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On a generalization of reversible rings

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Abstract We call a ring R *generalized reversible* if for any $a, b \in R - \{0\}$, $ab = 0$ implies there exists a positive integer n such that $b^n \neq 0$, $b^n a = 0$. It is proved that generalized reversible rings are NCI and abelian. It is also proved that the class of generalized reversible rings is closed under finite subdirect product. Some extensions of generalized reversible rings are looked into. Some conditions are provided for a generalized reversible ring to be reduced. Finally, a generalized reversible left (right) SF-ring is proved to be strongly regular.

Keywords reversible ring · abelian ring · generalized reversible ring

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1 Introduction

R denotes a ring and all rings considered in this paper are associative rings with identity. The symbols $E(R)$, $N(R)$, $Z(R)$ respectively stand for the set of all idempotent elements, the set of all nilpotent elements and the set of all central elements of R. We denote the left (right) annihilator of any element $a \in R$ by $l(a)$ $(r(a))$. R is *abelian* if all its idempotents are central. R is *reduced* if $N(R) = 0$. R is *reversible* ([1], [7]) if for any $a, b \in R$, $ab = 0$ implies $ba = 0$.

Various generalizations of reversible rings have been explored by many authors over the last several years. This paper presents a generalization of reversible rings which we call *generalized reversible rings*. We establish some properties of generalized reversible rings. Some results on reversible rings are extended to this generalized version.

2 Generalized reversible rings

Definition 2.1 *We call a ring R generalized reversible if for any $a, b \in R - \{0\}$, $ab = 0$ implies there exists a positive integer n such that $b^n \neq 0$, $b^n a = 0$.*

Every reversible ring is generalized reversible. A generalized reversible ring need not be a reversible ring as shown by the following example.

Example 2.1 Let $R = \mathbb{Z}_2[x, y]/I$ where $xy \neq yx$ and I is the ideal $< xy, y^2x, yx^2 >$ of $\mathbb{Z}_2[x, y]$. Then R is not reversible as $(x + I)(y + I) = 0$ but $(y + I)(x + I) \neq 0$. We claim that R is generalized reversible. If $0 \neq \overline{f(x, y)} \in R$, then without loss of generality we can assume that $f(x, y)$ is of the form x^m, y^n, yx, $x^m + y^n$, $x^m + y^n + yx$, $y^n + yx$, $x^m + yx$, $x^m + y^n + 1$, $x^m + y^n + yx + 1$, $x^m + yx + 1$, $y^n + yx + 1$, $x^m + 1$, $y^n + 1$, $yx + 1$, 1. For any $0 \neq \overline{f(x, y)} \in R$, if $0 \neq \overline{g(x, y)} \in R$ such that $f(x, y)g(x, y) \in I$, we have to prove that there exists a positive integer k such that $g(x, y)^k \notin I$ and $g(x, y)^k f(x, y) \in I$. We observe that if the constant term of $f(x, y) \notin I$ is 1, then there does not exist any $g(x, y) \notin I$ such that $f(x, y)g(x, y) \in I$. Hence, to establish the claim, it remains to check for each $f(x, y)$ of the above mentioned form, whose constant term is zero. In this example, all polynomials considered are from $\mathbb{Z}_2[x, y]$ and unless otherwise mentioned, m, n are arbitrary positive integers.
Case 1: $f(x, y) = x^m$ or $x^m + yx$. The corresponding polynomials $g(x, y)$ are y^n, yx,
$y^n + yx$. If $g(x, y) = y^n$ or $y^n + yx$, then $g(x, y)^2 \notin I$ and $g(x, y)^2 f(x, y) \in I$. If $g(x, y) = yx$, then $g(x, y)f(x, y) \in I$.
Case 2: $f(x, y) = y^n$ or $y^n + yx$. If $n = 1$, then $g(x, y) = yx$, $x^m\,(m > 1)$, $x^m + yx\,(m > 1)$ and in each case, $g(x, y)f(x, y) \in I$. If $n \geq 2$, then $g(x, y) = x^m$, yx, $x^m + yx$ and in each case, $g(x, y)f(x, y) \in I$.
Case 3: $f(x, y) = yx$. The corresponding polynomials $g(x, y)$ are x^m, y^n, yx, $x^m + y^n$, $x^m + y^n + yx$, $x^m + yx$, $y^n + yx$ and in each case, $g(x, y)f(x, y) \in I$.
Case 4: $f(x, y) = x^m + y^n$ or $x^m + y^n + yx$. Then $g(x, y) = yx$ and in this case, $g(x, y)f(x, y) \in I$.

