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Lightlike hypersurfaces of an indefinite nearly trans-Sasakian manifold

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Abstract In this paper, we study lightlike hypersurfaces M of an indefinite nearly trans-Sasakian manifold $\bar{M}$ or an indefinite nearly generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a tangential trans-Sasakian structure vector field ζ subject such that the induced structure tensor fields F of M is recurrent, nearly recurrent or Lie recurrent, or M itself is totally umbilical, screen totally umbilical or screen conformal.

Keywords recurrent · nearly recurrent · Lie recurrent · lightlike hypersurface · indefinite nearly trans-Sasakian manifold

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1 Introduction

The theory of lightlike submanifolds is an important topic of research in differential geometry and mathematical physics. In 1996, Duggal-Bejancu published a book [3] on the lightlike geometry of submanifolds. Since then the large number of papers published on null curves and lightlike submanifolds. In 2007, Duggal-Jin published a book [4] on the differential geometry of null curves and lightlike hypersurfaces which presented a comprehensive up to date information on their subject by that time. In 2010, Duggal-Sahin published a book [5] as an update on the study of lightlike submanifolds of indefinite Kaehler manifolds or indefinite Sasakian manifolds.

The objective of this paper is to study on the geometry of lightlike hypersurfaces M of an indefinite nearly trans-Sasakian manifold $\bar{M}$ or an indefinite nearly generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ subject such that the structure vector field ζ of $\bar{M}$ is tangent to M. The main results are characterization theorems for such a lightlike hypersurface which has recurrent, nearly recurrent or Lie recurrent structure tensor field F, or which M is totally umbilical, screen totally umbilical or screen conformal.

2 Indefinite almost contact metric manifolds

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be an *indefinite almost contact metric manifold* ([5], [6]) if there exists a structure set $\{J, \zeta, \theta, \bar{g}\}$, where J is a $(1,1)$-type tensor field, ζ is a vector field, θ is a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ such that

$$\begin{aligned} &J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta, \quad J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(\zeta) = 1, \\ &\theta(\bar{X}) = \bar{g}(\zeta, \bar{X}), \quad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \theta(\bar{X})\theta(\bar{Y}). \end{aligned} \tag{2.1}$$

In this paper, we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

A hypersurface M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called *lightlike hypersurface* if its normal bundle $TM^\perp$ is a subbundle of its tangent bundle TM, and coincides with the radical distribution $Rad(TM) = TM \cap TM^\perp$. Then there exists a non-degenerate complementary vector bundle $S(TM)$ of $TM^\perp$ in TM, which is called a *screen distribution* of M, such that

$$TM = TM^\perp \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also, denote by $(2.1)_i$ the i-th equation of (2.1). We use same notations for any others. It is well-known [3] that, for any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique lightlike vector bundle $tr(TM)$ of rank 1 in the orthogonal complement $S(TM)^\perp$ of $S(TM)$ in $\bar{M}$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

In this case, the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follows:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus_{orth} S(TM).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* with respect to the screen distribution $S(TM)$, respectively.

In the following, we denote by X, Y and Z the smooth vector fields on M, unless otherwise specified. Let $\bar{\nabla}$ be a unique Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulae of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.2}$$
$$\bar{\nabla}_X N = -A_{N} X + \tau(X)N; \tag{2.3}$$
$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.4}$$
$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.5}$$

where ∇ and ∇^* are the linear connections on TM and $S(TM)$, respectively, B and C are the local second fundamental forms on TM and $S(TM)$, respectively, A_{N} and A^*_ξ are the shape operators and τ is a 1-form on M.

The induced connection ∇ on M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{2.6}$$

where η is a 1-form such that $\eta(X) = \bar{g}(X,N)$. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. As $B(X,Y) = \bar{g}(\bar{\nabla}_X Y,\,\xi)$, we show that B is independent of the choice of the screen $S(TM)$ and satisfies

$$B(X,\xi) = 0. \tag{2.7}$$

The above two local second fundamental forms B and C of M and $S(TM)$, respectively, are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.8}$$

$$C(X,PY) = g(A_{_N} X, PY), \qquad \bar{g}(A_{_N} X, N) = 0. \tag{2.9}$$

From (2.8), the operator A^*_ξ is $S(TM)$-valued self-adjoint and $A^*_\xi \xi = 0$.

Let $\nabla^{\perp}_X N = \pi(\bar{\nabla}_X N)$, where π is the projection morphism of $T\bar{M}$ on $tr(TM)$, that is, $\nabla^{\perp}_X N = \tau(X)N$. In this case, $\nabla^{\perp}$ is a linear connection on the transversal vector bundle $tr(TM)$. We say that $\nabla^{\perp}$ is the *transversal connection* of M. We define a curvature tensor $R^{\perp}$ of $tr(TM)$ by

$$R^{\perp}(X,Y)N = \nabla^{\perp}_X \nabla^{\perp}_Y N - \nabla^{\perp}_Y \nabla^{\perp}_X N - \nabla^{\perp}_{[X,Y]} N.$$

The transversal connection $\nabla^{\perp}$ is called *flat* if $R^{\perp}$ vanishes identically [6].

