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Monotonicity of certain maps and Callebaut-type integral inequalities for continuous fields of operators

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Abstract In this paper, we investigate the monotonicity and the convexity of certain maps involving continuous fields of positive invertible operators, the Tracy-Singh product and the Khatri-Rao product. We obtain a number of integral inequalities of Callebaut-type for bounded continuous fields of operators involving certain kinds of operator products and weighted geometric means. Our results extend Callebaut-type inequalities for real numbers, matrices and operators.

Keywords Callebaut inequality · Tracy-Singh product · Khatri-Rao product · Bochner integral · weighted geometric mean · continuous field of operators

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1 Introduction

The famous Cauchy-Schwarz inequality states that

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right) \tag{1.1}$$

for all real numbers a_i's and b_i's. In 1965, this inequality was refined by Callebaut [5] as follows. For each sequence $x{=}(x_1,\ldots,x_k)$ and $y{=}(y_1,\ldots,y_k)$ of positive real numbers and $\alpha\in[0,1]$, let us denote $\mathcal{I}_\alpha^k(x,y)=\sum_{i=1}^k x_i\sharp_\alpha y_i$, here $\sharp_\alpha$ is the α-weighted geometric mean. The Callebaut inequality states that

$$\begin{aligned}\left(\mathcal{I}_{1/2}^k(x,y)\right)^2 &\leqslant \mathcal{I}_\alpha^k(x,y)\cdot\mathcal{I}_{1-\alpha}^k(x,y)\\ &\leqslant \mathcal{I}_\beta^k(x,y)\cdot\mathcal{I}_{1-\beta}^k(x,y)\\ &\leqslant \mathcal{I}_0^k(x,y)\cdot\mathcal{I}_1^k(x,y).\end{aligned} \tag{1.2}$$

for $0 \leqslant \beta \leqslant \alpha \leqslant \frac{1}{2}$ or $\frac{1}{2} \leqslant \alpha \leqslant \beta \leqslant 1$.

Matrix versions of (1.2) involving the tensor product and weighted geometric means were established in [9]. Indeed, for any positive invertible matrices A and B, the convexity of the map $\alpha \mapsto A^\alpha \otimes B^{1-\alpha} + A^{1-\alpha} \otimes B^\alpha$ implies that

$$\begin{aligned} 2\mathcal{I}^k_{1/2}(A,B) \otimes \mathcal{I}^k_{1/2}(A,B) & \\ &\leqslant \mathcal{I}^k_\alpha(A,B) \otimes \mathcal{I}^k_{1-\alpha}(A,B) + \mathcal{I}^k_{1-\alpha}(A,B) \otimes \mathcal{I}^k_\alpha(A,B) \\ &\leqslant \mathcal{I}^k_\beta(A,B) \otimes \mathcal{I}^k_{1-\beta}(A,B) + \mathcal{I}^k_{1-\beta}(A,B) \otimes \mathcal{I}^k_\beta(A,B) \\ &\leqslant \mathcal{I}^k_0(A,B) \otimes \mathcal{I}^k_1(A,B) + \mathcal{I}^k_1(A,B) \otimes \mathcal{I}^k_0(A,B), \end{aligned} \tag{1.3}$$

where $A = (A_1, \ldots, A_k), B = (B_1, \ldots, B_k)$ are sequences of positive invertible matrices and either $0 \leqslant \beta \leqslant \alpha \leqslant \frac{1}{2}$ or $\frac{1}{2} \leqslant \alpha \leqslant \beta \leqslant 1$. As a consequence, they established a Hadamard product version of the Callebaut inequality (1.3). A simple form of (1.2) for positive operators involving the tensor product, the geometric mean and a Kubo-Ando mean was settled in [14]. Some refinements and complementaries of (1.2) for operators involving the Hadamard product and weighted geometric means were presented in [2,3].

In this paper, first we investigate the monotonicity and convexity of certain maps involving continuous fields of positive operators and four kinds of operator products, namely, the Tracy-Singh product, Khatri-Rao product, tensor product and Hadamard product (see more information in Section 2). The methods in [9] inspire our proof techniques. The fields considered here are parametrized by a locally compact Hausdorff space endowed with a finite Radon measure. As consequences, we obtain integral Callecaut-type inequalities concerning the operator products and weighted geometric means. Our results generalize the previous works [2,3,5,9], and include results for Tracy-Singh product, Khatri-Rao product, Kronecker product, and Hadamard product of matrices.

This paper is organized as follows. In Section 2, we give preliminaries on operator products. In Section 3, we study the monotonicity and convexity of certain maps involving Tracy-Singh products, Khatri-Rao products, tensor products and Hadamard products. In order to establish the integral inequalities for continuous fields of operators, some useful operator inequalities were proved in Section 4. In Section 5, we establish some Callebaut-type inequalities involving the Tracy-Singh product, the Khatri-Rao product, and weighted operator geometric means.

2 Preliminaries on operator products

Throughout this paper, let $\mathbb{H}$ be a complex Hilbert space. When $\mathbb{X}$ and $\mathbb{Y}$ are Hilbert spaces, denote by $\mathfrak{B}(\mathbb{X},\mathbb{Y})$ the Banach space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$, and abbreviate $\mathfrak{B}(\mathbb{X},\mathbb{X})$ to $\mathfrak{B}(\mathbb{X})$. For Hermitian

operators A and B in $\mathfrak{B}(\mathbb{X})$, the situation $A \geqslant B$ means that $A - B$ is positive. Denote the set of all positive invertible operators on $\mathbb{X}$ by $\mathfrak{B}(\mathbb{X})^+$.

