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On porosity-convergence of real valued sequences

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Abstract In this paper, we define a new type of convergence called $\overline{p}_\mu$-convergence for real valued sequences by using the porosity notion for subsets of natural numbers at infinity. Besides the main properties of $\overline{p}_\mu$-convergence we showed that $\overline{p}_\mu$-convergence is a regular summability method.

Keywords Local upper porosity · Porosity of subsets of natural numbers · Porosity convergence

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1 Introduction

Porosity notion was defined in the papers of Denjoy [3], [4], Khintchine [10] and Dolzenko [5]. After these papers, this notion has many applications in the theory of free boundaries [9], generalized subharmonic functions [6], complex dynamics [11], quasisymmetric maps [14], infinitesimal geometry [2] and other fields of mathematics.

Let $A \subset \mathbb{R}^+$, the right upper porosity of A at 0 is defined as

$$p^+(A) = \limsup_{h \to 0^+} \frac{\lambda(A,h)}{h}$$

where $\lambda(A,h)$ is the length of the largest open subinterval of $(0,h)$ that contains no point of A (for more information [13]). The notion of right lower porosity of A at 0 is defined similarly.

In [1] the definition of porosity given for subsets of real numbers has been redefined for subsets of natural numbers by using a special function .

Let $\mu : \mathbb{N} \to \mathbb{R}^+$ be a strictly decreasing sequence such that $\lim_{n\to\infty} \mu(n) = 0$ and let E be a subset of $\mathbb{N}$. Let us define respectively the porosity numbers $\overline{p}_\mu(E)$ and $\underline{p}_\mu(E)$ as follows

$$\overline{p}_{\mu}(E) := \limsup_{n\to\infty} \frac{\lambda_{\mu}(E,n)}{\mu(n)} \tag{1.1}$$

and

$$\underline{p}_{\mu}(E) := \liminf_{n\to\infty} \frac{\lambda_{\mu}(E,n)}{\mu(n)} \tag{1.2}$$

where

$$\lambda_{\mu}(E,n) := \sup\{|\mu(n^{(1)}) - \mu(n^{(2)})| : n \leq n^{(1)} < n^{(2)},\ (n^{(1)}, n^{(2)}) \cap E = \emptyset\}.$$

The function μ is called a scaling function and $\overline{p}_{\mu}(E)$ and $\underline{p}_{\mu}(E)$ are called upper and lower porosity of the set E at infinity, respectively [1].

From the definition of upper and lower porosity we have the following trivial result.

Remark 1.1 ([1]) If E is a finite subset of $\mathbb{N}$, $|E| < \aleph_0$, then, for every $n \in \mathbb{N}$, $\lambda_{\mu}(E,n)$ is the length of the largest open subinterval of $(0, \mu(n))$ that contains no point of $\mu(E)$ and has a form $(\mu(n^{(2)}), \mu(n^{(1)}))$ with $n^{(1)} < n^{(2)}$. For the case of finite E we evidently have $\lambda_{\mu}(E,n) = \mu(n)$ for all sufficiently large n. Consequently the equality

$$\overline{p}_{\mu}(E) = \underline{p}_{\mu}(E) = 1$$

holds for every scaling function μ for all $E \subseteq \mathbb{N}$ with $|E| < \aleph_0$.

Thoughout this paper, we will use only the right upper porosity and the following terminology. A set $E \subseteq \mathbb{N}$ is called

(i) Porous at infinity if $\overline{p}_{\mu}(E) > 0$;

(ii) Strongly porous at infinity if $\overline{p}_{\mu}(E) = 1$;

(iii) Nonporous at infinity if $\overline{p}_{\mu}(E) = 0$.

Lemma 1.1 *Let A and B be arbitrary subsets of $\mathbb{N}$. If $A \subseteq B$, then*

$$\overline{p}_{\mu}(A) \geq \overline{p}_{\mu}(B)$$

holds for every scaling function μ.