Theorem 2.1 **[6]** *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the transversal connection is flat if and only if the 1-form τ is closed, i.e., $d\tau = 0$, on any neighborhood $\mathcal{U} \subset M$.*

Călin [2] proved that *if the structure vector field ζ of $\bar{M}$ is tangent to M, then it belongs to $S(TM)$* which we assume in this paper. It is known [6] that, for any lightlike hypersurface M of an indefinite almost contact metric manifold $\bar{M}$, $J(TM^{\perp})$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1, and $J(TM^{\perp}) \cap J(tr(TM)) = \{0\}$. Thus $J(TM^{\perp}) \oplus J(tr(TM))$ is a subbundle of $S(TM)$ of rank 2. Hence there exist two non-degenerate almost complex distributions D_o and D with respect to the indefinite almost contact structure tensor J, *i.e.*, $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = \{J(TM^{\perp}) \oplus J(tr(TM))\} \oplus_{orth} D_o,$$
$$D = TM^{\perp} \oplus_{orth} J(TM^{\perp}) \oplus_{orth} D_o.$$

In this case, the decomposition form of TM is reformed as follows:

$$TM = D \oplus J(tr(TM)).$$

Consider two lightlike vector fields U and V, and their 1-forms such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X,V), \quad v(X) = g(X,U). \tag{2.10}$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$JX = FX + u(X)N, \tag{2.11}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $FX = JSX$. Applying J to (2.11) and using $(2.1)_1$ and $(2.10)_1$, we have

$$F^2X = -X + u(X)U + \theta(X)\zeta. \tag{2.12}$$

Applying ∇_X to $F\xi = -V$ and $FV = \xi$ by turns and using (2.5), we get

$$(\nabla_X F)\xi = -\nabla_X V + F(A^*_\xi X) - \tau(X)V, \tag{2.13}$$
$$(\nabla_X F)V = -F\nabla_X V - A^*_\xi X - \tau(X)\xi. \tag{2.14}$$

Definition 2.2 [7] *The structure tensor field F of the lightlike hypersurfce M is said to be recurrent if there exists a 1-form ω on M such that*

$$(\nabla_X F)Y = \omega(X)FY. \tag{2.15}$$

Theorem 2.3 *Let M be a lightlike hypersurface of an indefinite almost contact metric manifold $\bar{M}$ such that ζ is tangent to M. If F is recurrent, then $\omega = 0$, i.e., F is parallel with respect to the induced connection ∇ on M.*

Proof. Comparing (2.14) with (2.15) [changed by $Y = V$], we have

$$F\nabla_X V + A^*_\xi X + \{\omega(X) + \tau(X)\}\xi = 0. \tag{2.16}$$

Also, comparing (2.13) with (2.15) [changed by $Y = \xi$], we obtain

$$\nabla_X V - F(A^*_\xi X) - \{\omega(X) - \tau(X)\}V = 0. \tag{2.17}$$

Taking the scalar product with V and ζ to (2.17) by turns, we have

$$u(\nabla_X V) = 0, \qquad \theta(\nabla_X V) = 0. \tag{2.18}$$

Applying F to (2.16) and using (2.12) and (2.18) and comparing this result with (2.17), we get $\omega = 0$. Thus F is parallel with respect to ∇.

□

Definition 2.4 [7] *The structure tensor field F of the lightlike hypersurfce M is said to be Lie recurrent if there exists a 1-form σ on M such that*

$$(\mathcal{L}_X F)Y = \sigma(X)FY, \tag{2.19}$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]. \tag{2.20}$$

The structure tensor field F is called Lie parallel if $\mathcal{L}_X F = 0$.

Theorem 2.5 *Let M be a lightlike hypersurface of an indefinite almost contact metric manifold $\bar{M}$ such that ζ is tangent to M. If the structure tensor field F is Lie recurrent, then $\sigma = 0$, i.e., F is Lie parallel.*

Proof. As ∇ is torsion-free, from (2.19) and (2.20) we have

$$(\nabla_X F)Y = \nabla_{FY}X - F\nabla_Y X + \sigma(X)FY. \tag{2.21}$$

Comparing (2.14) with (2.21) [changed by $Y = V$], we obtain

$$\nabla_\xi X = -F(\nabla_X V - \nabla_V X) - A^*_\xi X - \{\sigma(X) + \tau(X)\}\xi. \tag{2.22}$$

Also, comparing (2.13) with (2.21) [changed by $Y = \xi$], we obtain

$$F\nabla_\xi X = \nabla_X V - \nabla_V X - F(A^*_\xi X) - \{\sigma(X) - \tau(X)\}V. \tag{2.23}$$

Taking the scalar product with V and ζ to (2.23) by turns, we obtain

$$u(\nabla_X V - \nabla_V X) = 0, \qquad \theta(\nabla_X V - \nabla_V X) = 0. \tag{2.24}$$

Applying F to (2.22) and using (2.12) and $(2.24)_{1,2}$ and then, comparing this result with (2.23), we have $\sigma = 0$. Thus F is Lie parallel.

□

3 Indefinite nearly trans-Sasakian manifolds

Definition 3.1 [8] *An indefinite almost contact metric manifold $\bar{M}$ is called an indefinite nearly trans-Sasakian manifold if there exist two smooth functions α and β such that $\{J, \zeta, \theta, \bar{g}\}$ satisfies*

$$\begin{aligned}(\bar{\nabla}_{\bar{X}} J)\bar{Y} + (\bar{\nabla}_{\bar{Y}} J)\bar{X} = \alpha\{&2\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} \\ &- \beta\{\theta(\bar{Y})J\bar{X} + \theta(\bar{X})J\bar{Y}\},\end{aligned} \tag{3.1}$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$. We say that $\{J, \zeta, \theta, \bar{g}\}$ is an indefinite nearly trans-Sasakian structure, of type (α, β).