Recall that tensor product of $A, B \in \mathfrak{B}(\mathbb{H})$ is a unique bounded linear operator from $\mathbb{H} \otimes \mathbb{H}$ into itself such that for all $x, y \in \mathbb{H}$,

$$(A \otimes B)(x \otimes y) \;=\; Ax \otimes By.$$

Fix an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ for $\mathbb{H}$. Recall that the Hadamard product of $A, B \in \mathfrak{B}(\mathbb{H})$ is defined to be the bounded linear operator $A \odot B$ from $\mathbb{H}$ into itself such that for all $i, j \in \mathbb{N}$,

$$\langle (A \odot B)e_i, e_j\rangle \;=\; \langle Ae_i, e_j\rangle\langle Be_i, e_j\rangle.$$

In [7], the Hadamard product is independent on a choice of the orthogonal basis and can be expressed as

$$A \odot B \;=\; U^*(A \otimes B)U, \tag{2.1}$$

where $U : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ is the isometry defined by $Ue_i = e_i \otimes e_i$ for all $i \in \mathbb{N}$. For matrices, the Hadamard product of $A = [a_{ij}]$ and $B = [b_{ij}]$ is given by the entrywise product $A \odot B = [a_{ij}b_{ij}]$ which is a principal submatrix of the tensor product $A \otimes B = [a_{ij}B]_{ij}$.

In order to define the Tracy-Singh product and the Khatri-Rao product of operators, we apply the projection theorem to decompose

$$\mathbb{H} \;=\; \bigoplus_{i=1}^{n} \mathbb{H}_i \tag{2.2}$$

in which all $\mathbb{H}_i$'s are Hilbert spaces. The decomposition is fixed throughout this paper. For each $j \in \{1, \ldots, n\}$, let $E_j : \mathbb{H}_j \to \mathbb{H}$ be the canonical embedding. For each $i \in \{1, \ldots, n\}$, let $P_i : \mathbb{H} \to \mathbb{H}_i$ be the orthogonal projection. Thus, each operator $A \in \mathfrak{B}(\mathbb{H})$ can be expressed uniquely as operator matrix

$$A = [A_{ij}]_{i,j=1}^{n,n}$$

where $A_{ij} = P_iAE_j \in \mathfrak{B}(\mathbb{H}_j, \mathbb{H}_i)$ for each i, j.

Definition 2.1 *Let $A = [A_{ij}]_{i,j=1}^{n,n}, B = [B_{ij}]_{i,j=1}^{n,n}$ be operators in $\mathfrak{B}(\mathbb{H})$ partitioned according to the decomposition* (2.2)*. We define Tracy-Singh product of A and B to be the operator matrix*

$$A \boxtimes B \;=\; \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij}. \tag{2.3}$$

We define Khatri-Rao product of A and B to be the operator matrix

$$A \boxdot B \;=\; [A_{ij} \otimes B_{ij}]_{ij}. \tag{2.4}$$

Consider matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad A_{21} = [7\ 8], \quad A_{22} = [9],$$
$$B_{11} = [1], \quad B_{12} = [4\ 7], \quad B_{21} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 5 & 8 \\ 6 & 9 \end{bmatrix}.$$

The Tracy-Singh product of A and B is

$$A \boxtimes B = \begin{bmatrix} A_{11} \otimes B_{11} & A_{11} \otimes B_{12} & A_{12} \otimes B_{11} & A_{12} \otimes B_{12} \\ A_{11} \otimes B_{21} & A_{11} \otimes B_{22} & A_{12} \otimes B_{21} & A_{12} \otimes B_{22} \\ A_{21} \otimes B_{11} & A_{21} \otimes B_{12} & A_{22} \otimes B_{11} & A_{22} \otimes B_{12} \\ A_{21} \otimes B_{21} & A_{21} \otimes B_{22} & A_{22} \otimes B_{21} & A_{22} \otimes B_{22} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 & 4 & 7 & 8 & 14 & 3 & 12 & 21 \\ 4 & 5 & 16 & 28 & 20 & 35 & 6 & 24 & 42 \\ 2 & 4 & 5 & 8 & 10 & 16 & 6 & 15 & 24 \\ 3 & 6 & 6 & 9 & 12 & 18 & 9 & 18 & 27 \\ 8 & 10 & 20 & 32 & 25 & 40 & 18 & 30 & 48 \\ 12 & 15 & 24 & 36 & 30 & 45 & 18 & 36 & 54 \\ 7 & 8 & 28 & 49 & 32 & 56 & 9 & 36 & 63 \\ 14 & 16 & 35 & 56 & 40 & 64 & 18 & 45 & 72 \\ 21 & 24 & 42 & 63 & 48 & 72 & 27 & 54 & 81 \end{bmatrix}.$$

The Khatri-Rao product of A and B is

$$A \boxdot B = \begin{bmatrix} A_{11} \otimes B_{11} & A_{12} \otimes B_{12} \\ A_{21} \otimes B_{21} & A_{22} \otimes B_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 12 & 21 \\ 4 & 5 & 24 & 42 \\ 14 & 16 & 45 & 72 \\ 21 & 24 & 54 & 81 \end{bmatrix}.$$