Proof. Let us assume that $A \subseteq B$. Then, the inequality

$$\lambda_{\mu}(B,n) \leq \lambda_{\mu}(A,n)$$

holds for all $n \in \mathbb{N}$. Consequently, we obtain

$$\overline{p}_{\mu}(B) = \limsup_{n\to\infty} \frac{\lambda_{\mu}(B,n)}{\mu(n)} \leq \limsup_{n\to\infty} \frac{\lambda_{\mu}(A,n)}{\mu(n)} = \overline{p}_{\mu}(A).$$

□

Corollary 1.2 *If $A \subseteq B \subseteq \mathbb{N}$ and $\overline{p}_\mu(B) > 0$, then the inequality $\overline{p}_\mu(A) > 0$ holds.*

Remark 1.2 Following statements hold for subsets of $\mathbb{N}$.
(i) A porous set doesn't contain any nonporous set.
(ii) If $\overline{p}_\mu(A) > 0$ and $B \subset \mathbb{N}$ be any set, then $\overline{p}_\mu(A \cap B) > 0$.

Remark 1.3 If $A \triangle B$ is finite, then $\overline{p}_\mu(B) = \overline{p}_\mu(A)$ holds. Where $A \triangle B := (A \backslash B) \cup (B \backslash A)$.

Remark 1.4 The union of any collection of porous sets is not necessarily porous.

Example 1.1 Let $A_i = \{n \ : \ n = i^k, \ k \in \mathbb{N}\}$, for $i \in \mathbb{N}$. It is clear that for every $i \in \mathbb{N}$, $\overline{p}_\mu(A_i) > 0$, but $\cup_{i=1}^{\infty} A_i = \mathbb{N}$ and $\overline{p}_\mu(\mathbb{N}) = 0$.

Remark 1.5 The union of two strongly porous sets does not have to be strongly porous.

Proof. Let $E = \{n_1, n_2, ..., n_k, ...\}$ be a subset of $\mathbb{N}$ for which $1 = n_1 < n_2 < ... < n_k < n_{k+1}, ...$ and

$$\lim_{k \to \infty} \frac{\mu(n_k - 1)}{\mu(n_{k-1})} = 0 \tag{1.3}$$

Define $A, B \subseteq \mathbb{N}$ as

$$A = (\cup_{k=0}[n_{2k}, n_{2k+1} - 1]) \cap \mathbb{N}$$

and

$$B = (\cup_{k=1}[n_{2k+1}, n_{2k+2} - 1]) \cap \mathbb{N}.$$

For every infinite set $E = \{m_1, ..., m_k, ...\}$ with $m_i < m_{i+1}$, we have the equality

$$\overline{p}_\mu(E) = 1 - \liminf \frac{\mu(m_i)}{\mu(m_{i-1})}$$

(see Lemma 3.7 in [1]).

This equality and (1.3) imply that

$$\overline{p}_\mu(A) = \overline{p}_\mu(B) = 1.$$

Hence A and B are strongly porous at infinity. Since $A \cup B = \mathbb{N}$, we have that $A \cup B$ is nonporous at infinity.

□

Let $A = \{n_1, n_2, ..., n_k, ...\} \subset \mathbb{N}$ such that $n_k < n_{k+1}$ and $n_0 \cdot A = \{n_0 n_1, n_0 n_2, ..., n_0 n_k, ...\}$ for $n_0 \in \mathbb{N}$.

Lemma 1.3 *Let $A \subset \mathbb{N}$ be a porous set. Then, $n_0 \cdot A \subset \mathbb{N}$, $n_0 \in \mathbb{N}$ is a porous set.*

Proof. Let A be a porous set. From Lemma 3.7 in [1] we have

$$\overline{p}_\mu(n_0 \cdot A) = 1 - \liminf_{k\to\infty} \frac{\mu(n_0 n_k)}{\mu(n_0 n_{k-1})} = 1 - \liminf_{k\to\infty} \frac{\mu(n_k)}{\mu(n_{k-1})} = \overline{p}_\mu(A).$$

Hence $n_0 \cdot A$ is also a porous set at infinity.

□

2 $\overline{p}_\mu$-Convergence

The concept of limit is one of the main subject of analysis. In this section, the notion of convergence is generalized by using the upper porosity of the subsets of natural numbers at infinity.

Definition 2.1 *Let $x = (x_n)_{n\in\mathbb{N}}$ be a real valued sequence. We say that, $x = (x_n)_{n\in\mathbb{N}}$ is $\overline{p}_\mu$ convergent to $l \in \mathbb{R}$ if for each $\varepsilon > 0$,*

$$\overline{p}_\mu(A_\varepsilon) > 0 \text{ and } \overline{p}_\mu(A_\varepsilon^c) = 0,$$

holds, where $A_\varepsilon := \{n : |x_n - l| \geq \varepsilon\}$ and A_ε^c is complement of A_ε. It is denoted by $x \to l(\overline{p}_\mu)$ (or $\overline{p}_\mu - \lim_{n\to\infty} x_n = l$).