If $\beta = 0$, then $\bar{M}$ is said to be an *indefinite nearly α-Sasakian manifold*. Indefinite nearly Sasakian manifolds appear as examples of indefinite nearly α-Sasakian manifolds with $\alpha = 1$. If $\alpha = 0$, then $\bar{M}$ is said to be an *indefinite nearly β-Kenmotsu manifold*. Indefinite nearly Kenmotsu manifolds are particular examples of indefinite nearly β-Kenmotsu manifold with $\beta = 1$. If $\alpha = \beta = 0$, then $\bar{M}$ is an indefinite nearly cosymplectic manifold which is another kind of indefinite nearly trans-Sasakian manifold.

Theorem 3.2 *Let M be a lightlike hypersurfac of an indefinite nearly trans-Sasakian manifold $\bar{M}$ subject such that ζ is tangent to M. Then the integral curve of ζ is a spacelike geodesic of both $\bar{M}$ and M with respect to the connections $\bar{\nabla}$ and ∇, respectively.*

Proof. Taking $\bar{X} = \bar{Y} = \zeta$ to (3.1) and using (2.1), we get $\bar{\nabla}_\zeta \zeta = 0$. Thus

$$\nabla_\zeta \zeta = 0, \qquad B(\zeta, \zeta) = 0, \qquad C(\zeta, \zeta) = 0. \tag{3.2}$$

As $\bar{\nabla}_\zeta \zeta = \nabla_\zeta \zeta = 0$, the integral curve of ζ is a spacelike geodesic of both $\bar{M}$ and M with respect to the connections $\bar{\nabla}$ and ∇, respectively.

□

By using (2.2), (2.3), (2.11) and (3.1), we obtain

$$\begin{aligned}(\nabla_X F)Y + (\nabla_Y F)X = {}& \alpha\{2g(X,Y)\zeta - \theta(Y)X - \theta(X)Y\} \\ & - \beta\{\theta(Y)FX + \theta(X)FY\} \\ & + u(X)A_{_N}Y + u(Y)A_{_N}X - 2B(X,Y)U.\end{aligned} \tag{3.3}$$

Lemma 3.3 *Let M be a lightlike hypersurface of an indefinite nearly trans-Sasakian manifold $\bar{M}$ such that ζ is tangent to M. Then we have*

$$\begin{cases} B(U,\zeta) + C(V,\zeta) = -2\alpha, & B(U,V) = C(V,V), \\ B(U,U) = C(U,V), & v(\nabla_U V) = -\tau(U), \\ 2C(V,\zeta) + C(\zeta,V) = -3\alpha, & B(U,\zeta) = \theta(\nabla_\xi U). \end{cases} \tag{3.4}$$

Proof. By directed calculation from $FU = 0$ and $FV = \xi$, we obtain

$$(\nabla_U F)V + (\nabla_V F)U = -F(\nabla_U V + \nabla_V U) - A^*_\xi U - \tau(U)\xi.$$

Comparing this result with (3.3) [changed by $X = U$ and $Y = V$], we have

$$F(\nabla_U V + \nabla_V U) + A^*_\xi U + \tau(U)\xi = -2\alpha\zeta - A_{_N}V + 2B(U,V)U.$$

As $v(\nabla_V U) = 0$, taking the scalar product with ζ, V, U and N by turn, we get the first four equations of (3.4) [which are denoted by $(3.4)_{1\sim 4}$].

By directed calculation from $FU = 0$ and $F\zeta = 0$, we obtain

$$(\nabla_U F)\zeta + (\nabla_\zeta F)U = -F(\nabla_U \zeta + \nabla_\zeta U).$$

Comparing this result with (3.3) [changed by $X = U$ and $Y = \zeta$], we have

$$F(\nabla_U \zeta + \nabla_\zeta U) = -A_{_N}\zeta + \{\alpha + 2B(U,\zeta)\}U.$$

Taking the scalar product with V to this equation, we get

$$2B(U,\zeta) - C(\zeta,V) = -\alpha.$$

Substituting $(3.4)_1$ into the last equation, we have $(3.4)_5$.

Applying ∇_X to $v(Y) = g(Y,U)$ and using (2.6), we obtain

$$(\nabla_X v)(Y) = B(X,U)\eta(Y) + g(Y, \nabla_X U).$$

Applying $\bar{\nabla}_X$ to $v(Y) = g(FY,N)$ and using (2.1) and (2.2), we obtain

$$g((\nabla_X F)Y, N) = (\nabla_X v)(Y) - v(Y)\tau(X) + g(A_{_N}X, FY).$$

Taking the scalar product with N to (3.3) and using $(2.9)_2$ and the last two equations, we obtain

$$\begin{aligned}&B(X,U)\eta(Y)+B(Y,U)\eta(X)+g(Y,\nabla_X U)+g(X,\nabla_Y U)\\&=-\alpha\{\theta(Y)\eta(X)+\theta(X)\eta(Y)\}-\beta\{\theta(Y)v(X)+\theta(X)v(Y)\}\\&+v(Y)\tau(X)+v(X)\tau(Y)-g(A_{N}X,FY)-g(A_{N}Y,FX).\end{aligned}$$

Taking $X=\zeta$ and $Y=\xi$ to this equation and using (2.7), we have

$$B(U,\zeta)-C(\zeta,V)=-\alpha-\theta(\nabla_\xi U).$$

Substituting this into $2B(U,\zeta)-C(\zeta,V)=-\alpha$, we have $(3.4)_6$.