A ring R is *NCI* ([2]) if $N(R) = 0$ or there exists a non-zero ideal of R contained in $N(R)$. R is *NI* ([2]) if $N(R)$ is an ideal of R. It is evident that NI rings are NCI but the converse need not be true ([2]).

Proposition 2.2 *Let R be a generalized reversible ring. Then R is NCI and abelian ring.*

Proof. Let $N(R) \neq 0$ and $0 \neq a \in N(R)$. Let n be the least positive integer such that $a^n = 0$. Then $a^{n-1} \neq 0$. Now for any $r \in R$, $ra(a^{n-1}) = 0$. As R is generalized reversible, there exists a positive integer m such that $(a^{n-1})^m \neq 0$ and $(a^{n-1})^m ra = 0$. It follows that $m = 1$ and hence $a^{n-1}Ra^{n-1} = 0$. Therefore $Ra^{n-1}R$ is a non-zero ideal of R contained in $N(R)$. Hence R is NCI.

Let $0, 1 \neq e \in E(R)$. For any $r \in R$, $re(1-e) = 0$. Since R is generalized reversible, there exists a positive integer n such that $(1-e)^n re = (1-e)re = 0$, so $re = ere$. Similarly $r(1-e)e = 0$ leads to $er = ere$. This proves that R is abelian.

□

The following example shows that an abelian ring need not be generalized reversible.

Example 2.2 Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ a - d \equiv b \equiv c \equiv 0 (mod\, 2) \right\}$.

Then $E(R) = \{0, I\}$, where I is the 2×2 identity matrix over $\mathbb{Z}$. It follows that R is abelian. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in R$. Then $AB = 0$, $B^2 = 0$ and $BA \neq 0$. Therefore R is not generalized reversible.

An NCI ring need not be generalized reversible which is shown by the following example.

Example 2.3 Let $\mathbb{F}$ be a field and $R = \begin{bmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{bmatrix}$.

Then $N(R) = \begin{bmatrix} 0 & \mathbb{F} \\ 0 & 0 \end{bmatrix}$ which is an ideal of R which implies that R is an NI ring and hence R is an NCI ring.
Now $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in E(R)$ and A is not central. Therefore R is not abelian, hence R is not generalized reversible.

R is *weakly reversible* ([6]) if for any $a, b, r \in R, ab = 0$ implies $Rbra$ is a nil left ideal.

Proposition 2.3 *A generalized reversible ring is weakly reversible.*

Proof. Let $a, b, r \in R$ with $ab = 0$. For any $s \in R$, $(bra)(bras) = 0$. Since R is a generalized reversible ring, there exists a positive integer n such that $(bras)^n bra = 0$. This implies $(bras)^{n+1} = 0$, so $(sbra)^{n+2} = 0$. Hence $Rbra$ is a nil left ideal.

□

R is *weakly semicommutative* ([5]) if for any $a, b \in R, ab = 0$ implies $aRb \subseteq N(R)$.

Theorem 2.4 *A weakly reversible ring is weakly semicommutative.*

Proof. Let R be a weakly reversible ring and $a, b \in R$ with $ab = 0$. Then for any $r \in R$, $rba \in N(R)$ which implies $arb \in N(R)$.

□

Corollary 2.5 *A generalized reversible ring is weakly semicommutative.*

Recall that a ring R is *central reversible* ([3]) if for any $a, b \in R$, $ab = 0$ implies $ba \in Z(R)$. The following example shows that a central reversible ring need not be generalized reversible.