If $\overline{p}_\mu(A_\varepsilon) = 1$, we call $x = (x_n)$ is strongly $\overline{p}_\mu$ convergent to l and denoted by $x_n \to l(s - \overline{p}_\mu)$ or $s - \overline{p}_\mu - \lim_{n\to\infty} x_n = l$.

Theorem 2.2 *The $\overline{p}_\mu$-limit of the sequence $x = (x_n)$ is unique.*

Proof. Suppose that $x = (x_n)$ is $\overline{p}_\mu$-convergent to both l and L, i. e., $x_n \to l(\overline{p}_\mu)$, $x_n \to L(\overline{p}_\mu)$ and $l \neq L$.

If we consider $\varepsilon < |l - L|$, then from the definition of $\overline{p}_\mu$-convergence $\overline{p}_\mu(\{n : |x_n - l| \geq \varepsilon\}) > 0$ and $\overline{p}_\mu(\{n : |x_n - L| \geq \varepsilon\}) > 0$ hold. The inclusion

$$\{n : |x_n - l| < \varepsilon\} \subset \{n : |x_n - L| \geq \varepsilon\}$$

holds. From Corollary 1.2 we have $\overline{p}(\{n : |x_n - l| < \varepsilon\}) > 0$. But it is a contradiction to $x_n \to l(\overline{p}_\mu)$. Therefore $l = L$.

□

Theorem 2.3 *Let $x = (x_n)$ be a real valued sequence. If $x_n \to l$ $(n \to \infty)$, then $x_n \to l(\overline{p}_\mu)$.*

Proof. Since $x_n \to l$ $(n \to \infty)$, then for every $\varepsilon > 0$, there exists an $n_0 \equiv n_0(\varepsilon) \in \mathbb{N}$ such that $|x_n - l| < \varepsilon$ holds for all $n \geq n_0(\varepsilon)$.

It is clear that $\{n : |x_n - l| \geq \varepsilon\} \subseteq \{1, 2, ..., n_0\}$. Therefore, the desired result is obtained from Remark 1.1 and Lemma 1.1.

□

Theorem 2.3 shows that $\overline{p}_\mu$-convergence is a regular summability method.

Remark 2.1 The converse of Theorem 2.3 is not true, in general.

Example 2.1 Let $\mu : \mathbb{N} \to \mathbb{R}^+$ be a scaling function, and $x = (x_n)$ be a sequence as follows:

$$x_n := \begin{cases} n, n = 2^m, \quad m \in \mathbb{N}, \\ 0, \text{ otherwise} \end{cases}$$

and

$$\mu(n) = \frac{1}{n}, n \in \mathbb{N},$$

respectively.

This sequence is $\overline{p}_\mu$-convergent to 0, $(x_n \to 0(\overline{p}_\mu))$. Really for any $\varepsilon > 0$ the set

$$A_\varepsilon = \{n : |x_n - 0| \geq \varepsilon\} = \{n : |x_n| \geq \varepsilon\} = \{n : n = 2^m, m \in \mathbb{N}\}$$

is porous because of Lemma 3.7 in [1]. But it is not convergent to zero.

Theorem 2.4 *Let $x = (x_n)$ be a real valued sequence and (n_k) be a strictly increasing sequence of natural numbers such that the set $\{n_k : k \in \mathbb{N}\}$ is nonporous. If (x_n) is $\overline{p}_\mu$-convergent to $l \in \mathbb{R}$, then (x_{n_k}) is also $\overline{p}_\mu$-convergent to l.*

Proof. Let $x_n \to l(\overline{p}_\mu)$. Then, for each $\varepsilon > 0$, we have

$$\overline{p}_\mu(A_\varepsilon) > 0$$

where $A_\varepsilon := \{n \ : \ |x_n - l| \geq \varepsilon\}$.

Consider the set $A'_\varepsilon := \{n_k \ : \ |x_{n_k} - l| \geq \varepsilon\}$ for any $\varepsilon > 0$. It is clear that the inclusion

$$A'_\varepsilon \subset A_\varepsilon$$

holds. This inclusion and Lemma 1.1, give that

$$\overline{p}_\mu(A'_\varepsilon) > 0.$$

Hence, the subsequence (x_{n_k}) is also $\overline{p}_\mu$-convergent to l.