Lemma 2.2 [11,12] *Let $A, B, C, D \in \mathfrak{B}(\mathbb{H})$. Then*

1. *The map $(A, B) \mapsto A \boxtimes B$ is bilinear.*
2. *$(A \boxtimes B)(C \boxtimes D) = (AC) \boxtimes (BD)$.*
3. *$(A \boxtimes B)^* = A^* \boxtimes B^*$.*
4. *If $A \geqslant 0$ and $B \geqslant 0$, then $A \boxtimes B \geqslant 0$.*
5. *If $A, B \geqslant 0$, then $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ for any $\alpha \geqslant 0$.*

For each $r = 1, \ldots, n$, consider the operator matrix

$$E_r = \left[E_{ij}^{(r)}\right]_{i,j=1}^{n,n} : \bigoplus_{k=1}^{n} \mathbb{H}_k \otimes \mathbb{H}_k \to \bigoplus_{k=1}^{n} \mathbb{H}_r \otimes \mathbb{H}_k,$$

where $E_{ij}^{(r)}$ is the identity operator if $i = j = r$ and the others are zero operators. Now, construct

$$Z \;=\; \begin{bmatrix} E_1 \\ \vdots \\ E_n \end{bmatrix}. \tag{2.5}$$

Lemma 2.3 [10] *For any $A, B \in \mathfrak{B}(\mathbb{H})$, we have*

$$A \boxdot B \;=\; Z^*(A \boxtimes B)Z$$

where Z is the operator defined by (2.5).

Lemma 2.4 [12,13] *The maps $(A, B) \mapsto A \boxtimes B$ and $(A, B) \mapsto A \boxdot B$ are (jointly) continuous with respect to the operator-norm topology.*

3 Monotonicity of certain maps involving continuous fields of operators

From now on, let Ω be a locally compact Hausdorff space endowed with a finite Radon measure μ. A family $(A_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$ is said to be a continuous field if the parametrization $t \mapsto A_t$ is norm-continuous on Ω and the norm function $t \mapsto \|A_t\|$ is Lebesgue integrable on Ω, then we can form the Bochner integral $\int_\Omega A_t d\mu(t)$ as an element in $\mathfrak{B}(\mathbb{H})$.

In this section, we investigate the monotonicity of certain maps dealing with continuous fields of operators. We start with some auxiliary results.

Lemma 3.1 e.g. [1] *Let $f : \Gamma \to \mathbb{X}$ be a Bochner integrable function. If $\varphi : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator, then the composition $\varphi \circ f$ is Bochner integrable and*

$$\int_\Gamma (\varphi \circ f)d\upsilon \;=\; \varphi\left(\int_\Gamma f d\upsilon\right).$$

Lemma 3.2 *Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})$. Then for any $X \in \mathfrak{B}(\mathbb{H})$,*

$$\left(\int_\Omega A_t d\mu(t)\right) \boxtimes X \;=\; \int_\Omega (A_t \boxtimes X)d\mu(t).$$

Proof. Since the map $t \mapsto A_t$ is continuous and bounded, it is Bochner integrable on Ω. By using Lemma 3.1 and the fact that the map $A \mapsto A \boxtimes X$ is bounded linear operator, we get the result.

□

Lemma 3.3 [6] *For each $A \in \mathfrak{B}(\mathbb{H})^+$, the map $\alpha \mapsto A^\alpha + A^{-\alpha}$ is convex on $\mathbb{R}$, increasing on $[0,\infty)$, decreasing on $(-\infty, 0]$ and has its minimum at $\alpha = 0$.*

Lemma 3.4 *Let $A, B \in \mathcal{B}(\mathbb{H})^+$. Then the map*

$$\phi \ : \ [0,1] \ \to \ \mathfrak{B}\big(\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{n} \mathbb{H}_i \otimes \mathbb{H}_j\big), \ \phi(\alpha) \ = \ A^\alpha \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^\alpha \tag{3.1}$$

is convex on $[0,1]$, decreasing on $[0, \frac{1}{2}]$, increasing on $[\frac{1}{2}, 1]$, and attains its minimum at $\alpha = \frac{1}{2}$.

Proof. Since there is an affine transformation between $[0,1]$ and $[-1,1]$, it suffices to prove that the map

$$\varphi(\alpha) \ = \ A^{1+\alpha} \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^{1+\alpha} \tag{3.2}$$

is convex on $[-1,1]$, decreasing on $[-1,0]$, increasing on $[0,1]$, and attains its minimum at $\alpha = 0$. Let $0 \leqslant \alpha \leqslant \beta \leqslant 1$ with $\beta \neq 0$. In Lemma 3.3, replace α by α/β and A by $A^\beta \boxtimes B^{-\beta}$. Then Lemma 2.2 implies that

$$A^\alpha \boxtimes B^{-\alpha} + A^{-\alpha} \boxtimes B^\alpha \ \leqslant \ A^\beta \boxtimes B^{-\beta} + A^{-\beta} \boxtimes B^\beta. \tag{3.3}$$

Multiply both sides of (3.3) by $A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}$. Then, we have by Lemma 2.2 that

$$A^{1+\alpha} \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^{1+\alpha} \ \leqslant \ A^{1+\beta} \boxtimes B^{1-\beta} + A^{1-\beta} \boxtimes B^{1+\beta}.$$

Thus, φ is increasing on $[0,1]$. Note that $\varphi(\alpha) = \varphi(-\alpha)$, so φ is decreasing on $[-1,0]$. Consequently, ϕ is convex on $[-1,1]$ and has its minimum at $\alpha = 0$.

