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Concircularly flat Z-symmetric manifolds

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Abstract The object of the present paper is to study concircularly flat Z-symmetric manifold which is a type of non-flat Riemannian manifold. In the first section, we give the definition of a concircularly flat Z-symmetric manifold. In the second section, some theorems about these manifolds are proved and some properties of them are obtained. In the last section, we give an example for the existence of this manifold.

Keywords concircular curvature tensor · concircularly flat Z-symmetric manifold · recurrence vector field

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1 Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan [3], who, in particular, obtained a classification of those spaces. Let (M^n, g), $(n = dimM)$ be a Riemannian manifold, i.e., a manifold M with the Riemannian metric g, and let ∇ be the Levi-Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric [3] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [11]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature. During the last eight decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to different extent such as conformally symmetric manifolds by Chaki and Gupta [5], recurrent manifolds introduced by Walker [15], conformally recurrent manifolds by Adati and Miyazawa [1], pseudo symmetric manifolds by Chaki [4], weakly symmetric manifolds by Tamassy and Binh [14] etc.

The notion of weakly Z-symmetric manifolds was introduced by Mantica and Molinari [9]. It was a generalization of the notion of weakly Ricci symmetric manifolds, pseudo Ricci symmetric manifolds, pseudo projective Ricci symmetric manifolds. Many authors have been studied on Z-symmetric manifolds such as weakly cyclic Z symmetric manifolds [6], pseudo Z-symmetric Riemannian manifolds with harmonic curvature tensors [10], on almost pseudo Z-symmetric manifolds [7].

A $(0,2)$ symmetric tensor is generalized Z tensor if

$$Z(X,Y) = S(X,Y) + \phi g(X,Y), \tag{1.1}$$

where ϕ denotes an arbitrary scalar function. The scalar $\bar{Z}$ is obtained by contracting (1.1) over X and Y as follows:

$$\bar{Z} = r + n\phi. \tag{1.2}$$

The classical Z tensor is obtained with the choice $\phi = -\frac{1}{n}r$. Hereafter we refer to the generalized Z tensor simply as the Z tensor. The Z tensor allows us to reinterpret several well known structures on Riemannian manifolds.

If $Z(X,Y) = 0$, the (Z-flat) manifold is an Einstein space [2],

$$S(X,Y) = (\frac{r}{n})g(X,Y).$$

A. Gray [8] introduced the notion of Codazzi type of the Ricci tensor. A Riemannian manifold is said to satisfy Codazzi type of Ricci tensor if its Ricci tensor S of type (0,2) is non-zero and satisfies the following condition

$$(\nabla_X S)(Y,W) = (\nabla_Y S)(X,W). \tag{1.3}$$

One can easily find that this condition is equivalent to

$$(\nabla_X S)(Y,W) - (\nabla_Y S)(X,W) = (\nabla_Y \phi)g(X,W) - (\nabla_X \phi)g(Y,W). \tag{1.4}$$

Contracting (1.4) over Y and W, we get $\nabla_X(r+2(n-1)\phi) = 0$ and finally

$$(\nabla_X S)(Y,W) - (\nabla_Y S)(X,W) = \frac{1}{2(n-1)}\Big[(\nabla_X r)g(Y,W) - (\nabla_Y r)g(X,W)\Big].$$

A manifold with Z-tensor of Codazzi type is thus a nearly conformal symmetric manifold $(NCS)_n$. This condition was introduced and studied by Roter [12] and Suh, Kwon and Yang [13]. Conversely, an $(NCS)_n$ manifold has a Z-tensor of Codazzi type, if the condition $\nabla_X(r + 2(n-1)\phi) = 0$ is satisfied.

A (0,4) type tensor $K(X,Y,Z,W)$ which remains invariant under concircular transformation, for an n-dimensional Riemannian space V_n, is defined by Yano and Kon (1984) as

$$K(X,Y,Z,W) = R(X,Y,Z,W) - \frac{r}{n(n-1)}[g(X,W)g(Y,Z) - g(Y,W)g(X,Z)] \quad (1.5)$$

where $R(X,Y,Z,W)$ is the Riemannian curvature tensor and r is the Ricci scalar. The curvature tensor $K(X,Y,Z,W)$ is known as concircular curvature tensor. The contraction of (1.5) over X and W leads to

$$K(Y,Z) = S(Y,Z) - \frac{r}{n}g(Y,Z) \quad (1.6)$$

which is also invariant under concircular transformation.