□

Remark 2.2 The converse of Theorem 2.4 is not true.

Example 2.2 Let a sequence $x = (x_n)$ and its subsequence (x_{2m}) as follows:

$$x_n = \begin{cases} 2^k, n = 2^k, \\ 3^k, n = 2k - 1, \\ 0, \quad otherwise, \end{cases}$$

and

$$x_{2m} = \begin{cases} 2^k, m = 2^{k-1}, \\ 0, \quad otherwise. \end{cases}$$

It is clear from Lemma 3.7, [1] that the set $\{2m : m \in \mathbb{N}\}$ is nonporous. So, the condition of Theorem 2.4 holds. Also, $x_{2m} \to 0(\overline{p}_\mu)$ because the set $\{2^k : k \in \mathbb{N}\}$ is porous. But the sequence (x_n) is not $\overline{p}_\mu$ convergent to 0.

Remark 2.3 Nonporosity of the set $\{n_k : k \in \mathbb{N}\}$ can not be omitted in Theorem 2.4.

Let us consider the sequence in Example 2.1 and the set $\{2^m : m \in \mathbb{N}\}$. It is clear that $\overline{p}_\mu(\{2^m : m \in \mathbb{N}\}) > 0$. If we take the subsequence (x_{2^m}) of (x_n), then $x_n \to l(\overline{p}_\mu)$, but $x_{2^m} \nrightarrow 0(\overline{p}_\mu)$.

Theorem 2.5 *Let $x = (x_n)$ be a real valued sequence. If $x_n \to l(\overline{p}_\mu)$ then there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) converges to l.*

Proof. Let $x_n \to l(\overline{p}_\mu)$. Then $A_\varepsilon = \{n : |x_n - l| \geq \varepsilon\}$ is a porous set. This set can be written as follows:

$$A_\varepsilon = \{n_1, n_2, ..., n_k, n_{k+1}, ...\}$$

such that $n_k < n_{k+1}$ and $n_k \to \infty$ for $k \to \infty$.

The interval (n_k, n_{k+1}) may include a finite number of natural numbers or infinite number of natural number. Let us assume that n_k^1 is the smallest natural number in the first interval which contains a finite number of natural numbers, n_k^2 is the smallest natural number in the second interval which contains a finite number of natural numbers.

If so continue, we construct a sequence $(n_k^i)_{i\in\mathbb{N}}$ of natural numbers such that for each $i \in \mathbb{N}$ $n_k^i \notin A_\varepsilon$ and $n_k^i < n_k^{i+1}$. Using the sequence $(n_k^i)_{i\in\mathbb{N}}$ we obtain a subsequence $(x_{n_k^i})$ of $x = (x_n)$ such that $(x_{n_k^i})$ converges to l.

□

Theorem 2.6 *Let $x = (x_n)$ and $y = (y_n)$ be real valued sequences. If $x_n \to l(\overline{p}_\mu)$ and $y_n \to t(\overline{p}_\mu)$, then*

$$x_n + y_n \to l + t(\overline{p}_\mu).$$

Proof. Since $x_n \to l(\overline{p}_\mu)$ and $y_n \to t(\overline{p}_\mu)$, then for any $\varepsilon > 0$ the sets $\{n : |x_n - l| \geq \varepsilon\}$ and $\{n : |y_n - t| \geq \varepsilon\}$ are porous. The following inclusion

$$\{n : |x_n + y_n - (l + t)| \geq \varepsilon\} \subseteq \{n : |x_n - l| \geq \frac{\varepsilon}{2} \wedge |y_n - t| \geq \frac{\varepsilon}{2}\} \tag{2.1}$$

holds. It is clear that the inclusion

$$\left\{n : |x_n - l| \geq \frac{\varepsilon}{2} \wedge |y_n - t| \geq \frac{\varepsilon}{2}\right\} \subseteq \{n : |x_n - l| \geq \frac{\varepsilon}{2}\}$$

hold and from Lemma 1.1 the inequality

$$\overline{p}_\mu(\{n : |x_n - l| \geq \frac{\varepsilon}{2}\}) \leq \overline{p}_\mu(\{n : |x_n - l| \geq \frac{\varepsilon}{2} \wedge |y_n - t| \geq \frac{\varepsilon}{2}\})$$

satisfies. This gives that

$$\overline{p}_\mu(\{n : |x_n - l| \geq \frac{\varepsilon}{2} \wedge |y_n - t| \geq \frac{\varepsilon}{2}\}) > 0.$$

From (2.1) and Lemma 1.1, we have

$$\overline{p}_\mu(\{n : |x_n + y_n - (l+t)| \geq \varepsilon\}) > 0.$$

Consequently, $x_n + y_n \to l + t(\overline{p}_\mu)$.

