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Algorithmic problems for free Poisson algebras

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Abstract It is well known that the ideal membership problem for free Lie algebras is undecidable which implies the undecidability of the ideal membership problem for free Poisson algebras. Using that fact we prove that the subalgebra membership problem is algorithmically undecidable for free Poisson algebras.

Keywords Free Poisson algebras · the ideal membership problem · the subalgebra membership problem

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1 Introduction

Let $K = k[x_1, x_2, \ldots, x_n]$ be the polynomial algebra in the variables x_1, x_2, $\ldots$, x_n over a constructive field k. The subalgebra problem for the polynomial algebra K refers to the algorithmic question of deciding whether or not a given element $f \in K$ belongs to the subalgebra $\langle f_1, \ldots, f_m \rangle$ of K generated by a given set $\{f_1, \ldots, f_m\}$ of elements of K. Gröbner bases [3] gives the decidability of the subalgebra membership problem for K.

The subalgebra membership problem is decidable for free algebras of Nielsen-Schreier varieties of algebras [2] and for free Lie algebras [5,8]. Yet, it is algorithmically undecidable for free associative and free Jordan algebras [6]. Using a method of interpreting [6], we prove the subalgebra membership problem is algorithmically undecidable for free Poisson algebras.

2 Preliminaries

Let k be an arbitrary field of characteristic 0. A vector space P over k endowed with two bilinear operations $x.y$ (a multiplication) and $\{x, y\}$ (a

Poisson bracket) is called Poisson algebra if P is a commutative associative algebra under $x.y$, P is a Lie algebra under $\{x, y\}$, and P satisfies the following identity (the Leibniz identity):

$$\{x, y.z\} = \{x, y\}.z + y.\{x, z\}.$$

A Poisson subalgebra of P is a subalgebra A of P such that $\{a, b\} \in A$ for all $a, b \in A$. A Poisson ideal of P is an ideal I of P such that $\{a, x\} \in I$, for $a \in I$, $x \in P$.

The universal enveloping algebra P^e of P is an associative algebra for which the category of Poisson modules over P and of (left) modules over P^e are equivalent. It can be given by generators and defining relations. Let $m_P = \{m_a : a \in P\}$ and $h_P = \{h_a : a \in P\}$ be two copies of the vector space P endowed with two linear isomorphisms $m : P \to m_P, (a \to m_a)$ and $h : P \to h_P, (a \to h_a)$. Then P^e is an associative algebra over k, with an identity 1, generated by two bilinear m_P and h_P and defined by the relations

$$\begin{gathered} m_{xy} = m_x m_y, \\ h_{\{x,y\}} = h_x h_y - h_y h_x \\ h_{xy} = m_y h_x + m_x h_y \\ m_{\{x,y\}} = h_x m_y - m_y h_x \end{gathered}$$

for all $x, y \in P$,[7]. Additionally we put $m_1 = 1$ since we consider only unitary Poisson modules.

Let an arbitrary vector space V be a Poisson P-module, then V becomes a left P^e- module under the actions

$$m_x v = x.v, h_x v = \{x, v\},$$

for all $x \in P$ and $v \in V$. Conversely, if V is a left P^e-module then the same formulas turn V to a Poisson P-module.

Let g be a Lie algebra with a linear basis $e_1, e_2, \ldots, e_m, \ldots$. The Poisson symmetric algebra $PS(g)$ of g is the usual polynomial algebra $k[e_1, e_2, \ldots, e_m, \ldots]$ endowed with the Poisson bracket defined by

$$\{e_i, e_j\} = [e_i, e_j],$$

for all i, j where $[x, y]$ is the multiplication in the Lie algebra g.

Denote by $P = P\langle x_1, x_2, \ldots, x_n\rangle$ the free Poisson algebra over a field k of characteristic 0 in variables $x_1, x_2, \ldots, x_n$. Let $g = Lie\langle x_1, x_2, \ldots, x_n\rangle$ be the free Lie algebra with free (Lie) generators $x_1, x_2, \ldots, x_n$. The Poisson symmetric algebra $PS(g)$ is the free Poisson algebra $P\langle x_1, x_2, \ldots, x_n\rangle$ in the variables $x_1, x_2, \ldots, x_n$ [4].

