This proves the theorem as $\sigma_i = scal_i(x)$. □

Recall that $scal_1 = scal$ is the usual scalar curvature. Our computation of certain cases of the universal polynomials $A_j^{n,k}$ gives the following explicit formulas in low dimensions:

Corollary 5.2 *A 3-manifold has 2-positive Ricci curvature if and only if the following inequalities hold for the higher scalar curvatures:*

$$\begin{aligned} 0 &< scal \\ 0 &< scal_2 + scal^2 \\ 0 &< -scal_3 + scal_2 scal \end{aligned}$$

Corollary 5.3 *A 4-manifold has 2-positive Ricci curvature if and only if the following inequalities hold for the higher scalar curvatures:*

$$\begin{aligned} 0 &< scal \\ 0 &< 2scal_2 + 3scal^2 \\ 0 &< 4scal_2 scal + scal^3 \\ 0 &< -4scal_4 + scal_3 scal + scal_2^2 + 2scal_2 scal^2 \\ 0 &< -3scal_4 scal + scal_3 scal^2 + scal_2^2 scal \\ 0 &< -scal_4 scal^2 - scal_3^2 + scal_3 scal_2 scal \end{aligned}$$

Corollary 5.4 *A 4-manifold has 3-positive Ricci curvature if and only if the following inequalities hold for the higher scalar curvatures:*

$$\begin{aligned} 0 &< scal \\ 0 &< scal_2 + 3scal^2 \\ 0 &< -scal_3 + 2scal_2 scal + scal^3 \\ 0 &< scal_4 - scal_3 scal + scal_2 scal^2 \end{aligned}$$

6 Application to Shen's *k*-th Ricci curvature

The same method also applies to the positivity of Shen's k-th Ricci curvature. The proof works again as this condition can also be formulated using a sum of eigenvalues.

Theorem 6.1 *In dimension n, Shen's k-th Ricci curvature is positive if and only if the nonlinear system of inequalities given by the $A_j^{n-1,k}$ applied to the higher Ricci curvatures holds:*

$$A_j^{n-1,k}(Ric_1(v), .., Ric_{k-1}(v)) > 0$$

for all $j = 1, 2, \ldots \binom{n-1}{k}$ and for all unit vectors $v \in S(TM)$.

Proof. Because of $Ric^{(k)}(v) = tr^{(k)}(Q^v|_{v^\perp})$, the condition $Ric^{(k)}(v) > 0$ for all unit vectors v is equivalent to

$$tr^{(k)}(Q^v|_{v^\perp}) = \mu_1^v + \mu_2^v + \ldots + \mu_k^v > 0$$

where we have ordered the eigenvalues of Q^v such that $\mu_1^v \leq \mu_2^v \leq \ldots \leq \mu_{n-1}^v$ and $\mu_n^v = 0$ corresponds to the degenerate direction v of Q^v. As $\sigma_j(Q^v|_{v^\perp}) = \sigma_j(Q^v) = Ric_j(v)$ for $j \leq n-1$, the statement follows. □

Recall that $Ric_1 = Ric$ is the usual Ricci curvature. Again, this gives the following low-dimensional cases:

Corollary 6.2 *A 4-manifold has positive Shen's 2-nd Ricci curvature if and only if the following inequalities hold for the higher Ricci curvatures for all $v \in S(TM)$:*

$$\begin{aligned} 0 &< Ric(v) \\ 0 &< Ric_2(v) + Ric(v)^2 \\ 0 &< -Ric_3(v) + Ric_2(v)Ric(v) \end{aligned}$$

Corollary 6.3 *A 5-manifold has positive Shen's 2-nd Ricci curvature if and only if the following inequalities hold for the higher Ricci curvatures for all $v \in S(TM)$:*

$$\begin{aligned} 0 &< Ric(v) \\ 0 &< 2Ric_2(v) + 3Ric(v)^2 \\ 0 &< 4Ric_2(v)Ric(v) + Ric(v)^3 \\ 0 &< -4Ric_4(v) + Ric_3(v)Ric(v) + Ric_2(v)^2 + 2Ric_2(v)Ric(v)^2 \\ 0 &< -3Ric_4(v)Ric(v) + Ric_3(v)Ric(v)^2 + Ric_2(v)^2 Ric(v) \\ 0 &< -Ric_4(v)Ric(v)^2 - Ric_3(v)^2 + Ric_3(v)Ric_2(v)Ric(v) \end{aligned}$$

Corollary 6.4 *A 5-manifold has positive Shen's 3-rd Ricci curvature if and only if the following inequalities hold for the higher Ricci curvatures for all $v \in S(TM)$:*

$$\begin{aligned} 0 &< Ric(v) \\ 0 &< Ric_2(v) + 3Ric(v)^2 \\ 0 &< -Ric_3(v) + 2Ric_2(v)Ric(v) + Ric(v)^3 \\ 0 &< Ric_4(v) - Ric_3(v)Ric(v) + Ric_2(v)Ric(v)^2 \end{aligned}$$

Acknowledgements The author would like to cordially thank Prof. Anand Dessai and Prof. Wilderich Tuschmann for many helpful discussions on this subject during a joint research stay at CIRM (Centre International de Rencontres Mathématiques in Luminy, France) in March 2015, which enabled this work. Moreover, the author would like to thank CIRM for excellent working conditions.

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Received: 30.V.2017 / Accepted: 12.IX.2017

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