□

In the next theorem, we generalize Lemma 3.4 to that for an integral map dealing with a continuous field of operators as a weighted function. For each bounded continuous field $\mathcal{X} = (X_t)_{t\in\Omega}$ and $\mathcal{W} = (W_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$, we set

$$\mathcal{T}_{\mathcal{W}}(\mathcal{X}) \ = \ \int_\Omega W_t^* X_t W_t d\mu(t).$$

The operator $\mathcal{T}_{\mathcal{W}}(\mathcal{X})$ is well-defined since the map $t \mapsto W_t^* X_t^\alpha W_t$ is Bochner integrable due to its boundedness and continuity. For any $\alpha \in [0,1]$ and a continuous field $\mathcal{X} = (X_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})^+$, we set $\mathcal{X}^\alpha = (X_t^\alpha)_{t\in\Omega}$. When X_t is the identity operator for all $t \in \Omega$, we write $\mathcal{X}$ to be $\mathcal{I}$.

Theorem 3.5 *Let $\mathcal{X} = (X_t)_{t\in\Omega}$ and $\mathcal{W} = (W_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$ and $\mathfrak{B}(\mathbb{H})$, respectively. Then the map*

$$\begin{aligned}\psi \;:\; [0,1] \;&\to\; \mathfrak{B}\big(\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{n}\mathbb{H}_i\otimes\mathbb{H}_j\big),\\ \psi(\alpha) \;&=\; \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha}),\end{aligned}\tag{3.4}$$

is continuous and convex on $[0,1]$, decreasing on $[0,\frac{1}{2}]$, increasing on $[\frac{1}{2},1]$ and attains its minimum at $\alpha=\frac{1}{2}$.

Proof. Since the map $\alpha\mapsto A^{\alpha}$ is continuous, we have that the map $\alpha\mapsto\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})$ is also continuous. Thus, the map ψ is continuous by the continuity of the Tracy-Singh product (Lemma 2.4). We have by Lemmas 2.2 and 3.2, and Fubini's theorem for Bochner integrals [4] that for any $\alpha\in[0,1]$,

$$\begin{aligned}\psi(\alpha) \;&=\; \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})\\ &=\iint_{\Omega^2}\big(W_t^*X_t^{\alpha}W_t\boxtimes W_s^*X_s^{1-\alpha}W_s + W_t^*X_t^{1-\alpha}W_t\boxtimes W_s^*X_s^{\alpha}W_s\big)d\mu(t)d\mu(s)\\ &=\iint_{\Omega^2}(W_t\boxtimes W_s)^*\big(X_t^{\alpha}\boxtimes X_s^{1-\alpha} + X_t^{1-\alpha}\boxtimes X_s^{\alpha}\big)(W_t\boxtimes W_s)d\mu(t)d\mu(s).\end{aligned}$$

Now, for each $\alpha,\beta\in[0,1]$ and $r\in(0,1)$, we have by Lemmas 2.2 and 3.4 that

$$\begin{aligned}&\psi((1-r)\alpha+r\beta)\\ &=\iint_{\Omega^2}(W_t\boxtimes W_s)^*\Big(X_t^{(1-r)\alpha+r\beta}\boxtimes X_s^{1-((1-r)\alpha+r\beta)}\\ &\quad + X_t^{1-((1-r)\alpha+r\beta)}\boxtimes X_s^{(1-r)\alpha+r\beta)}\Big)(W_t\boxtimes W_s)d\mu(t)d\mu(s)\\ &\leqslant\iint_{\Omega^2}(W_t\boxtimes W_s)^*\Big[(1-r)\left(X_t^{\alpha}\boxtimes X_s^{1-\alpha}+X_t^{1-\alpha}\boxtimes X_s^{\alpha}\right)\\ &\quad + r\left(X_t^{\beta}\boxtimes X_s^{1-\beta}+X_t^{1-\beta}\boxtimes X_s^{\beta}\right)\Big](W_t\boxtimes W_s)d\mu(t)d\mu(s)\\ &=(1-r)\iint_{\Omega^2}(W_t\boxtimes W_s)^*\big(X_t^{\alpha}\boxtimes X_s^{1-\alpha}+X_t^{1-\alpha}\boxtimes X_s^{\alpha}\big)(W_t\boxtimes W_s)d\mu(t)d\mu(s)\\ &\quad + r\iint_{\Omega^2}(W_t\boxtimes W_s)^*\Big(X_t^{\beta}\boxtimes X_s^{1-\beta}+X_t^{1-\beta}\boxtimes X_s^{\beta}\Big)(W_t\boxtimes W_s)d\mu(t)d\mu(s)\\ &=(1-r)\psi(\alpha)+r\psi(\beta).\end{aligned}$$

Therefore, ψ is convex on $[0,1]$. For $0 \leqslant \alpha \leqslant \beta \leqslant \frac{1}{2}$, we have

$$\begin{aligned}\psi(\alpha) &= \iint_{\Omega^2} (W_t \boxtimes W_s)^* \left(X_t^\alpha \boxtimes X_s^{1-\alpha} + X_t^{1-\alpha} \boxtimes A_s^\alpha\right)(W_t \boxtimes W_s) d\mu(t) d\mu(s) \\ &\geqslant \iint_{\Omega^2} (W_t \boxtimes W_s)^* \Big(X_t^\beta \boxtimes X_s^{1-\beta} + X_t^{1-\beta} \boxtimes X_s^\beta\Big)(W_t \boxtimes W_s) d\mu(t) d\mu(s) \\ &= \psi(\beta).\end{aligned}$$

Thus, ψ is decreasing on $[0, \frac{1}{2}]$. Similarly, ψ is increasing on $[\frac{1}{2}, 1]$.