Let us choose a multihomogeneous linear basis

$$x_1, \ldots, x_n, [x_1, x_2], \ldots, [x_1, x_n], \ldots, [x_{n-1}, x_n], [[x_1, x_2], x_3], \ldots$$

of a free Lie algebra g and denote the elements of this basis by

$$e_1, e_2, \ldots, e_m, \ldots. \tag{2.1}$$

The algebra $P\langle x_1, x_2, \ldots, x_n\rangle$ coincides with the polynomial algebra on the elements (2.1). Consequently, the set of all words of the form

$$u = e^{\sigma} = e_1^{i_1} e_2^{i_2} \ldots e_m^{i_m}, \tag{2.2}$$

where $0 \leq i_k, 1 \leq k \leq m$ and $m \geq 0$, forms linear basis of P.

By the deg, we denote the Poisson degree function on $P\langle x_1, x_2, \ldots, x_n\rangle$ as the polynomial algebra on the elements (2.1), i.e. $deg(e_i)$ is the Lie degree of this element. Note that

$$deg\{f, g\} = degf + degg,$$

if f and g are homogeneous and $\{f, g\} \neq 0$. By deg_{x_i}, we denote the degree function on P with respect to x_i, i.e. $deg_{x_i}(e_j)$ is the number of appearance of x_i in the Lie monomial e_j. If f is homogenous with respect to each deg_{x_i}, where $1 \leq i \leq n$, then f is called multihomogeneous. The basis (2.2) is multihomogeneous since so is (2.1).

3 The Ideal Membership Problem

Lemma 3.1 *Let L be a free Lie algebra over a field k characteristic zero and $P = PS(L)$ be the free Poisson algebra. Denote an ideal of L by I and an Poisson ideal of P by $(I)_P$. Then $(I)_P \cap L = I$.*

Proof. Let $f_1, ..., f_n, ...$ be a basis of I as a vector space. Let us complete this basis to a basis $f_1, \ldots, f_n, \ldots, e_1, \ldots, e_m, \ldots$ of L. Then

$$P = k[f_1, \ldots, f_n, \ldots, e_1, \ldots, e_m, \ldots].$$

We want to show that $I \subset (I)_P \cap L$ and $(I)_P \cap L \subset I$. Since I is an ideal of L and $(I)_P$ is generated by I, then $I \subset (I)_P \cap L$. Consider an element $f \in (I)_P \cap L$. Since $f \in L$, then

$$f = \textstyle\sum_{i \in I} \alpha_i f_i + \sum_{j \in J} \beta_j e_j,$$

where $\alpha_i, \beta_j \in k$. On the other hand $f \in (I)_P$. Thus $\beta_j = 0$, for all $j \in J$. So we have

$$f = \textstyle\sum_{i \in I} \alpha_i f_i.$$

Moreover,

$$\left[\textstyle\sum_{i \in I} \alpha_i f_i, t\right] = \sum_{i \in I} \alpha_i \left[f_i, t\right],$$

for $\forall t \in L$. Since I is a Lie ideal of L, then for all i, $[f_i, t] \in I$ and so $f \in I$. □

Corollary 3.2 *The ideal membership problem is undecidable for free Poisson algebras.*

Proof. The ideal membership problem for free Lie algebras is undecidable [1]. Then the proof follows by Lemma 3.1.

□

4 The Subalgebra Membership Problem

Let k be an arbitrary field of characteristic 0 and $P = P\langle x_1, x_2, \ldots, x_n\rangle$ be the free Poisson algebra over k in the variables $x_1, \ldots, x_n$. Let $I = (f_1, \ldots, f_r)_P$ be an ideal of P generated by $f_1, \ldots, f_r \in P$. Let $B = P\langle x_1, x_2, \ldots, x_n, t\rangle$ be a free Poisson algebra with one more variable t over the same field. We embed the free Poisson algebra P in B. Therefore each $f_i \in P$ has an image $\overline{f_i} \in B$, where $i = 1, 2, \ldots, r$. For short, we will identify $\overline{f_i}$ with f_i, $i = 1, 2, \ldots, r$. Denote by J, the subalgebra of B generated by

$$x_1, x_2, \ldots, x_n, [x_1, t], \ldots, [x_n, t], tf_1, tf_2, \ldots, tf_r.$$

Proposition 4.1 *Let $f \in P$. Then $f \in I$ if and only if $tf \in J$.*

Proof. Let P^e be the universal enveloping algebra of P. Then every element of P^e is a linear combination of elements of the form

$$pw,$$

where $0 \neq p \in P$ and $w = h_{x_{j_1}} \ldots h_{x_{j_k}}$, [7].