Example 2.4 Let x, y, z be indeterminates and consider the set

$$R = \{a_0 + a_1x + a_2y + a_3z : a_0, a_1, a_2, a_3 \in \mathbb{Z}\}$$

with the usual componentwise addition and the following multiplication:
$(a_0 + a_1x + a_2y + a_3z)(b_0 + b_1x + b_2y + b_3z) =$
$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_2b_0)y + (a_0b_3 + a_3b_0 + a_1b_2)z$.
R is a ring with 1 as the identity ([10], Example 5.1). From this multiplication it can be seen that $x^2 = 0$, $y^2 = 0$, $z^2 = 0$, $yz = 0$, $zy = 0$, $xz = 0$, $zx = 0$, $yx = 0$ and $xy = z$ and z is central. Therefore R is central reversible. Now $yx = 0$ but $xy = z \neq 0$ and $x^2 = 0$ which proves that R is not generalized reversible.

Theorem 2.6 *Finite subdirect product of generalized reversible rings is generalized reversible.*

Proof. Let R be a subdirect product of two generalized reversible rings A and B. It suffices to prove that R is generalized reversible. It is clear that there exists ideals I, J of R such that $A \simeq R/I$, $B \simeq R/J$ and $I \cap J = 0$. Let $0 \neq a$, $0 \neq b \in R$ such that $ab = 0$. Let $a \notin I \cup J$ and $b \notin I \cup J$. Then $a, b \notin I$, $a, b \notin J$. Now $\overline{a}\overline{b} = 0$ in R/I. As R/I is generalized reversible, there exists a positive integer m such that $\overline{b}^m \neq 0$, $\overline{b}^m\overline{a} = 0$ which implies that $b^m a \in I$. Similarly there exists a positive integer k such that $\overline{b}^k \neq 0$, $b^k a \in J$. Let n= max (m, k). Then $b^n \neq 0$, $b^n a \in I \cap J = 0$. Let $a \in I \cup J$ or $b \in I \cup J$. Then $ba \in I \cup J$. If $ba \in I \cap J$, then $ba = 0$, so assume $ba \in I$ and $ba \notin J$. Now $\overline{a}\overline{b} = 0$ in R/J which implies that there exists a positive integer l such that $b^l \neq 0$, $b^l a \in J$ and so $b^l a \in I \cap J = 0$. This proves that R is generalized reversible.

□

Proposition 2.7 *Let I be an ideal of R containing no non-zero nilpotent elements. If R/I is generalized reversible, then R is generalized reversible.*

Proof. Let $0 \neq a, 0 \neq b \in R$ such that $ab = 0$. Then $\overline{a}\overline{b} = \overline{0}$. If $a, b \notin I$, then $\overline{a} \neq \overline{0}, \overline{b} \neq \overline{0}$. As R/I is generalized reversible, there exists a positive integer n such that $\overline{b}^n \neq \overline{0}$ and $\overline{b}^n\overline{a} = \overline{0}$. So $b^n \neq 0$ and $b^n a \in I$. Now $(b^n a)^2 = 0$. As I does not contain any non-zero nilpotent elements, we have $b^n a = 0$. If $a \in I$ or $b \in I$, then $ba \in I$. Then $(ba)^2 = 0$ leads to $ba = 0$.

□

Example 2.5 A semicommutative ring need not be generalized reversible. Let $R = S_3(\mathbb{R}) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$. Then by ([4], Proposition

1.2), R is semicommutative.
Let $A = \begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&0&0 \end{pmatrix}$, $B = \begin{pmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}$. Then $AB = 0$, $B^2 = 0$ and $BA \neq 0$. Therefore R is not generalized reversible.

Proposition 2.8 *Let R be a generalized reversible ring which is not reversible. Then*

$$V_n(R) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ 0 & a_0 & a_1 & \dots & a_{n-1} \\ 0 & 0 & a_0 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_0 \end{pmatrix} : a_i \in R, i = 0, 1, 2, \cdots, n \right\}$$

is not generalized reversible for any $n \geq 1$.