□

Corollary 3.6 *Let $\mathcal{X} = (X_t)_{t\in\Omega}$ a be bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})^+$ and $w : \Omega \to [0,\infty)$ a bounded continuous function. Then the map*

$$\alpha \mapsto \int w(t) X_t^\alpha d\mu(t) \boxtimes \int w(t) X_t^{1-\alpha} d\mu(t) + \int w(t) X_t^{1-\alpha} d\mu(t) \boxtimes \int w(t) X_t^\alpha d\mu(t)$$

is convex on $[0,1]$, decreasing on $[0, \frac{1}{2}]$, increasing on $[\frac{1}{2}, 1]$ and attains its minimum at $\alpha = \frac{1}{2}$. Here, all integrals are taken over Ω.

Proof. From Theorem 3.5, set $W_t = \sqrt{w_t} I$ where I is the identity operator.

□

Corollary 3.7 *Let $\mathcal{X} = (X_t)_{t\in\Omega}$ and $\mathcal{W} = (W_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$ and $\mathfrak{B}(\mathbb{H})$, respectively. Then the map*

$$\begin{aligned}\tau &: [0,1] \to \mathfrak{B}\Big(\bigoplus_{i=1}^n \mathbb{H}_i \otimes \mathbb{H}_i\Big), \\ \tau(\alpha) &= \mathcal{T}_{\mathcal{W}}(\mathcal{X}^\alpha) \boxdot \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) \boxdot \mathcal{T}_{\mathcal{W}}(\mathcal{X}^\alpha),\end{aligned}$$

is continuous and convex on $[0,1]$, decreasing on $[0, \frac{1}{2}]$, increasing on $[\frac{1}{2}, 1]$ and attains its minimum at $\alpha = \frac{1}{2}$.

Proof. We have by the continuity of the Khatri-Rao product (Lemma 2.4) that the map τ is continuous. Let Z be the isometry in Lemma 2.3. Then the map

$$F : \mathfrak{B}\Big(\bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathbb{H}_i \otimes \mathbb{H}_j\Big) \to \mathfrak{B}\Big(\bigoplus_{i=1}^n \mathbb{H}_i \otimes \mathbb{H}_i\Big), \quad X \mapsto Z^* X Z$$

is linear and order-preserving. Recall that the composition between a linear map and a convex map results in a convex map. Using the properties of the map ψ in Theorem 3.5, we observe that the map

$$\tau(\alpha) = (F \circ \psi)(\alpha) = \mathcal{T}_{\mathcal{W}}(\mathcal{X}^\alpha) \boxdot \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) \boxdot \mathcal{T}_{\mathcal{W}}(\mathcal{X}^\alpha)$$

is convex on $[0,1]$, decreasing on $[0,\frac{1}{2}]$, increasing on $[\frac{1}{2},1]$. It follows from the monotonicity of the map F that $\tau(\frac{1}{2}) = F(\psi(\frac{1}{2})) \leqslant F(\psi(\alpha)) = \tau(\alpha)$ for all $\alpha \in [0,1]$. Thus τ attains its minimum at $\alpha = \frac{1}{2}$.

□

Corollary 3.8 *Let $\mathcal{X} = (X_t)_{t\in\Omega}$ and $\mathcal{W} = (W_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$ and $\mathfrak{B}(\mathbb{H})$, respectively. Then the maps*

$$\alpha \mapsto \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha}) \otimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) \otimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha}), \tag{3.5}$$

$$\alpha \mapsto \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha}) \odot \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}), \tag{3.6}$$

are convex on $[0,1]$, decreasing on $[0,\frac{1}{2}]$, increasing on $[\frac{1}{2},1]$ and attain its minima at $\alpha = \frac{1}{2}$.

Proof. The map (3.5) is a special case of the Tracy-Singh product map in Theorem 3.5. We get the result for Hadamard products by using the map (3.5) and the commutative property of Hadamard product, and the fact that the map $X \mapsto U^*XU$ is a positive linear map.

□

4 Some auxiliary operator inequalities

From Lemma 3.4, for either $0 \leqslant \beta \leqslant \alpha \leqslant \frac{1}{2}$ or $\frac{1}{2} \leqslant \alpha \leqslant \beta \leqslant 1$, we have that

$$A^{\alpha} \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^{\alpha} \leqslant A^{\beta} \boxtimes B^{1-\beta} + A^{1-\beta} \boxtimes B^{\beta}. \tag{4.1}$$

In this section, we provide some refinements of (4.1). These results will be used in the next section.

Lemma 4.1 *Let $x, y \geqslant 0$ and $r \in [0,1]$. Let $p = \min\{r, 1-r\}$. Then*

$$x^r y^{1-r} + x^{1-r} y^r + 2p\left(\sqrt{x} - \sqrt{y}\right)^2 \leqslant x + y. \tag{4.2}$$

Proof. From [8], we have the inequalities

$$x^{1-r}y^r + p\left(\sqrt{x} - \sqrt{y}\right)^2 \leqslant (1-r)x + ry,$$
$$y^{1-r}x^r + p\left(\sqrt{y} - \sqrt{x}\right)^2 \leqslant (1-r)y + rx.$$

Summing these inequalities, we get the desired result.

□

The next result is a refinement of (4.1).