If the tensor $K(X,Y,Z,W)$ of a Riemannian manifold of dimension n ($n > 2$) vanishes i.e.,$K(X,Y,Z,W) = 0$ then this manifold is said to be concircularly flat.

If our manifold is concircularly flat then from (1.6), we find

$$S(Y,Z) = \frac{r}{n}g(Y,Z). \quad (1.7)$$

This paper is organized as follows: after defining concircularly flat Z-symmetric manifolds in section 2, we study Codazzi type and Cyclic parallel Z-symmetric tensor in a concircularly flat manifold. Considering special vector fields in this manifold, we prove some theorems. In section 3, we give an example of a concircularly flat Z-symmetric manifold to verify the result of a theorem obtained in section 2.

2 Concircularly flat Z-symmetric manifold

Let S and r denote the Ricci tensor of type (0,2) and the scalar curvature respectively. L denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is,

$$g(LX,Y) = S(X,Y) \quad (2.1)$$

for any vectors X and Y.

We have $Z(X,Y) = Z(Y,X)$ and $Z(Y,Q) = S(Y,Q) + \phi g(Y,Q)$ from (1.1).

Theorem 2.1 *A concircularly flat Z-symmetric manifold is an Einstein manifold, whose Z-symmetric tensor is in the form* $Z(X,Y) = (\frac{r}{n} + \phi)g(X,Y)$.

Proof. Now, we assume that our manifold is concircularly flat Z-symmetric then from (1.1) and (1.7), we obtain

$$Z(X,Y) = (\frac{r}{n} + \phi)g(X,Y). \tag{2.2}$$

□

Theorem 2.2 *A concircularly flat Z-symmetric tensor is of non-zero trace.*

Proof. Contracting (2.2) over X and Y, we find $\bar{Z} = r + n\phi$, where $\bar{Z}$ is the trace of $Z(X,Y)$.

If $\bar{Z} = 0$ then $r = -n\phi$. By putting this relation in (2.2), we get $Z(X,Y) = 0$.

□

Theorem 2.3 *If we assume that the concircularly flat Z-symmetric curvature tensor is chosen either Codazzi type or cyclic then this tensor must be covariantly constant.*

Proof. Now, if we differentiate the equation (2.2), we get

$$(\nabla_X Z)(U,V) = (\frac{1}{n}\nabla_X r + \phi(X))g(U,V) \tag{2.3}$$

where $\phi(X) = (\nabla_X \phi)$.

Assume that the concircularly flat Z-symmetric tensor is Codazzi type. In this case, from (2.3), we find

$$\begin{aligned} 0 &= (\nabla_X Z)(U,V) - (\nabla_V Z)(U,X) \\ &= (\frac{1}{n}\nabla_X r + \phi(X))g(U,V) - (\frac{1}{n}\nabla_V r + \phi(V))g(U,X). \end{aligned} \tag{2.4}$$

Hence, contracting (2.4) over U and V, we obtain

$$(n-1)(\nabla_X r + n\phi(X)) = 0. \tag{2.5}$$

Thus, we get

$$\nabla_X r = -n\phi(X). \tag{2.6}$$

By putting (2.6) in (2.3), one can easily obtain that

$$(\nabla_X Z)(U,V) = 0. \tag{2.7}$$

As a similar, if we assume that the concircularly flat Z-symmetric tensor is cyclic then, we find

$$(\nabla_X Z)(U,V) + (\nabla_U Z)(V,X) + (\nabla_V Z)(X,U) = 0. \tag{2.8}$$

From (2.3) and (2.8), it can be written as

$$(\frac{1}{n}\nabla_X r + \phi(X))g(U,V) + (\frac{1}{n}\nabla_U r + \phi(U))g(V,X) + (\frac{1}{n}\nabla_V r + \phi(V))g(X,U) = 0. \tag{2.9}$$

Contracting (2.9) over U and V, we find

$$\nabla_X r + n\phi(X) = 0. \tag{2.10}$$

Finally, we get from (2.3) and (2.10)

$$(\nabla_X Z)(U,V) = 0. \tag{2.11}$$

□

Definition 2.4 *[16] A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the following condition*

$$\nabla_X \xi = \rho X + \phi(X)\xi \tag{2.12}$$

where $X \in TM$, $\phi(X)$ is a linear form and ρ is a function. In local transcription, this reads

$$\xi^h_{,i} = \rho\delta^h_i + \xi^h\phi_i \tag{2.13}$$

where ξ^h and ϕ_i are the components of ξ and ϕ respectively, and δ^h_i is the Kronecker symbol. A torse-forming vector field ξ is called
i) recurrent, if $\rho = 0$,
ii) concircular, if the form ϕ_i is gradient covector, i.e., there is a function ψ such that $\phi = d\psi(X)$,
iii) convergent, if it is concircular and $\rho = const.exp(\phi)$.