Proof. As R is not reversible, there exists $a, b \in R$ such that $ab = 0$, $ba \neq 0$.
Then $A = \begin{pmatrix} a&0&0&\dots&0\\ 0&a&0&\dots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&\dots&a \end{pmatrix}$, $B = \begin{pmatrix} 0&0&0&\dots&b\\ 0&0&0&\dots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&0 \end{pmatrix} \in V_n(R)$. But $AB = 0$, $B^2 = 0$ and $BA \neq 0$ as $ba \neq 0$.

□

Theorem 2.9 *Let R be a ring and S be a multiplicatively closed subset of R consisting of central regular elements. Then R is generalized reversible if and only if $S^{-1}R$ is generalized reversible.*

Proof. Let R be generalized reversible and $0 \neq \alpha, 0 \neq \beta \in S^{-1}R$ such that $\alpha\beta = 0$. Let $\alpha = m^{-1}a, \beta = n^{-1}b$ where $m, n \in S, a(\neq 0), b(\neq 0) \in R$. Now $S \subseteq Z(R)$ leads to $0 = \alpha\beta = m^{-1}an^{-1}b = (m^{-1}n^{-1})ab = (mn)^{-1}ab$, so $ab = 0$. By hypothesis, there exists a positive integer k such that $b^k \neq 0$, $b^k a = 0$. Therefore $\beta^k \neq 0$ and $\beta^k\alpha = (n^{-1})^k m^{-1}b^k a = 0$. Hence $S^{-1}R$ is generalized reversible. The converse easily follows.

□

Corollary 2.10 *For any ring R, $R[x]$ is generalized reversible if and only if the ring of Laurent polynomials $R[x; x^{-1}]$ is generalized reversible.*

Proof. Assume $R[x]$ is generalized reversible. Let $S = \{1, x, x^2, \cdots, x^n, \cdots\}$. Then S is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. Therefore by Theorem 2.9, $S^{-1}R[x]$ is generalized reversible. As $R[x; x^{-1}] \simeq S^{-1}R[x]$, the result follows. Converse is trivial.

□

The Dorroh extension, denoted by $D(R,\mathbb{Z})$, of a ring R over $\mathbb{Z}$ is the ring $R\times\mathbb{Z}$ with the usual componentwise addition and the following multiplication:
$(r_1,s_1)(r_2,s_2)=(r_1r_2+s_1r_2+s_2r_1,s_1s_2)$ where $r_1,r_2\in R,\ s_1,s_2\in\mathbb{Z}$.

Lemma 2.11 *For any $(r,s)\in D(R,\mathbb{Z})$,*

$$(r,s)^k=\left(\sum_{i=0}^{i=k-1}{}^kC_is^ir^{k-i},s^k\right)$$

for any positive integer k.

Proof. We will prove the result by induction on k, the power of (r,s).
For $k=2$, $(r,s)^2=(r^2+sr+sr,\ s^2)=(r^2+{}^2C_1sr,\ s^2)$, the result is true.
Let the result be true for $k=n$. Now,

$$(r,s)^{n+1}=(r,s)^n(r,s)=\left(\sum_{i=0}^{i=n-1}{}^nC_is^ir^{n-i},s^n\right)(r,s)$$

$$=\left(\left(\sum_{i=0}^{i=n-1}{}^nC_is^ir^{n-i}\right)(r+s)+s^nr,\ s^{n+1}\right)$$

$$=\left(r^{n+1}+\sum_{i=1}^{i=n}({}^nC_i+{}^nC_{i-1})s^ir^{n+1-i},\ s^{n+1}\right)$$

$$=\left({}^{n+1}C_0r^{n+1}+\sum_{i=1}^{i=n}{}^{n+1}C_is^ir^{n+1-i},\ s^{n+1}\right)=\left(\sum_{i=0}^{i=n}{}^{n+1}C_is^ir^{n+1-i},s^{n+1}\right).$$

So the result is true for $k=n+1$ if it is true for $k=n$.