Proposition 4.2 *Let $A, B \in \mathfrak{B}(\mathbb{H})^+$ and either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$. Denote $\gamma = \frac{\beta-\alpha}{\beta-1/2}$. Then*

$$A^\alpha \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^\alpha + \gamma\big(A^\beta \boxtimes B^{1-\beta} + A^{1-\beta} \boxtimes B^\beta - 2A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\big) \leqslant A^\beta \boxtimes B^{1-\beta} + A^{1-\beta} \boxtimes B^\beta. \tag{4.3}$$

Proof. In Lemma 4.1, replacing y by x^{-1} and putting $r = \frac{1-u}{2}$, we have

$$x^u + x^{-u} + (1-u)\left(x + x^{-1} - 2\right) \leqslant x + x^{-1}.$$

Consider $v, w \in \mathbb{R}$ such that $u := \frac{v}{w} \in (0, 1]$. Applying the functional calculus on the spectrum of $A \boxtimes B$ and Lemma 2.2, we get

$$A^v \boxtimes B^{-v} + A^{-v} \boxtimes B^v + \left(1 - \frac{v}{w}\right)\left(A^w \boxtimes B^{-w} + A^{-w} \boxtimes B^w - 2I \boxtimes I\right) \leqslant A^w \boxtimes B^{-w} + A^{-w} \boxtimes B^w. \tag{4.4}$$

Multiplying both sides of (4.4) by $A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}$, we obtain

$$\begin{aligned} A^{1+w} \boxtimes B^{1-w} + A^{1-w} \boxtimes B^{1+w} &\geqslant A^{1+v} \boxtimes B^{1-v} + A^{1-v} \boxtimes B^{1+v} + \left(1 - \frac{v}{w}\right)\big(A^{1+w} \boxtimes B^{1-w} \\ &\quad + A^{1-w} \boxtimes B^{1+w} - 2A \boxtimes B\big). \end{aligned}$$

To get the desired result, simply replace v, w, A, B by $2\alpha - 1, 2\beta - 1, A^{\frac{1}{2}}, B^{\frac{1}{2}}$, respectively.

□

Lemma 4.3 **[2]** *Let $x > 0$ and $r \geqslant 1$. Then*

$$x + x^{-1} + (r-1)(x + x^{-1} - 2) \leqslant x^r + x^{-r}. \tag{4.5}$$

The next result is another refinement of (4.1).

Proposition 4.4 *Let $A, B \in \mathfrak{B}(\mathbb{H})^+$ and either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$. Denote $\delta = \frac{\beta-\alpha}{\alpha-1/2}$. Then*

$$A^\alpha \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^\alpha + \delta\Big(A^\alpha \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^\alpha - 2A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\Big) \leqslant A^\beta \boxtimes B^{1-\beta} + A^{1-\beta} \boxtimes B^\beta. \tag{4.6}$$

Proof. Consider positive real numbers v, w such that $v \leqslant w$. From Lemma 4.3, applying the functional calculus on the spectrum of $A \boxtimes B$ and taking $r = \frac{w}{v}$, we get by Lemma 2.2 that

$$A^v \boxtimes B^{-v} + A^{-v} \boxtimes B^v + \left(\frac{w-v}{v}\right)\left(A^v \boxtimes B^{-v} + A^{-v} \boxtimes B^v - 2I \boxtimes I\right) \leqslant A^w \boxtimes B^{-w} + A^{-w} \boxtimes B^w. \tag{4.7}$$

Multiply both sides of (4.7) by $A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}$. Finally, replace v, w, A, B by $2\alpha - 1$, $2\beta - 1$, $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, respectively.

□

Corollary 4.5 *The results in Propositions 4.2 and 4.4 still hold if we replace the Tracy-Singh products by the Khatri-Rao products or the tensor products or the Hadamard products.*

5 Callebaut-type integral inequalities

In this section, we establish Callebaut-type integral inequalities involving certain kinds of operator products and weighted geometric means.

Theorem 5.1 *Let $\mathcal{X} = (X_t)_{t\in\Omega}$ and $\mathcal{W} = (W_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$ and $\mathfrak{B}(\mathbb{H})$, respectively. Assume that $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$. Denote $\gamma = \frac{\beta-\alpha}{\beta-1/2}$. Then*

$$\begin{aligned}
&2\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \\
&\quad\leqslant \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha}) \\
&\quad\leqslant \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) + \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha}) \\
&\qquad+ \gamma\big[\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta}) \\
&\qquad- 2\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\big] \\
&\quad\leqslant \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) + \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta}) \\
&\quad\leqslant \mathcal{T}_{\mathcal{W}}(\mathcal{X}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{I}) + \mathcal{T}_{\mathcal{W}}(\mathcal{I}) \boxtimes \mathcal{T}_{\mathcal{W}}(\mathcal{X}).
\end{aligned} \tag{5.1}$$

Proof. From Theorem 3.5, using the fact that $\psi(\frac{1}{2}) \leqslant \psi(\alpha) \leqslant \psi(\beta) \leqslant \psi(1)$ for $\frac{1}{2} \leqslant \alpha \leqslant \beta \leqslant 1$, or $\psi(\frac{1}{2}) \leqslant \psi(\alpha) \leqslant \psi(\beta) \leqslant \psi(0)$ for $\frac{1}{2} \geqslant \alpha \geqslant \beta \geqslant 0$, we get the first and forth inequalities. Since $\gamma \geqslant 0$, we have that $\gamma[\psi(\beta) - \psi(\frac{1}{2})]$ is a positive operator. This implies that the second inequality holds. Applying