□

Theorem 3.4 *Let M be a lightlike hypersurfac of an indefinite nearly trans-Sasakian manifold $\bar{M}$ such that ζ is tangent to M. If one of the following three conditions is satisfied,*

(1) $(\nabla_X F)Y+(\nabla_Y F)X=0$,
(2) *F is parallel with respect to the induced connection ∇ on M,*
(3) *F is recurrent,*

then $\alpha=\beta=0$. Thus $\bar{M}$ is an indefinite nearly cosymplectic manifold. In this case, the shape operators A^*_ξ and A_{N} satisfy

$$\begin{cases} A^*_\xi V=0, & A^*_\xi\zeta=0, \quad A_{N}\xi=0,\\ A_{N}V=0, & A_{N}\zeta=0, \quad A_{N}X=C(X,V)U.\end{cases} \tag{3.5}$$

Proof. (1) Assume that $(\nabla_X F)Y+(\nabla_Y F)X=0$. In this case, taking the scalar product with N to (4.2) and using $(2.9)_2$, we have

$$\alpha\{\theta(Y)\eta(X)+\theta(X)\eta(Y)\}+\beta\{\theta(Y)v(X)+\theta(X)v(Y)\}=0.$$

Taking $Y=\xi$ and $Y=V$ to this equation by turns, we get $\alpha\theta(X)=0$ and $\beta\theta(X)=0$. Taking $X=\zeta$ to these two results, we have $\alpha=\beta=0$. Therefore, $\bar{M}$ is an indefinite nearly cosymplectic manifold.

As $\alpha=\beta=0$, the equation (3.3) is reduced to

$$2B(X,Y)U=u(X)A_{N}Y+u(Y)A_{N}X. \tag{3.6}$$

Taking the scalar product with V to (3.6), we have

$$2B(X,Y)=u(Y)u(A_{N}X)+u(X)u(A_{N}Y). \tag{3.7}$$

Taking $Y=V$ to this equation and using $(2.9)_1$, we obtain

$$2B(X,V)=u(X)C(V,V).$$

Replacing X by U to this equation, we have $2B(U,V)=C(V,V)$. Comparing this result with $(3.4)_2$, we have $C(V,V)=0$. Thus we obtain

$$C(V,V)=B(U,V)=0, \qquad B(X,V)=0. \tag{3.8}$$

From (2.8), $(3.8)_2$ and the facts that B is symmetric and $S(TM)$ is non-degenerate, we have $(3.5)_1$: $A^*_\xi V = 0$. Taking $X = U$ and $Y = V$ to (3.6) and using $(3.8)_1$, we get $(3.5)_4$: $A_{_N} V = 0$. Taking the scalar product with ζ to $A_{_N} V = 0$, we have $C(V, \zeta) = 0$. From this equation, $(3.4)_{1,5}$, we get

$$B(U, \zeta) = C(V, \zeta) = C(\zeta, V) = 0.$$

Taking $X = U, Y = \zeta$ and $X = U, Y = \xi$ to (3.6) by turns, we have $(3.5)_5$: $A_{_N}\zeta = 0$ and $(3.5)_3$: $A_{_N}\xi = 0$, respectively. Taking $Y = \zeta$ to (3.7) and using the fact that $C(\zeta, V) = 0$, we have $B(X, \zeta) = 0$. It follows that $A^*_\xi \zeta = 0$.

Taking $Y = U$ to (3.7), we obtain

$$2B(X, U) = u(A_{_N} X) + u(X)u(A_{_N} U).$$

Replacing Y by U to (3.6) and using the last equation, we get

$$A_{_N} X - u(A_{_N} X)U + u(X)\{A_{_N} U - u(A_{_N} U)U\} = 0.$$

Taking $X = U$ to this, we have $A_{_N} U = u(A_{_N} U)U$. Thus we have $(3.5)_6$:

$$A_{_N} X = u(A_{_N} X)U.$$

(2) If F is parallel with respect to ∇, then $(\nabla_X F)Y + (\nabla_Y F)X = 0$. Thus $\alpha = \beta = 0$, and A^*_ξ and $A_{_N}$ satisfy all equations in (3.5) by (1).

(3) If F is recurrent, then F is parallel with respect to ∇ by Theorem 2.3. Thus $\alpha = \beta = 0$, and A^*_ξ and $A_{_N}$ satisfy all equations in (3.5) by (2). □

Definition 3.5 *The structure tensor field F of the lightlike hypersurface M is said to be nearly recurrent if there exists a 1-form ω on M such that*

$$(\nabla_X F)Y + (\nabla_Y F)X = \omega(X)FY + \omega(Y)FX. \tag{3.9}$$

Theorem 3.6 *Let M be a lightlike hypersurface of an indefinite nearly trans-Sasakian manifold $\bar{M}$ such that ζ is tangent to M. If F is nearly recurrent, then we have $\alpha = 0$ and $\beta\theta = -\omega$ on M. In this case, the shape operators A^*_ξ and $A_{_N}$ satisfy all equations in* (3.5).