□

Theorem 2.12 *R is generalized reversible if and only if $D(R,\mathbb{Z})$ is generalized reversible.*

Proof. Let $(0,0)\neq(r_1,s_1),(0,0)\neq(r_2,s_2)\in D(R,\mathbb{Z})$ such that $(r_1,s_1)(r_2,s_2)=0$. Then $r_1r_2+s_1r_2+s_2r_1=0$, $s_1s_2=0$ so that $s_1=0$ or $s_2=0$. If $s_1=0$, then $r_1r_2+s_2r_1=r_1(r_2+s_2)=0$.
If $(r_2+s_2)\neq0$, by hypothesis there exists a positive integer n such that $(r_2+s_2)^n\neq0$, $(r_2+s_2)^nr_1=0$. By Lemma 2.11, we see that the sum of the components of $(r_2,s_2)^n$ is $(r_2+s_2)^n$. Therefore it follows that $(r_2,s_2)^n\neq(0,0)$.

$$\text{Now, } (r_2,s_2)^n(r_1,s_1)=\left(\sum_{i=0}^{i=n-1}{}^nC_is_2^ir_2^{n-i},s_2^n\right)(r_1,0)$$

$$=\left(\left(\sum_{i=0}^{i=n-1}{}^nC_is_2^ir_2^{n-i}\right)r_1+s_2{}^nr_1,0\right)$$

$= ((r_2 + s_2)^n r_1, 0) = (0, 0)$. If $(r_2 + s_2) = 0$, then $r_2 = -s_2$ so that $(r_2, s_2)(r_1, s_1) = (r_2, s_2)(r_1, 0) = ((r_2 + s_2)r_1, 0) = (0, 0)$.

If $s_2 = 0$, then $r_1 r_2 + s_1 r_2 = 0 = (r_1 + s_1)r_2$. Now $s_2 = 0$ and $(r_2, s_2) \neq (0, 0)$ which implies that $r_2 \neq 0$. By hypothesis, there exists a positive integer n such that ${r_2}^n \neq 0$ and ${r_2}^n(r_1 + s_1) = 0$. Therefore $(r_2, s_2)^n = ({r_2}^n, 0) \neq (0, 0)$ and $(r_2, s_2)^n(r_1, s_1) = ({r_2}^n, 0)(r_1, s_1) = ({r_2}^n r_1 + s_1 {r_2}^n, 0) = ({r_2}^n(r_1 + s_1), 0) = (0, 0)$. Hence $D(R, \mathbb{Z})$ is generalized reversible.
Converse follows trivially.

□

Let A be an arbitrary ring with identity and B be a subring of A containing the identity element of A. $R[A, B]$ denote the set
$\{(a_1, a_2, \cdots, a_n, b, b, \cdots) : a_i \in A,\ b \in B, n \geq 1,\ 1 \leq i \leq n\}$. Then $R[A, B]$ is a ring under the componentwise addition and multiplication.

Theorem 2.13 *Let A be an arbitrary ring and B be a subring of A containing the identity element of A. Then the following conditions are equivalent:*