□

Theorem 2.7 *Let $x = (x_n)$ be real valued sequence. If $x_n \to l(\overline{p}_\mu)$, then $\alpha x_n \to \alpha l(\overline{p}_\mu)$, such that $\alpha \in \mathbb{R}$.*

Proof. Let us assume that $x_n \to l(\overline{p}_\mu)$. Then $\overline{p}_\mu(A_\varepsilon) > 0$ such that $A_\varepsilon = \{n \ : \ |x_n - l| \geq \varepsilon\}$. We have

$$\{n \ : \ |\alpha x_n - \alpha l| \geq \varepsilon\} = \{n \ : \ |x_n - l| \geq \frac{\varepsilon}{|\alpha|}\} = A_{\frac{\varepsilon}{|\alpha|}}$$

for the sequence $\alpha x = (\alpha x_n)$. From the $\overline{p}_\mu$-convergency of $x = (x_n)$ we obtain

$$\overline{p}_\mu(A_{\frac{\varepsilon}{|\alpha|}}) > 0.$$

Hence $\alpha x_n \to \alpha l(\overline{p}_\mu)$.

□

Following two theorems gives a characterization that a given sequence is $\overline{p}_\mu$-convergent (strongly $\overline{p}_\mu$-convergent) or not $\overline{p}_\mu$-convergent (strongly $\overline{p}_\mu$-convergent).

Theorem 2.8 *Let $x = (x_n)$ be a real valued sequence and $\mu : \mathbb{N} \to \mathbb{R}^+$ be scaling function. $x_n \nrightarrow l(\overline{p}_\mu)$ if and only if*

$$\lim_{k\to\infty} \frac{\mu(n_{k+1})}{\mu(n_k)} = 1 \tag{2.2}$$

where $n_k, n_{k+1} \in A_\varepsilon$ such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$.

Proof. Suppose that (2.2) holds. Then, by Lemma 3.7 in [1], we obtain

$$\overline{p}_\mu(A_\varepsilon) = 1 - \liminf_{k\to\infty} \frac{\mu(n_{k+1})}{\mu(n_k)} = 1 - \lim_{k\to\infty} \frac{\mu(n_{k+1})}{\mu(n_k)} = 0.$$

Thus A_ε is nonporous at infinity w.r.t. μ, that is $x \nrightarrow l(\overline{p}_\mu)$.

Conversely, assume $x \nrightarrow l(\overline{p}_\mu)$. Then using Lemma 3.7 in [1] we have

$$1 = \liminf_{k\to\infty} \frac{\mu(n_{k+1})}{\mu(n_k)}.$$

Since μ is decreasing the inequality

$$\limsup_{k\to\infty} \frac{\mu(n_{k+1})}{\mu(n_k)} \leq 1$$

holds. Thus

$$1 = \liminf_{k\to\infty} \frac{\mu(n_{k+1})}{\mu(n_k)} \leq \limsup_{k\to\infty} \frac{\mu(n_{k+1})}{\mu(n_k)} \leq 1$$

that implies (2.2).

□

Theorem 2.9 *Let $x = (x_n)$ be a real valued sequence and $\mu : \mathbb{N} \to \mathbb{R}^+$ be scaling function. The following statements hold.*

(i) $x_n \to l(\overline{p}_\mu)$ if and only if

$$\limsup_{k\to\infty} \frac{\mu(n_{k+1})}{\mu(n_k)} < 1$$

(ii) $x_n \to l(s - \overline{p}_\mu)$ if and only if

$$\limsup_{k\to\infty} \frac{\mu(n_{k+1})}{\mu(n_k)} = 0.$$

Definition 2.10 *A sequence $x = (x_n)$ is said to be $\overline{p}_\mu$-Cauchy if for every $\varepsilon > 0$ and sufficiently large n, $m \in \mathbb{N}$, $\overline{p}_\mu(A(\varepsilon)) > 0$ and $\overline{p}_\mu(A(\varepsilon)^c) = 0$ where $A(\varepsilon) = \{n : |x_n - x_m| \geq \varepsilon\}$ and $A(\varepsilon)^c$ is complement of $A(\varepsilon)$.*