Proof. If F is nearly recurrent, then, substituting (3.9) into (3.3), we get

$$\begin{aligned} &\alpha\{\theta(Y)X + \theta(Y)X - 2g(X, Y)\zeta\} \\ &+ \{\omega(Y) + \beta\theta(Y)\}FX + \{\omega(X) + \beta\theta(X)\}FY \\ &= u(X)A_{_N} Y + u(Y)A_{_N} X - 2B(X, Y)U. \end{aligned} \tag{3.10}$$

Taking the scalar product with N to this and using $(2.9)_2$, we have

$$\begin{aligned} &\{\omega(Y) + \beta\theta(Y)\}v(X) + \{\omega(X) + \beta\theta(X)\}v(Y) \\ &+ \alpha\{\theta(Y)\eta(X) + \theta(X)\eta(Y)\} = 0. \end{aligned} \tag{3.11}$$

Taking $X = \zeta$ and $Y = \xi$ to (3.11) by turns, we have $\alpha = 0$. Also, taking $X = Y = V$ to (3.11), we get $\omega(V) = 0$. Replacing Y by V to (3.11), we get

$$\omega(X) + \beta\theta(X) = 0.$$

As $\alpha = 0$ and $\omega(X) + \beta\theta(X) = 0$, taking the scalar product with V to (3.10), we have (3.6). By the course of the proof of (1) in Theorem 3.4, we have all equations in (3.5). Thus we have our theorem.

□

Theorem 3.7 *Let M be a lightlike hypersurface of an indefinite nearly trans-Sasakian manifold $\bar{M}$ such that ζ is tangent to M. If F is Lie recurrent, then $\alpha = 0$, i.e., $\bar{M}$ is an indefinite nearly β-Kenmotsu manifold.*

Proof. If F is Lie recurrent, then F is Lie parallel, *i.e.*, $\sigma = 0$, by Theorem 2.5. Replacing Y by U to (2.21), we have $(\nabla_X F)U = -F\nabla_U X$. Applying ∇_X to $FU = 0$, we get $(\nabla_X F)U = -F\nabla_X U$. Therefore we have

$$F(\nabla_X U - \nabla_U X) = 0. \tag{3.12}$$

Taking the scalar product with N to this equation, we obtain

$$v(\nabla_X U - \nabla_U X) = 0. \tag{3.13}$$

Applying ∇_X to $g(U, U) = 0$ and using (2.6), $(3.4)_4$ and (3.13), we get

$$v(\nabla_X U) = 0, \qquad v(\nabla_U X) = 0, \qquad \tau(U) = 0. \tag{3.14}$$

Taking $X = U$ to (2.22) and using (3.12) and $(3.14)_3$, we obtain

$$\nabla_\xi U = -A^*_\xi U. \tag{3.15}$$

Substituting (3.15) into the right term of $(3.4)_6$: $B(U, \zeta) = \theta(\nabla_\xi U)$, we have

$$B(U, \zeta) = 0, \qquad v(\nabla_\xi \zeta) = -\theta(\nabla_\xi U) = 0. \tag{3.16}$$

Taking the scalar product with U to (2.22), we have

$$v(\nabla_\xi X) = \eta(\nabla_X V - \nabla_V X) - B(X, U).$$

Replacing X by ζ to this and using (2.4) and $(3.16)_{1,2}$, we obtain

$$C(\zeta, V) = C(V, \zeta).$$

From this result and $(3.4)_5$, we have $C(V, \zeta) = -\alpha$. Substituting this result into $(3.4)_1$, we get $B(U, \zeta) = -\alpha$. From this and $(3.16)_1$, we have $\alpha = 0$.

□

Definition 3.8 **[3]** *A lightlike hypersurface M of $(\bar{M}, \bar{g})$ is said to be*

(1) *totally umbilical if there is a smooth function ρ on any coordinate neighborhood $\mathcal{U}$ in M such that $A^*_\xi X = \rho PX$, or equivalently,*

$$B(X,Y) = \rho g(X,Y). \tag{3.17}$$

In case $\rho = 0$ on $\mathcal{U}$, we say that M is totally geodesic.

(2) *screen totally umbilical if there exist a smooth function γ on any $\mathcal{U}$ such that $A_{_N} X = \gamma PX$, or equivalently,*

$$C(X,PY) = \gamma g(X,PY). \tag{3.18}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that M is screen totally geodesic.

Theorem 3.9 *Let M be a lightlike hypersurface of an indefinite nearly trans-Sasakian manifold $\bar{M}$ such that ζ is tangent to M.*

(1) *If M is totally umbilical, then M is totally geodesic.*
(2) *If M is screen totally umbilical, then M is screen totally geodesic.*

Proof. (1) Taking $X = Y = \zeta$ to (3.17) and using $(3.2)_2$, we have $\rho = 0$. Thus M is totally geodesic.

(2) Taking $X = PY = \zeta$ to (3.18) and using $(3.2)_3$, we have $\gamma = 0$. Thus M is screen totally geodesic.