1. *A is generalized reversible.*
2. *$R[A, B]$ is generalized reversible.*

Proof. (1)⇒(2). Let $f=(a_1, a_2, \cdots, a_{n_1}, a, a, \cdots)$, $g=(b_1, b_2, \cdots, b_{n_2}, b, b, \cdots) \in R[A, B]$ be non-zero elements with $fg = 0$ where $a_i, b_j \in A$, $a, b \in B$, $n_1, n_2 \geq 1$, $1 \leq i \leq n_1$, $1 \leq j \leq n_2$. Take $n = max\{n_1, n_2\}$. If n_1 is maximum, let $b_i = b$ for any i, $n_2 + 1 \leq i \leq n_1$ and if n_2 is maximum, let $a_i = a$ for any i, $n_1 + 1 \leq i \leq n_2$. Then $ab = 0$ and for any i, $1 \leq i \leq n$, $a_i b_i = 0$. By hypothesis, there exists positive integers k and $k_i\ (1 \leq i \leq n)$ such that $b^k a = 0 = b_i^{k_i} a_i$ where $b^k \neq 0$ if $b \neq 0$ and $b_i^{k_i} \neq 0$ if $b_i \neq 0$. Let $m = max\{k_1, k_2, \cdots, k_n, k\}$. Then $g^m \neq 0$, $g^m f = 0$. This implies that $R[A, B]$ is generalized reversible.
(2) ⇒ (1) is trivial.

□

3 Consequences of generalized reversibility on some classes of rings

A left R-module M is *Wnil-injective* ([11]) if for any $0 \neq a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and every left R-homomorphism from Ra^n to M extends to one from R to M.

Theorem 3.1 *Let R be a generalized reversible ring. Then R is reduced if R satisfies any of the following conditions:*

1. *R is semiprime.*
2. *Every simple singular left R-module is Wnil-injective.*
3. *Every simple singular right R-module is Wnil-injective.*
4. *Every maximal essential left ideal of R is Wnil-injective.*

5. Every maximal essential right ideal of R is Wnil-injective.

Proof. Let R be a generalized reversible ring which satisfies condition (1) and $0 \neq a \in R$ with $a^2 = 0$. Then for any $r \in R$, $(ra)a = 0$. Since R is generalized reversible, we obtain $ara = 0$ and hence $aRa = 0$. As R is semiprime, this leads to $a = 0$.

Let R be a generalized reversible ring satisfying condition (2). If $0 \neq a \in R$ with $a^2 = 0$, then there exists a maximal left ideal $L \subseteq R$ containing $l(a)$. Since R is abelian, it is easy to see that L is essential so that R/L is a simple singular left R-module and so it is Wnil-injective by hypothesis. Hence every left R-homomorphism from Ra to R/L extends to a left R-homomorphism from R to R/L. Consider the left R-homomorphism $f : Ra \to R/L$ defined by $f(ra) = r + L$ and let $g : R \to R/L$ be the left R-homomorphism which extends f. Then $1 + L = f(a) = g(a) = ag(1) = ab + L$ for some $b \in R$ so that $1 - ab \in L$. As R is generalized reversible and $baa = 0$, we obtain $aba = 0$ so that $ab \in l(a) \subseteq L$ which leads to $1 \in L$, a contradiction. This proves that R is reduced.

Let R be a generalized reversible ring satisfying condition (3). If $0 \neq a \in R$ with $a^2 = 0$, then there exists a maximal essential right ideal K of R containing $r(a)$. Arguing as in last paragraph, we see that there exists some $b \in R$ such that $1 - ba \in K$. As R is generalized reversible and $baa = 0$, we have $aba = 0$ and so $(ba)^2 = 0$ so that $1 - ba$ is a unit. This leads to a contradiction as $K \neq R$.

Let R be a generalized reversible ring which satisfies condition (4). Let $0 \neq a \in R$ such that $a^2 = 0$. There exists a maximal left ideal M of R such that $l(a) \subseteq M$. We claim that $r(a) \subseteq M$. If this is not true, then there exists $x \in r(a)$ such that $x \notin M$. This implies that $M + Rx = R$ which yields $u + vx = 1$ for some $u \in M$, $v \in R$. Now $xaxv = 0$. As R is generalized reversible, there exists some positive integer n such that $(xv)^n xa = 0$ so that $v(xv)^n xa = 0$. This implies that $(vx)^{n+1}a = 0$ and hence $(1 - u)^{n+1} = (vx)^{n+1} \in l(a) \subseteq M$. As $u \in M$, this leads to $1 \in M$, a contradiction. Therefore $r(a) \subseteq M$. It is easy to see that M is essential. By hypothesis, every left R- homomorphism from Ra to M extends to a left R-homomorphism from R to M. It follows that there exists some $b \in M$ such that $a = ab$, whence $1 - b \in r(a) \subseteq M$ which leads to $1 \in M$, a contradiction. This proves that R is reduced.