Since any ideal of P is an P^e-submodule of P, if $f \in I$, then

$$f = \textstyle\sum_{p,w,i} \lambda_{p,w,i} pw f_i$$

for some $\lambda_{p,w,i} \in k$, $1 \leq i \leq r$, $p \in P$, $w = h_{x_{j_1}} \ldots h_{x_{j_k}}$. Let T be the subalgebra of J generated by elements

$$x_1, x_2, \ldots, x_n, [x_1, t], \ldots, [x_n, t].$$

Notice that $P \subset T$. Now we must show that

$$w(tf_i) = tw(f_i) + g, g \in T. \tag{4.1}$$

Let $w = h_{x_1} \ldots h_{x_k}, k \leq 1$. By induction on k,

$w(tf_i) = h_{x_1}(tf_i) = [x_1, tf_i] = t[x_1, f_i] + [x_1, t]f_i = t[x_1, f_i] + g = tw(f_i) + g,$

where $g \in T, f_i \in I, 1 \leq i \leq r$. For $k = 2$,

$w(tf_i) = h_{x_1} h_{x_2}(tf_i)$

$= h_{x_1}[x_2, tf_i] = h_{x_1}(t[x_2, f_i] + g) = h_{x_1}(t[x_2, f_i]) + h_{x_1}(g)$

$= h_{x_1}(t)[x_2, f_i] + th_{x_1}[x_2, f_i] + h_{x_1}(g) = th_{x_1} h_{x_2}(f_i) + [x_1, t][x_2, f_i] + [x_1, g]$

$= th_{x_1} h_{x_2}(f_i) + g_1 = tw(f_i) + g_1,$

where $g_1 \in T$. Assume that the equality (4.1) satisfies $w = h_{x_2} \ldots h_{x_{n+1}}$, then

$h_{x_1}(w(tf_i)) = [x_1, w(tf_i)] = [x_1, tw(f_i) + g] = [x_1, tw(f_i)] + [x_1, g]$

$= t[x_1, w(f_i)] + [x_1, t]w(f_i) + [x_1, g] = th_{x_1}(w(f_i)) + g_2,$

where $g_2 \in T$. Consequently,

$$tf = \sum_{p,w,i} \lambda_{p,w,i} tpw(f_i) = \sum_{p,w,i} \lambda_{p,w,i} pw(tf_i) + g, \quad g \in T.$$

Therefore $tf \in J$.

Denote by $\deg_t$, the standard degree function on B such that

$$\deg_t([x_i, t]) = 1 \text{ and } \deg_t([x_i, x_j]) = 0$$

for $1 \leq i < j \leq n$. Denote by H, the subspace of all homogeneous elements of degree 1 of J with respect to $\deg_t$. Then every element of H is a linear combination of elements of the form

$$p[x_i, t], pw(tf_j),$$

where $p \in P, 1 \leq i \leq n, 1 \leq j \leq r$.

Taking into account (4.1), we may assume that every element of H is a linear combination of elements

$$p[x_i, t], ptw(f_j).$$

Moreover, every element of H divisible by t is a linear combination of elements

$$ptw(f_i),$$

where $p \in P, 1 \leq i \leq r$.

Assume that $tf \in J$ for some $f \in P$. We have $tf \in H$ and tf divisible by t. Then

$$tf = \sum_{p,w,i} \lambda_{p,w,i} ptw(f_i).$$

Consequently,

$$f = \sum_{p,w,i} \lambda_{p,w,i} pw(f_i) \in I.$$

□

Theorem 4.2 *The subalgebra membership problem for free Poisson algebras is algorithmically undecidable.*

Proof. Recall that the ideal membership problem for free Lie algebras is undecidable [1]. Using that fact and Lemma 3.1 we get the undecidability of the ideal membership problem for free Poisson algebras. Then the proof follows by Proposition 4.1.

□

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