□

4 Indefinite nearly generalized Sasakian space forms

Denote by $\bar{R}$, R and R^* the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$ and the induced connection ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae for M and $S(TM)$, we obtain the following two Gauss equations for M and $S(TM)$ such that

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ B(X,Z)A_{_N}Y - B(Y,Z)A_{_N}X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N,\end{aligned} \tag{4.1}$$

$$\begin{aligned}R(X,Y)PZ \;=\; & R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X \\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &\quad - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ)\}\xi.\end{aligned} \tag{4.2}$$

Definition 4.1 *An indefinite nearly trans-Sasakian manifold $\bar{M}$ is called an indefinite nearly generalized Sasakian space form, denoted by $\bar{M}(f_1, f_2, f_3)$, if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that*

$$\begin{aligned}\bar{R}(\bar{X},\bar{Y})\bar{Z} \;=\; & f_1\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y}\} \\ &+ f_2\{\bar{g}(\bar{X},J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y},J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\} \\ &+ f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X} \\ &\quad + \bar{g}(\bar{X},\bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y},\bar{Z})\theta(\bar{X})\zeta\}.\end{aligned} \tag{4.3}$$

The notion of (Riemannian) generalized Sasakian space form was introduced by Alegre *et. al.* [1]. Sasakian, Kenmotsu and cosymplectic space form are important kinds of generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4}$$

respectively, where c is a constant J-sectional curvature of each space forms.

Comparing the tangential, transversal and radical components of the left-right terms of (4.3) such that $\bar{X} = X, \bar{Y} = Y$ and $\bar{Z} = Z$ and using $(2.9)_2$, (2.10), (2.11), (4.1) and (4.2), we obtain the following equations:

$$\begin{aligned} R(X,Y)Z = {} & B(Y,Z)A_{_N}X - B(X,Z)A_{_N}Y \\ & + f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{\bar{g}(X,JZ)FY - \bar{g}(Y,JZ)FX + 2\bar{g}(X,JY)FZ\} \\ & + f_3\{[\theta(X)Y - \theta(Y)X]\theta(Z) \\ & \qquad + [g(X,Z)\theta(Y) - g(Y,Z)\theta(X)]\zeta\}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} & (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ & = f_2\{\bar{g}(X,JZ)u(Y) - \bar{g}(Y,JZ)u(X) + 2\bar{g}(X,JY)u(Z)\}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} & (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ & \qquad - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ) \\ & = f_1\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\} \\ & + f_2\{\bar{g}(X,JPZ)v(Y) - \bar{g}(Y,JPZ)v(X) + 2\bar{g}(X,JY)v(PZ) \\ & + f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ). \end{aligned} \tag{4.6}$$

Using the Gauss-Weingarten formulae (2.4) and (2.5) for $S(TM)$, we obtain the following Codazzi equation for $S(TM)$ such that

$$\begin{aligned} R(X,Y)\xi = {} & -\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y] \\ & - \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X \\ & + \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X,Y)\}\xi. \end{aligned}$$

Replacing Z by ξ to (4.4) and using (2.7) and (2.10), we have

$$R(X,Y)\xi = f_2\{u(Y)FX - u(X)FY - 2\bar{g}(X,JY)V\}.$$

Comparing the radical components of the last two equations, we get

$$\begin{aligned} & f_2\{u(Y)v(X) - u(X)v(Y)\} \\ & = g(A_{_N}Y, A^*_\xi X) - g(A_{_N}X, A^*_\xi Y) - 2d\tau(X,Y). \end{aligned} \tag{4.7}$$

Theorem 4.2 *Let M be a lightlike hypersurface of an indefinite nearly generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ such that ζ is tangent to M. If one of the following four conditions is satisfied,*

(1) $(\nabla_X F)Y + (\nabla_Y F)X = 0$,
(2) *F is parallel with respect to the connection ∇,*
(3) *F is recurrent,*
(4) *F is nearly recurrent,*

then $f_1 = -f_2 = f_3$. Moreover, if $d\tau(U, V) = 0$, then $f_1 = f_2 = f_3 = 0$.

Proof. If one of the items (1) $\sim$ (4) is satisfied, then A^*_ξ and A_N satisfy (3.5). Taking the scalar product with ζ to $(3.5)_6$ and using $(2.9)_1$, we have

$$C(X, \zeta) = 0.$$

Applying ∇_X to $C(Y, \zeta) = 0$ and using the last equation, we have

$$(\nabla_X C)(Y, \zeta) = -C(Y, \nabla_X \zeta).$$

Substituting the last two equations into (4.6) with $PZ = \zeta$, we obtain

$$C(X, \nabla_Y \zeta) - C(Y, \nabla_X \zeta) = (f_1 - f_3)\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}.$$

Taking $Y = \zeta$ and $X = \xi$ and using $(3.2)_1$ and $(3.5)_5$, we get $f_1 = f_3$.

Taking the scalar product with U to $(3.5)_6$ and using $(2.9)_1$, we have

$$C(X, U) = 0.$$

Applying ∇_X to $C(Y, U) = 0$ and using the last equation, we have

$$(\nabla_X C)(Y, U) = -C(Y, \nabla_X U).$$

Substituting the last two equations into (4.6) with $PZ = U$, we obtain

$$C(X, \nabla_Y U) - C(Y, \nabla_X U) = (f_1 + f_2)\{v(Y)\eta(X) - v(X)\eta(Y)\}.$$

Taking $Y = V$ and $X = \xi$ and using $(2.9)_1$ and $(3.5)_{3,4}$, we get $f_1 + f_2 = 0$.

Taking $Y = U$ and $X = V$ to (4.7) and using $(3.5)_{1,4}$, we obtain

$$f_2 = 2d\tau(U, V).$$

If $d\tau(U, V) = 0$, then we have $f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$.