Lemmas 2.2 and 3.2, Proposition 4.2, and Fubini's theorem, we get

$$\begin{aligned}
&\mathcal{T}_\alpha(\mathcal{X})\boxtimes\mathcal{T}_{1-\alpha}(\mathcal{X})+\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})\\
&+\gamma\Big[\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta})-2\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\Big]\\
&= \iint_{\Omega^2}\Big[W_t^*X_t^\alpha W_t\boxtimes W_s^*X_s^{1-\alpha}W_s+W_t^*X_t^{1-\alpha}W_t\boxtimes W_s^*X_s^\alpha W_s\\
&\quad+\gamma\Big(W_t^*X_t^\beta W_t\boxtimes W_s^*X_s^{1-\beta}W_s+W_t^*X_t^{1-\beta}W_t\boxtimes W_s^*X_s^\beta W_s\\
&\quad-2W_t^*X_t^{\frac{1}{2}}W_t\boxtimes W_s^*X_s^{\frac{1}{2}}W_s\Big)\Big]d\mu(t)\mu(s)\\
&= \iint_{\Omega^2}(W_t\boxtimes W_s)^*\Big[X_t^\alpha\boxtimes X_s^{1-\alpha}+X_t^{1-\alpha}\boxtimes X_s^\alpha\\
&\quad+\gamma\Big(X_t^\beta\boxtimes X_s^{1-\beta}+X_t^{1-\beta}\boxtimes X_s^\beta-2X_t^{\frac{1}{2}}\boxtimes X_s^{\frac{1}{2}}\Big)\Big](W_t\boxtimes W_s)\;d\mu(t)\mu(s)\\
&\leqslant \iint_{\Omega^2}(W_t\boxtimes W_s)^*\left(X_t^\beta\boxtimes X_s^{1-\beta}+X_t^{1-\beta}\boxtimes X_s^\beta\right)(W_t\boxtimes W_s)\;d\mu(t)\mu(s)\\
&= \iint_{\Omega^2}\Big(W_t^*X_t^\beta W_t\boxtimes W_s^*X_s^{1-\beta}W_s+W_t^*X_t^{1-\beta}W_t\boxtimes W_s^*X_s^\beta W_s\Big)d\mu(t)\mu(s)\\
&= \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta}).
\end{aligned}$$

□

Recall that for any $\alpha\in[0,1]$, the α-weighted geometric mean of $A,B\in\mathfrak{B}(\mathbb{H})^+$ is defined by

$$A\sharp_\alpha B \;=\; A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}}.$$

For two bounded continuous fields $\mathcal{A}=(A_t)_{t\in\Omega},\mathcal{B}=(B_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})^+$ and $\alpha\in[0,1]$, we set

$$\mathcal{I}_\alpha(\mathcal{A},\mathcal{B}) \;=\; \int_\Omega A_t\sharp_\alpha B_t d\mu(t). \tag{5.2}$$

In particular, $\mathcal{I}_0(\mathcal{A},\mathcal{B})=\int_\Omega A_t d\mu(t)$ and $\mathcal{I}_1(\mathcal{A},\mathcal{B})=\int_\Omega B_t d\mu(t)$. The operator $\mathcal{I}_\alpha(\mathcal{A},\mathcal{B})$ is well-defined since for any $t\in\Omega$, we have

$$\|A_t\sharp_\alpha B_t\| \;\leqslant\; \|A_t\|\sharp_\alpha\|B_t\|.$$

The next result is a continuous version of the Callebaut inequality (1.2) for operators concerning weighted geometric mean and the Tracy-Singh product.

Corollary 5.2 *Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$. If either $0 \leqslant \beta \leqslant \alpha < \frac{1}{2}$ or $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, then*

$$\begin{aligned}
&2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \\
&\quad\leqslant \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \\
&\quad\leqslant \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \\
&\qquad+ \gamma\big[\mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \\
&\qquad- 2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\big] \\
&\quad\leqslant \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}) \\
&\quad\leqslant \mathcal{I}_{0}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{1}(\mathcal{A},\mathcal{B}) + \mathcal{I}_{1}(\mathcal{A},\mathcal{B}) \boxtimes \mathcal{I}_{0}(\mathcal{A},\mathcal{B}),
\end{aligned} \tag{5.3}$$

where $\gamma = \frac{\beta-\alpha}{\beta-1/2}$.

Proof. When we set $X_t = A_t^{-\frac{1}{2}} B_t A_t^{-\frac{1}{2}}$ and $W_t = A_t^{\frac{1}{2}}$ for all $t \in \Omega$, we have for any $\alpha \in [0,1]$ that

$$\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha}) = \int_{\Omega} A^{\frac{1}{2}} \big(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\big)^{\alpha} A^{\frac{1}{2}} d\mu(t) = \int_{\Omega} A_t \sharp_{\alpha} B_t d\mu(t) = \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B})$$

Using Theorem 5.1, we reach the result.

□

Corollary 5.3 *The results in Theorem 5.1 and Corollary 5.2 are still valid if we replace the Tracy-Singh products by the Khatri-Rao products or the tensor products or the Hadamard products.*

Remark 5.1 For the discrete space $\Omega = \{1,\ldots,k\}$ equipped with the counting measure, the integral (5.2) reduces to the finite sum

$$\mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B}) \;=\; \sum_{i=1}^{k} A_i \sharp_{\alpha} B_i.$$

From previous corollaries, we obtain discrete versions of Callebaut-type inequalities involving Tracy-Singh products, Khatri-Rao products, tensor products and Hadamard products, respectively. For a particular case when $\mathbb{H} = \mathbb{C}^n$, we get a matrix versions of Callebaut-type inequality concerning Tracy-Singh products, Khatri-Rao products, Kronecker products and Hadamard products. Such matrix results are refinements of matrix Callebaut inequalities in the literature, including [9, Theorem 3.4 and Corollary 3.5].