Theorem 2.11 *Let $x = (x_n)$ be a $\overline{p}_\mu$-convergent sequence. Then, $x = (x_n)$ is a $\overline{p}_\mu$-Cauchy sequence.*

Proof. Let us assume that $x = (x_n) \to l(\overline{p}_\mu)$. It is clear that the inclusion

$$\{n : |x_n - x_m| \geq \varepsilon\} \subseteq \{n : |x_n - l| \geq \frac{\varepsilon}{2} \wedge |x_m - l| \geq \frac{\varepsilon}{2}\}$$

holds. It is also true that the inclusion

$$\{n : |x_n - l| \geq \frac{\varepsilon}{2} \wedge |x_m - l| \geq \frac{\varepsilon}{2}\} \subset \{n : |x_n - l| \geq \frac{\varepsilon}{2}\}$$

holds. From Lemma 1.1 we have

$$\overline{p}_\mu(\{n : |x_n - l| \geq \frac{\varepsilon}{2} \wedge |x_m - l| \geq \frac{\varepsilon}{2}\}) > 0.$$

This gives that $\overline{p}_\mu(\{n : |x_n - x_m| \geq \varepsilon\}) > 0$. Therefore, $x = (x_n)$ is a $\overline{p}_\mu$-Cauchy sequence.

□

Definition 2.12 *Let $x = (x_n)$ be a real valued sequence and $M > 0$ be a real number. Let us consider $S = \{n : |x_n| \geq M\}$. If $\overline{p}_\mu(S) > 0$ and $\overline{p}_\mu(S^c) = 0$, then $x = (x_n)$ is $\overline{p}_\mu$-bounded sequence.*

It is clear that every bounded sequence is also $\overline{p}_\mu$-bounded. But the converse is not true.

Example 2.3 Let us take account of Example 2.1. It is clear that $x = (x_n)$ is $\overline{p}_\mu$-bounded sequence but it is not bounded.

Theorem 2.13 *Every $\overline{p}_\mu$-convergent sequence is $\overline{p}_\mu$-bounded sequence.*

Proof. Let us assume that $x = (x_n)$ is a $\overline{p}_\mu$-convergent sequence. That is, $x_n \to l(\overline{p}_\mu)$ holds. Then, there exists $M > 0$ such that the inclusion

$$\{n : |x_n| > M\} \subset \{n : |x_n - l| \geq \varepsilon\}$$

holds. From Lemma 1.1 the set $\{n : |x_n| > M\}$ is also porous. Hence $x = (x_n)$ is $\overline{p}_\mu$-bounded.

□

Theorem 2.14 *Every $\overline{p}_\mu$-Cauchy sequence is also $\overline{p}_\mu$-bounded sequence.*

Proof. Let $x = (x_n)$ be a $\overline{p}_\mu$-Cauchy sequence. Then, there exists $M > 0$ such that the inclusion

$$\begin{aligned}\{n : |x_n| > M\} &= \{n : |x_n - x_m + x_m| > M\} \\ &\subset \{n : |x_n - x_m| > \frac{M}{2} \wedge |x_m| > \frac{M}{2}\} \\ &\subset \{n : |x_n - x_m| > \frac{M}{2}\} \\ &\subset \{n : |x_n - x_m| > \varepsilon\}\end{aligned}$$

holds. From this inclusion and Lemma 1.1 the set $\{n : |x_n| > M\}$ is porous. Consequently $x = (x_n)$ is a $\overline{p}_\mu$-bounded sequence.