□

Definition 4.3 **[7]** *A lightlike hypersurface M is said to be a Hopf lightlike hypersurface if the structure vector field U is an eigenvector of A^*_ξ.*

Theorem 4.4 *Let M be a lightlike hypersurface of an indefinite nearly generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ such that ζ is tangent to M. If the structure tensor field F is Lie recurrent, then*

$$3f_2 = g(A^*_\xi U, A^*_\xi U), \qquad f_2 = -g(A_N V, A^*_\xi U).$$

Moreover, if M is a Hopf lightlike hypersurface of $\bar{M}(c)$, then $f_2 = 0$.

Proof. Taking the scalar product with U to (3.15) and using $(3.14)_1$, we get

$$B(U,U) = 0. \tag{4.8}$$

Applying ∇_ξ to (4.8): $B(U,U) = 0$ and using (3.15), we see that

$$(\nabla_\xi B)(U,U) = 2g(A^*_\xi U, A^*_\xi U).$$

Also, applying ∇_U to $B(\xi, U) = 0$ and using (2.5) and (2.7), we see that

$$(\nabla_U B)(\xi, U) = g(A^*_\xi U, A^*_\xi U).$$

Taking $X = \xi$, $Y = U$ and $Z = U$ to (4.5) and using (2.7), $(3.14)_3$, (4.8) and the last two equations, we obtain

$$3f_2 = g(A^*_\xi U, A^*_\xi U).$$

Substituting (2.21) into (3.3) and using the facts: $\sigma = \alpha = 0$, we have

$$\begin{aligned}&\nabla_{FY}X + \nabla_{FX}Y - F(\nabla_Y X + \nabla_X Y)\\ &= -\beta\{\theta(Y)FX + \theta(X)FY\}\\ &+ u(X)A_{N}Y + u(Y)A_{N}X - 2B(X,Y)U.\end{aligned}$$

Replacing Y by ξ to this and using (2.5), (2.7) and (3.9), we obtain

$$\begin{aligned}&\nabla_X V + A^*_\xi FX + \tau(FX)\xi - 2F(A^*_\xi X) + 2\tau(X)V\\ &= -\beta\theta(X)V - u(X)A_{N}\xi.\end{aligned} \tag{4.9}$$

Taking the scalar product with N to this and using (2.4), we have

$$C(X,V) + \tau(FX) = 2B(X,U).$$

Taking X by V and ξ to this by turns and using $(3.4)_2$, we have

$$B(U,V) = \tau(\xi), \qquad C(\xi,V) = \tau(V). \tag{4.10}$$

Taking the scalar product with U to (4.9), we have

$$v(\nabla_X V) + B(FX,U) + 2\tau(X) = -\beta\theta(X) - u(X)C(\xi,U).$$

Replacing X by ξ to this equation and using $(4.10)_1$, we have

$$v(\nabla_\xi V) = -\tau(\xi).$$

Substituting this and (3.15) into the equation: $v(\nabla_\xi V) + u(\nabla_\xi U) = 0$ and using $(4.10)_1$: $B(U,V) = \tau(\xi)$, we have $B(U,V) = C(V,V) = 0$ and

$$\tau(\xi) = 0.$$

Taking the scalar product with V to (4.9), as $g(\nabla_X V, V) = 0$, we have

$$B(FX,V) = -u(X)C(\xi,V).$$

Taking $X = U$ to this, we have $C(\xi, V) = 0$. Comparing this with $(4.10)_2$, we get $\tau(V) = 0$. Taking $X = V$ to (2.22) and using $\tau(V) = 0$, we obtain

$$\nabla_\xi V = -A^*_\xi V.$$

Taking $X = \xi$ to (4.9) and using the last equation, we obtain

$$A^*_\xi V = 0,$$

due to $A^*_\xi \xi = \tau(V) = \tau(\xi) = 0$. As $\tau(U) = \tau(V) = 0$, we see that

$$2d\tau(U, V) = -\tau([U,\, V]),$$

where the symbol $[U,\, V]$ denotes the Lie bracket of U and V. Taking $X = U$ and $Y = \xi$ to (2.20): $[X, FY] - F[X, Y] = 0$, we have

$$[U,\, V] = -F[U,\, \xi].$$

As ∇ is torsion-free, using (2.5), $(3.14)_3$ and (3.15), we see that

$$[U,\, \xi] = \nabla_U \xi - \nabla_\xi U = 0,$$

Thus we get $[U,\, V] = 0$. Consequently, we have

$$d\tau(U, V) = 0.$$

As $A^*_\xi V = 0$ and $d\tau(U, V) = 0$, taking $Y = U$ and $X = V$ to (4.7), we get

$$f_2 = -g(A_{\scriptscriptstyle N} V, A^*_\xi U).$$

If M is a Hopf lightlike hypersurface of $\bar{M}(f_1, f_2, f_3)$, that is, $A^*_\xi U = \lambda U$ for some smooth function λ, then $g(A^*_\xi U, A^*_\xi U) = 0$. Thus $f_2 = 0$.

□

Theorem 4.5 *Let M be a totally umbilical lightlike hypersurface of an indefinite nearly generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ such that ζ is tangent to M. Then $f_2 = 0$ and the transversal connection is flat.*

Proof. If M is totally umbilical, then M is totally geodesic by (1) of Theorem 3.9. Taking $X = \xi$ and $Y = Z = U$ to (4.5), we see that $f_2 = 0$.