Let R be a generalized reversible ring which satisfies condition (5). Let $0 \neq a \in R$ with $a^2 = 0$. There exists a maximal essential right ideal $K \subseteq R$ containing $r(a)$. By hypothesis, every right R-homomorphism from aR to K extends to one from R to K. It follows that there exists some $b \in K$ such that $a = ba$ which implies that $(1 - b)a = 0$. As R is generalized reversible, we obtain $a(1 - b) = 0$ and so $1 - b \in r(a) \subseteq K$ which leads to $1 \in K$, a contradiction.

□

Corollary 3.2 *Let R be a generalized reversible ring. Then R is reversible if R satisfies any of the following conditions:*

1. *R is semiprime.*
2. *Every simple singular left R-module is Wnil-injective.*
3. *Every simple singular right R-module is Wnil-injective.*
4. *Every maximal essential left ideal of R is Wnil-injective.*
5. *Every maximal essential right ideal of R is Wnil-injective.*

Theorem 3.3 *A ring R is a domain if and only if R is prime and generalized reversible.*

Proof. If R is a domain, then clearly R is prime and generalized reversible. Conversely, let R be a prime, generalized reversible ring and $a, b \in R$ with $ab = 0$. Then for any $r \in R$, $(bra)^2 = br(ab)ra = 0$. Since R is generalized reversible, $(bra)R(bra) = 0$. As R is prime, this leads to $bRa = 0$ which further leads to $a = 0$ or $b = 0$.

□

R is *(von Neumann) regular* if for any $a \in R$, there exists some $b \in R$ such that $a = aba$. R is *strongly regular* if for any $a \in R$, there exists some $b \in R$ such that $a = a^2b$. It is known that R is strongly regular if and only if R is reduced regular. A ring R is *right SF* ([8]) if its simple right modules are flat. *Left SF-rings* are defined similarly. It is known that regular rings are left (right) SF-rings. However, till date, it is unknown whether left (right) SF-rings are regular. The following theorem proves the strong regularity of a generalized reversible left (right) SF-ring.

Theorem 3.4 *Let R be a generalized reversible ring. The following conditions are equivalent:*

1. *R is a strongly regular ring.*
2. *R is a left SF-ring.*
3. *R is a right SF-ring.*

Proof. It is well known that a regular ring is a left and right SF-ring. So $(1) \Longrightarrow (2), (3)$ holds.

$(2) \Longrightarrow (1)$. As a reduced left (right) SF-ring is strongly regular ([9]), it is enough to prove that R is reduced. Let $0 \neq a \in R$ such that $a^2 = 0$. There exists a maximal left ideal M of R such that $l(a) \subseteq M$. Arguing as in the proof of Theorem 3.1, we have $r(a) \subseteq M$. As R is left SF and $a \in l(a) \subseteq M$, by ([9], Lemma 3.14) there exists some $b \in M$ such that $a = ab$ and hence $1 - b \in r(a) \subseteq M$. Then using $b \in M$ we get $1 \in M$, a contradiction. So R is reduced.

$(3) \Longrightarrow (1)$. It is sufficient to prove that R is reduced. Let $0 \neq a \in R$ with $a^2 = 0$. If $x \in l(a)$, then $xa = 0$ which implies that $ax = 0$ as R is generalized reversible. This implies that $l(a) \subseteq r(a)$. Since $r(a) \neq R$, there exists a maximal right ideal M of R such that $r(a) \subseteq M$. As $a \in r(a) \subseteq M$ and R is right SF, by ([9], Lemma 3.14) there exists $b \in M$ such that $a = ba$. Therefore $(1 - b) \in l(a) \subseteq r(a) \subseteq M$. Then $1 = (1 - b) + b \in M$, a contradiction.

□

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