□

The class of all $\overline{p}_\mu$-convergent, strongly $\overline{p}_\mu$-convergent, $\overline{p}_\mu$-bounded and $\overline{p}_\mu$-Cauchy sequences are denoted by $c_\mu(x)$, $c^s_\mu(x)$, $B_\mu(x)$ and $C_\mu(x)$, respectively.

$$\begin{aligned}c_\mu(x) &:= \{x = (x_n) \ : \ \exists l \in \mathbb{R} \text{ such that } x_n \to l(\overline{p}_\mu)\}, \\ c^s_\mu(x) &:= \{x = (x_n) \ : \ \exists l \in \mathbb{R} \text{ such that } x_n \to l(s - \overline{p}_\mu)\}, \\ B_\mu(x) &:= \{x = (x_n) \ : \ (x_n) \text{ is } \overline{p}_\mu - \text{bounded}\}, \\ C_\mu(x) &:= \{x = (x_n) \ : \ (x_n) \text{ is } \overline{p}_\mu - \text{Cauchy sequence}\}.\end{aligned}$$

Remark 2.4 The following inclusion

$$c^s_\mu(x) \subset c_\mu(x) \subset C_\mu(x) \subset B_\mu(x)$$

holds for any scaling function μ.

Let μ_1 and μ_2 be two scaling functions. In the following theorem, we will give a necessary condition for the equality of C_{μ_1} and C_{μ_2}.

Theorem 2.15 *Let μ_1 and μ_2 be two scaling functions. If*

$$\lim_{n,m\to\infty} \frac{\mu_1(n)\mu_2(m)}{\mu_2(n)\mu_1(m)} = 1, \tag{2.3}$$

holds, then we have the equality

$$C_{\mu_1}(x) = C_{\mu_2}(x) \text{ and } C^s_{\mu_1}(x) = C^s_{\mu_2}(x) \tag{2.4}$$

Proof. Let us assume that $x_n \to l(\overline{p}_{\mu_1})$ and the limit relation (2.3) hold. By hypothesis

$$\overline{p}_{\mu_1}(A_\varepsilon) > 0, \tag{2.5}$$

holds. If $|A_\varepsilon| < \aleph_0$, since $\overline{p}_{\mu_1}(A_\varepsilon) = \overline{p}_{\mu_2}(A_\varepsilon) = 1$, then (2.4) hold.

Thus, without losing generality, it can be assumed that $|A_\varepsilon| = \aleph_0$. Then, for every $n \in \mathbb{N}$ there are $n^{(1)},\ n^{(2)} \in \mathbb{N}$ such that $n \le n^{(1)} < n^{(2)}$ and

$$\lambda_{\mu_1}(A_\varepsilon, n) = \mu_1(n^{(1)}) - \mu_1(n^{(2)}). \tag{2.6}$$

Equality (2.3) implies that

$$(1-\varepsilon)\frac{\mu_2(n)}{\mu_2(m)} \le \frac{\mu_1(n)}{\mu_1(m)} \le (1+\varepsilon)\frac{\mu_2(n)}{\mu_2(m)} \tag{2.7}$$

holds for sufficiently large m and n.

Using (2.6) and (2.7) we obtain

$$\begin{aligned}
\frac{\lambda_{\mu_1}(A_\varepsilon, n)}{\mu_1(n)} &= \frac{\mu_1(n^{(1)})}{\mu_1(n)} - \frac{\mu_1(n^{(2)})}{\mu_1(n)} \\
&\le (1+\varepsilon)\frac{\mu_2(n^{(1)})}{\mu_2(n)} - (1-\varepsilon)\frac{\mu_2(n^{(2)})}{\mu_2(n)} \\
&\le \frac{\mu_2(n^{(1)}) - \mu_2(n^{(2)})}{\mu_2(n)} + 2\varepsilon \\
&\le \frac{\lambda_{\mu_2}(A_\varepsilon, n)}{\mu_2(n)} + 2\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, the following inequality

$$\frac{\lambda_{\mu_1}(A_\varepsilon, n)}{\mu_1(n)} \le \frac{\lambda_{\mu_2}(A_\varepsilon, n)}{\mu_2(n)}$$

is obtained. From the last inequality and (2.5), we obtain

$$\overline{p}_{\mu_2}(A_\varepsilon) > 0.$$

So, $x_n \to l(\overline{p}_{\mu_2})$. Consequently $C_{\mu_1} \subseteq C_{\mu_2}$. Similar arguments show that $C_{\mu_2} \subseteq C_{\mu_1}$. Equality (2.4) follows.

□

Corollary 2.16 *Let $c > 0$ and Let $\mu_1 : \mathbb{N} \to \mathbb{R}^+$ and $\mu_2 : \mathbb{N} \to \mathbb{R}^+$ be scaling functions satisfying $\lim_{n\to\infty} \frac{\mu_1(n)}{\mu_2(n)} = c$. Then, the equality*

$$C_{\mu_1} = C_{\mu_2}$$

holds.