Theorem 2.5 *For a concircularly flat Z-symmetric manifold admitting constant scalar curvature and a torse forming vector field associated by the 1-form ϕ, the recurrence vector field generated by the 1-form λ and the vector field generated by ϕ are collinear.*

Proof. Now, we assume that our manifold has constant scalar curvature, we can find from the equation (2.3),

$$(\nabla_X Z)(U,V) = \phi(X)g(U,V). \tag{2.14}$$

Differentiating covariantly of the equation (2.14), we obtain

$$(\nabla_Y \nabla_X Z)(U,V) = (\nabla_Y \phi)(X) g(U,V). \tag{2.15}$$

Now, if we assume that $\phi(X)$ is a torse-forming vector field from (2.12), the following relation holds

$$(\nabla_Y \phi)(X) = \rho g(X,Y) + \lambda(Y)\phi(X). \tag{2.16}$$

Putting (2.16) in (2.15), it can be found that

$$(\nabla_Y \nabla_X Z)(U,V) = (\rho g(X,Y) + \lambda(Y)\phi(X)) g(U,V). \tag{2.17}$$

From the equation (2.15), we get

$$(\nabla_Y \nabla_X Z)(U,V) = (\nabla_X \nabla_Y Z)(U,V). \tag{2.18}$$

In this case, subtracting the equations (2.17) and (2.18) side by side, we get finally

$$\lambda(Y)\phi(X) = \lambda(X)\phi(Y). \tag{2.19}$$

Hence, we can say that $\lambda(Y)$ and $\phi(Y)$ are collinear.

□

Theorem 2.6 *For a concircularly flat Z-symmetric manifold admitting constant scalar curvature and a torse forming vector field associated by the 1-form ϕ, the flux density of the gradient flow of the trace function $\bar{Z}$ is $n(n\rho + (U\phi)\lambda(U))$.*

Proof. Contracting (2.17) over U and V and contracting over again X and Y, we obtain

$$\Delta \bar{Z} = n^2\rho + n(U\phi)\lambda(U). \tag{2.20}$$

In this case, from the equation (2.20), we can see that, the flux density of the gradient flow of the trace function $\bar{Z}$ is $n(n\rho + (U\phi)\lambda(U))$.

□

Theorem 2.7 *For a concircularly flat Z-symmetric manifold admitting constant scalar curvature and a torse forming vector field associated by the 1-form ϕ and with a scalar function ρ and a vector λ a necessary and sufficient condition the trace $\bar{Z}$ of the Z-symmetric tensor be harmonic is that the scalar function ρ be as the form $-\frac{1}{n}(U\phi)\lambda(U)$.*

Proof. If we have the equation (2.20) and if we assume that the trace function $\bar{Z}$ is harmonic then the scalar function ρ must be $-\frac{1}{n}(U\phi)\lambda(U)$.

□

Theorem 2.8 *For a concircularly flat Z-symmetric manifold admitting constant scalar curvature and a concircular vector field generated by the 1-form ϕ, the flux density of the generated flow of the trace function $\bar{Z}$ is $n^2\rho$.*

Proof. If the vector field generated by the 1-form ϕ is concircular then we have

$$(\nabla_Y \phi)(X) = \rho g(X, Y). \tag{2.21}$$

In this case, by using the equation (2.21), the equation (2.15) takes the form

$$(\nabla_U \nabla_V Z)(X, Y) = \rho g(X, Y) g(U, V). \tag{2.22}$$

From (2.22), the Laplacian of the function $\bar{Z}$ can be found as

$$\Delta \bar{Z} = n^2 \rho.$$

Thus the Laplacian of the function $\bar{Z}$, i.e., the flux density of the gradient flow of the trace function $\bar{Z}$ is $n^2\rho$.