As $f_2 = 0$ and $A^*_\xi = 0$, from (4.7) we see that $d\tau = 0$. Thus, by Theorem 2.1 the transversal connection of M is flat.

□

Theorem 4.6 *Let M be a screen totally umbilical lightlike hypersurface of an indefinite nearly generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ such that ζ is tangent to M. Then $f_1 = f_2 = f_3 = 0$, i.e., $\bar{M}(f_1, f_2, f_3)$ is flat, M is also flat, and $d\tau = 0$, i.e., the transversal connection is flat.*

Proof. If M is screen totally umbilical, then M is screen totally geodesic by (2) of Theorem 3.9. Taking $X = \xi$ and $Y = PZ = \zeta$ to (4.6), we obtain $f_1 = f_3$. Taking $X = \xi$, $Y = V$ and $PZ = U$ to (4.6), we have $f_1 + f_2 = 0$. Also, taking $X = \xi$, $Y = U$ and $PZ = V$ to (4.6), we obtain $f_1 + 2f_2 = 0$. Comparing these two results, we get $f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. As $f_1 = f_2 = f_3 = 0$ and $A_{\scriptscriptstyle N} = 0$, from (4.4) we get $R = 0$, and from (4.7), we have $d\tau = 0$. Thus M is also flat and and the transversal connection $\nabla^{\perp}$ on $tr(TM)$ is flat.

□

Definition 4.7 *A lightlike hypersurface M is said to be screen conformal [6] if there exist a non-vanishing smooth function φ on any coordinate neighborhood $\mathcal{U}$ in M such that $A_{\scriptscriptstyle N} = \varphi A^*_{\xi}$, or equivalently,*

$$C(X, PY) = \varphi B(X, PY). \tag{4.11}$$

Theorem 4.8 *Let M be a screen conformal lightlike hypersurface of an indefinite nearly generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ such that ζ is tangent to M. Then $f_1 = f_2 = f_3 = 0$ and $d\tau = 0$, i.e., $\bar{M}(f_1, f_2, f_3)$ is flat and the transversal connection $\nabla^{\perp}$ on $tr(TM)$ is flat.*

Proof. Let $\mu = U - \varphi V$. From $(3.4)_{2,3}$ and (4.11), we obtain

$$B(V, \mu) = 0, \qquad\qquad B(U, \mu) = 0. \tag{4.12}$$

Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this equation and (4.11) into (4.6) and using (4.5), we have

$$\begin{aligned}
&\{X\varphi - 2\varphi\tau(X)\}B(Y, PZ) - \{Y\varphi - 2\varphi\tau(Y)\}B(X, PZ)\\
&= \ f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}\\
&+ f_2\{\bar{g}(X, JPZ)g(Y, \mu) - \bar{g}(Y, JPZ)g(X, \mu) + 2\bar{g}(X, JY)g(PZ, \mu)\\
&+ f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).
\end{aligned}$$

Taking $X = \xi$ to this equation and using (2.7), we have

$$\begin{aligned}
\{\xi\varphi - 2\varphi\tau(\xi)\}B(Y, PZ) = \ & f_1 g(Y, PZ)\\
&+ f_2\{u(PZ)g(Y, \mu) + 2u(Y)g(PZ, \mu)\} - f_3\theta(Y)\theta(PZ).
\end{aligned} \tag{4.13}$$

Taking $Y = PZ = \zeta$ to (4.13) and using $(3.2)_2$, we get $f_1 = f_3$. Also, taking $Y = V$ and $PZ = \mu$ to (4.13) and using $(4.12)_1$, we get $f_1 + f_2 = 0$. On the other hand, taking $Y = U$ and $PZ = \mu$ to (4.13), we obtain $\varphi(f_1 + 3f_2) = 0$. From these results, we get $f_1 = f_2 = f_3 = 0$.

As $f_2 = 0$ and $A_{\scriptscriptstyle N} = \varphi A^*_{\xi}$, from (4.7) we see that $d\tau = 0$. Thus the transversal connection is flat by Theorem 2.1.

□

References

1. Alegre, P.; Blair, D.E.; Carriazo, A. – *Generalized Sasakian-space-forms*, Israel J. Math. **141** (2004), 157-183.
2. Călin, C. – *Contributions to geometry of CR-submanifold*, Thesis, University of Iaşi (Romania, 1998).
3. Duggal, K.L.; Bejancu, A. – *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Kluwer Academic Publishers Group, Dordrecht, 1996.
4. Duggal, K.L.; Jin, D.H. – *Null curves and hypersurfaces of semi-Riemannian manifolds,* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.
5. Duggal, K.L.; Sahin, B. – *Differential geometry of lightlike submanifolds*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2010.
6. Jin, D.H. – *Geometry of lightlike hypersurfaces of an indefinite Sasakian manifold*, Indian J. Pure Appl. Math. **41** (2010), no. 4, 569-581.
7. Jin, D.H. – *Special lightlike hypersurfaces of indefinite Kaehler manifolds*, Filomat, **30** (2016), no. 7, 1919-1930.
8. Mustafa, A.; Uddin, S.; Khan, V.A.; Wong, B.R. – *Contact CR-warped product submanifolds of nearly trans-Sasakian manifolds*, Taiwanese J. Math. **17** (2013), no. 4, 1473-1486.

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