Remark 5.2 In the previous corollaries, if we put $A_t = x_t I$ and $B_t = y_t I$ for all $t \in \Omega$ where x_t and y_t are positive real numbers, then we obtain a continuous version of the refined Callebaut-type inequality for real numbers.

We mention that if Ω is the finite set $\{1,\dots,k\}$ equipped with the counting measure, we get a discrete version of this integral Callebaut-type inequality which is a refinement of the classical Callebaut inequality (1.2).

Theorem 5.4 *Let $\mathcal{X}=(X_t)_{t\in\Omega}$ and $\mathcal{W}=(W_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$ and $\mathfrak{B}(\mathbb{H})$, respectively. If either $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ or $\frac{1}{2}<\alpha\leqslant\beta\leqslant 1$, then*

$$\begin{aligned}
&\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})+\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})\\
&\quad\leqslant\ \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})+\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})\\
&\qquad+\delta\big[\mathcal{T}_{\alpha}(\mathcal{X})\boxtimes\mathcal{T}_{1-\alpha}(\mathcal{X})+\mathcal{T}_{1-\alpha}(\mathcal{X})\boxtimes\mathcal{T}_{\alpha}(\mathcal{X})\\
&\qquad-2\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\big]\\
&\quad\leqslant\ \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta}),
\end{aligned}\tag{5.4}$$

where $\delta=\frac{\beta-\alpha}{\alpha-1/2}$.

Proof. By using Theorem 3.5 and the fact that $\delta\geqslant 0$, we have $\delta[\psi(\alpha)-\psi(\frac{1}{2})]$ is positive. This implies that the first inequality holds. We have by applying Lemmas 2.2 and 3.2, Proposition 4.4, and Fubini's theorem that

$$\begin{aligned}
&\mathcal{T}_{\alpha}(\mathcal{X})\boxtimes\mathcal{T}_{1-\alpha}(\mathcal{X})+\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})\\
&+\delta\Big[\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})+\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\alpha})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\alpha})-2\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\frac{1}{2}})\Big]\\
&=\ \iint_{\Omega^2}(W_t\boxtimes W_s)^*\Big[X_t^{\alpha}\boxtimes X_s^{1-\alpha}+X_t^{1-\alpha}\boxtimes X_s^{\alpha}\\
&\quad+\delta\Big(X_t^{\alpha}\boxtimes X_s^{1-\alpha}+X_t^{1-\alpha}\boxtimes X_s^{\alpha}-2X_t^{\frac{1}{2}}\boxtimes X_s^{\frac{1}{2}}\Big)\Big](W_t\boxtimes W_s)\ d\mu(t)\mu(s)\\
&\leqslant\ \iint_{\Omega^2}(W_t\boxtimes W_s)^*\left(X_t^{\beta}\boxtimes X_s^{1-\beta}+X_t^{1-\beta}\boxtimes X_s^{\beta}\right)(W_t\boxtimes W_s)\ d\mu(t)\mu(s)\\
&=\ \mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta})+\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{1-\beta})\boxtimes\mathcal{T}_{\mathcal{W}}(\mathcal{X}^{\beta}).
\end{aligned}$$

□

The following theorem is another Callebaut integral inequality and its refinement involving the Tracy-Singh product of operators.

Corollary 5.5 *Let $\mathcal{A}=(A_t)_{t\in\Omega}$ and $\mathcal{B}=(B_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$. If either $0\leqslant\beta\leqslant\alpha<\frac{1}{2}$ or $\frac{1}{2}<\alpha\leqslant\beta\leqslant 1$, then*

$$\begin{aligned}
&\mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B})\\
&\quad\leqslant\ \mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B})\\
&\qquad+\delta\big[\mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\alpha}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{\alpha}(\mathcal{A},\mathcal{B})\\
&\qquad-2\mathcal{I}_{1/2}(\mathcal{A},\mathcal{B})\boxtimes I_{1/2}(\mathcal{A},\mathcal{B})\big]\\
&\quad\leqslant\ \mathcal{I}_{\beta}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})+\mathcal{I}_{1-\beta}(\mathcal{A},\mathcal{B})\boxtimes\mathcal{I}_{\beta}(\mathcal{A},\mathcal{B}),
\end{aligned}\tag{5.5}$$

where $\delta = \frac{\beta-\alpha}{\alpha-1/2}$.

Corollary 5.6 *We can replace the Tracy-Singh products in* (5.5) *by the following products:*

(i) the Khatri-Rao products,
(ii) the tensor products,
(iii) the Hadamard products.

Remark 5.3 Discrete versions of Corollaries 5.5 and 5.6 are obtained when we set Ω to be a finite space with the counting measure. Thus, Corollary 5.6 includes the Callebaut-type inequality for Hadamard products in [2, Theorem 3.3]. In particular, when $\mathbb{H} = \mathbb{C}^n$, we get matrix versions involving Tracy-Singh products, Khatri-Rao products, Kronecker products and Hadamard products, including [9]. Moreover, if we put each operator to be a scalar multiple of the identity, we get a refinement of the classical Callebaut inequality (1.2).

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