Definition 2.17 *Let $x = (x_n)_{n\in\mathbb{N}}$ be a real valued sequence. The sequence $x = (x_n)$ is a $\overline{p}_\mu$-monotone sequence if there exists a subset $H \subset \mathbb{N}$ such that $\overline{p}_\mu(H) = 0$, and $x = (x_n)_{n\in H}$ is monotone.*

Theorem 2.18 *Every monotone sequence is also $\overline{p}_\mu$-monotone.*

Proof. Let us assume that $x = (x_n)$ be a monotone sequence. Then for every $n \in \mathbb{N}$, $x_n \leq x_{n+1}$ (or $x_{n+1} \leq x_n$) holds. Since $x = (x_n)$ is monotone in $\mathbb{N}$ and $\overline{p}_\mu(\mathbb{N}) = 0$, then $x = (x_n)$ is $\overline{p}_\mu$-monotone sequence.

□

Remark 2.5 The converse of Theorem 2.18 is not true, in general.

Example 2.4 Let us define $x = (x_n)$ and the scaling function $\mu : \mathbb{N} \to \mathbb{R}^+$ as follows:

$$x_n = \begin{cases} n, \ n \text{ is even}, \\ \frac{1}{n}, \ n \text{ is odd} \end{cases}$$

and

$$\mu(n) = \frac{1}{n}.$$

This sequence is $\overline{p}_\mu$-monotone, but it is not a monotone sequence.

Definition 2.19 *Let $x = (x_n)$ and $y = (y_n)$ be two sequences. (x_n) is eventually in (y_n) if there exists $N \in \mathbb{N}$ such that x_n is a term of (y_n) for all $n \geq N$.*

Theorem 2.20 *Let $x = (x_n)$ and $y = (y_n)$ be two sequences such that (x_n) is eventually in (y_n). Then, $y_n \to l(\overline{p}_\mu)$ implies that $x_n \to l(\overline{p}_\mu)$.*

Proof. Let (x_n) be an eventually in (y_n) and $y_n \to l(\overline{p}_\mu)$. Then the set $\{n : |y_n - l| \geq \varepsilon\}$ is porous. From the assumption the inclusion $\{n : |x_n - l| \geq \varepsilon\} \subseteq \{n : |y_n - l| \geq \varepsilon\}$ holds. Also, from Lemma 1.1 we have $\{n : |x_n - l| \geq \varepsilon\}$ is a porous set.

□

Example 2.5 Let $x = (x_n)$ and $y = (y_n)$ be sequences as follows:

$$x_n = \begin{cases} 2^k, \ n = 3^k, \\ 0, \ \text{ otherwise} \end{cases}$$

and

$$y_n = \begin{cases} 3^k, \ n = 2^k, \\ 0, \ \text{ otherwise.} \end{cases}$$

$x_n \to 0(\overline{p}_\mu)$ and $y_n \to 0(\overline{p}_\mu)$ but (x_n) does not eventually in (y_n), vice versa.

3 Conclusion

One of the main problem in summability theory is the extension of set of usual convergence of sequences. For this aim, many different summability methods were defined such as Cesàro, Nörlund, Abel, e.t.c. which can be represented as a matrix. Statistical convergence which has no matrix representation is most popular than others [7], [8], [12].

In this paper, porosity convergence is defined by considering upper porosity of the subsets of natural numbers at infinity. It is obtained that this new convergence method is a regular summability method.

Originating the definition of density and porosity there is a great difference between density and porosity of a given subset of $\mathbb{N}$. For example, the density of finite set (or $\mathbb{N}$) is zero (or 1) while the porosity is 1 (or zero). That's why the following question is important to understand the relations between the set of statistical convergence and porosity convergence of real valued sequences. Give a characterization for the subset A of $\mathbb{N}$ such that:

(1) $\overline{p}_\mu(A) > 0$ and $\overline{p}_\mu(A^c) = 0$ when density of A is zero.

(2) $\overline{p}_\mu(A) > 0$ and $\overline{p}_\mu(A^c) = 0$ when density of A is not zero.

(3) Density of A is zero when $\overline{p}_\mu(A) = 0$ or $\overline{p}_\mu(A^c) > 0$.

In connection with the above questions, the following problems are also open:

Under which condition on scaling function,

(1) the converse of Theorem 2.11 is true?

(2) the set of statistical convergent and porosity convergence coincide?

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