□

Theorem 2.9 *In a concircularly flat Z-symmetric manifold with constant scalar curvature and the recurrent vector field generated by the 1-form ϕ with a recurrent vector field generated by λ, a necessary and sufficient condition the trace of the Z-symmetric tensor be harmonic is that these vector fields be orthogonal.*

Proof. If the vector field generated by the 1-form ϕ is recurrent then we have

$$(\nabla_Y \phi)(X) = \lambda(Y)\phi(X). \tag{2.23}$$

In this case, from (2.15) and (2.23), we obtain

$$(\nabla_U \nabla_V Z)(X, Y) = \lambda(U)\phi(V) g(X, Y). \tag{2.24}$$

After that, contracting (2.24) over U and V and contracting over again X and Y, we obtain

$$\Delta \bar{Z} = n(U\phi)\lambda(U). \tag{2.25}$$

In this case, we can say that if the vector fields generated by the 1-forms λ and ϕ are orthogonal then the trace function $\bar{Z}$ is harmonic. Also, the converse is true.

□

3 An example for concircularly flat Z-symmetric manifold

In this section we will give an example for Theorem 2.3. about concircularly flat Z-symmetric manifold.

We define a Riemannian metric on $\mathbb{R}^4$ by the formula,

$$ds^2 = g_{ij}dx^i dx^j = (1+2e^{x^1})[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2] \quad (3.1)$$

where i,j=1,2,3,4 and x^1, x^2, x^3, x^4 are the standard coordinates of $\mathbb{R}^4$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature are given by, respectively

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = -\Gamma^1_{22} = -\Gamma^1_{33} = -\Gamma^1_{44} = \frac{e^{x^1}}{1+2e^{x^1}},$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{e^{x^1}}{1+2e^{x^1}}, \quad R_{3223} = R_{4224} = R_{4334} = \frac{e^{2x^1}}{1+2e^{x^1}}$$

$$S_{11} = \frac{3e^{x^1}}{(1+2e^{x^1})^2}, \quad S_{22} = S_{33} = S_{44} = \frac{e^{x^1}}{1+2e^{x^1}}$$

$$r = \frac{6e^{x^1}(1+e^{x^1})}{(1+2e^{x^1})^3} \quad (3.2)$$

and the components which can be obtained from these by the symmetry properties. Therefore $\mathbb{R}^4$ with the considered metric is a Riemannian manifold (M^4, g) of non-vanishing scalar curvature.

From (3.2), the only non-vanishing components of the concircularly flat Z-symmetric tensor Z_{ij} are

$$Z_{11} = Z_{22} = Z_{33} = Z_{44} = \frac{3}{2}\frac{e^{x^1}(1+e^{x^1})}{(1+2e^{x^1})^2} + \phi(1+2e^{x^1}). \quad (3.3)$$

To verify the Theorem 2.3, it is sufficient to check the relations. We assume that

$$Z_{ij,k} - Z_{ik,j} = 0 \qquad i = 1,2,3,4 \quad (3.4)$$

where "," denotes the covariant differentiation with respect to the metric tensor g in local coordinates.

From (3.3) and (3.4), we obtain the following relations

$$Z_{11,j} = (1 + 2e^{x^1})\phi_j = 0 \qquad j = 2, 3, 4$$

$$Z_{22,j} = (1 + 2e^{x^1})\phi_j = 0 \qquad j = 3, 4$$

$$Z_{33,j} = (1 + 2e^{x^1})\phi_j = 0 \qquad j = 2, 4$$

$$Z_{44,j} = (1 + 2e^{x^1})\phi_j = 0 \qquad j = 2, 3 \tag{3.5}$$

The relations in (3.5) imply that the scalar function ϕ is independent of the coordinates x^2, x^3, x^4. On the other hand, from (3.3) and (3.4) and for j=1, it follows that

$$Z_{ii,1} - Z_{i1,i} = 0 \qquad i = 2, 3, 4 \tag{3.6}$$

From (3.2),(3.3) and (3.6),we have

$$Z_{22,1} = Z_{33,1} = Z_{44,1} = \frac{3}{2}\Big[\frac{e^{x^1} - 2e^{2x^1} - 2e^{3x^1}}{(1 + 2e^{x^1})^3}\Big] + (1 + 2e^{x^1})\phi_1 = 0 \tag{3.7}$$

where $\phi_1 = \frac{d\phi}{dx^1}$. Then solving the equation (3.7) with respect to x^1, we find

$$\phi = -\frac{3}{2}\frac{e^{x^1}(1 + e^{x^1})}{(1 + 2e^{x^1})^3} + c \tag{3.8}$$

Now putting (3.8) in the equation (3.3), we get $Z_{ii} = (1 + 2e^{x^1})c$ where i=1,2,3,4. If we take the covariant derivative of Z_{ii} then we find $Z_{ii,k} = 0$. Finally, this shows that we can find such a scalar function ϕ satisfies Theorem 2.3.

Thus, the manifold under this consideration is concircularly flat Z-symmetric manifold.

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