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Explicit non-algebraic limit cycle of a family of polynomial differential systems of degree even

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Abstract Up to now all the examples of polynomial differential systems for which non-algebraic limit cycles are known explicitly have degree odd. Here we show that already there are polynomial differential systems of degree even exhibiting explicit non-algebraic limit cycles.

Keywords planar polynomial differential system · first integral · non-algebraic limit cycle

Mathematics Subject Classification (2010) 34A05 · 34C05 · 34CO7 · 34C25

1 Introduction

A polynomial differential system is a system of the form

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = P(x,y), \\ \dot{y} &= \frac{dy}{dt} = Q\left(x,y\right), \end{aligned} \tag{1.1}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials in the variables x and y. The degree of the system is the maximum of the degrees of the polynomials P and Q.

A limit cycle of system (1.1) is an isolated periodic solution in the set of all periodic solutions of system (1.1). If a limit cycle is contained in an algebraic curve of the plane, then we say that it is algebraic, otherwise it is called non–algebraic.

In the qualitative theory of planar dynamical systems see [[3], [9], [15], [10]], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem. There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their

stability and rare are papers concerned by giving them explicitly see ([4], [5], [6], [11], [12]).

Until recently, the only limit cycles known in an explicit way were algebraic (see for instance [1], [4], [5], [8], [13] and references therein). On the other hand, it seems intuitively clear that "most" limit cycles of planar polynomial vector fields have to be non algebraic. Nevertheless, until 1995 it was not proved that the limit cycle of the van der Pol equation is not algebraic, see K. Odani [14]. The van der Pol system can be written as a polynomial differential system (1.1) of degree 3, but its limit cycle is not known explicitly. Until recently, the only limit cycles known in an explicit way were algebraic. In the chronological order the first examples were explicit non-algebraic limit cycles appeared are those of A. Gasull and all [11], and J. Giné and M. Grau [12] and by Al-Dosary, Khalil I. T. [2] for $n = 5$. In [7], an example of an explicit limit cycle which is not algebraic is given for $n = 3$.

In this paper, we consider the family of the polynomial differential system of the form

$$\begin{aligned}\dot{x}=&n((\alpha x-y)(wx+vy+l)+vx^2-y(wx+l))((a+b)x^2+(a-b)y^2+2cxy)^n\\&+x(wx+vy+l)^{n+1},\\\dot{y}=&n((x+\alpha y)(wx+vy+l)-wy^2+x(vy+l))((a+b)x^2+(a-b)y^2+2cxy)^n\\&+y(wx+vy+l)^{n+1},\end{aligned}\tag{1.2}$$

where a, b, c, w, v, l and α are real constants and n is strictly positive integer ($n \in \mathbb{N}^*$). We determine sufficient conditions for a polynomial differential system to possess an explicit non-algebraic limit cycles. Concrete examples exhibiting the applicability of our result are introduced.

2 Main result

Our main result is contained in the following theorem.

Theorem 2.1 *If $\alpha < 0$, $w \geq 0$, $l > 0, a > |c| + |b|$, then system (1.2) has an explicit non algebraic limit cycle, given in polar coordinates (r, θ) by*

$$r(\theta) = \frac{1}{2}\left(g(\theta)\rho(\theta, r_*) + \sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta, r_*)\right)^2 + 4l\rho(\theta, r_*)}\right),$$

where $\rho(\theta, r_*) = e^{\alpha n\theta}\left(\dfrac{r_*^{2n}}{(wr_*+l)^n} + f(\theta)\right)$, $f(\theta) = \int_0^\theta \frac{e^{-\alpha ns}}{(a+c\sin 2s+b\cos 2s)^n}\,ds$, $g(\theta) = w\cos\theta + v\sin\theta$ *and*

$$r_* = \frac{1}{2}\left(\sqrt{(we^{2\pi\alpha}(\tfrac{-f(2\pi)}{e^{2\pi n\alpha}-1})^{\frac{1}{n}})^2 + 4le^{2\pi\alpha}(\tfrac{-f(2\pi)}{e^{2\pi n\alpha}-1})^{\frac{1}{n}}} + we^{2\pi\alpha}(\tfrac{-f(2\pi)}{e^{2\pi n\alpha}-1})^{\frac{1}{n}}\right).$$

Moreover, this limit cycle is a stable hyperbolic limit cycle.

Proof. Firstly, we have

$$x\dot{y} - y\dot{x} = n\left(x^2+y^2\right)(vy+wx+2l)\left((a+b)\,x^2+(a-b)\,y^2+2cxy\right)^n,$$

thus, since $a > |c| + |b|$ then, the equilibrium points of system (1.2) are present in the straight line

$$vy + wx + 2l = 0 \tag{2.1}$$

and the origin of coordinates which is an unstable node because its eigenvalues are $l^{n+1} > 0$ with multiplicity two, for more details see for instance [[10], Theorem 2.15].

In polar coordinates system (1.2) reads as

$$\begin{aligned}\dot{r} &= n\,(a+b\cos 2\theta+c\sin 2\theta)^n\,(((v+w\alpha)\cos\theta+(v\alpha-w)\sin\theta)\,r+\alpha l)\,r^{2n+1}\\ &\quad+((w\cos\theta+v\sin\theta)\,r+l)^{n+1}\,r\\ \dot{\theta} &= n\,(a+b\cos 2\theta+c\sin 2\theta)^n\,((w\cos\theta+v\sin\theta)\,r+2l)\,r^{2n}.\end{aligned} \tag{2.2}$$

Taking as new independent variable the coordinate θ, this differential system reads

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{(a+b\cos 2\theta+c\sin 2\theta)^n(((v+w\alpha)\cos\theta+(v\alpha-w)\sin\theta)\,r+\alpha l)\,r^{2n+1}}{(a+b\cos 2\theta+c\sin 2\theta)^n\,((w\cos\theta+v\sin\theta)\,r+2l)\,r^{2n}}\\ &\quad+\frac{((w\cos\theta+v\sin\theta)\,r+l)^{n+1}\,r}{n\,(a+b\cos 2\theta+c\sin 2\theta)^n\,((w\cos\theta+v\sin\theta)\,r+2l)\,r^{2n}}.\end{aligned} \tag{2.3}$$

Via the change of variables $\rho = \dfrac{r^{2n}}{((w\cos\theta+v\sin\theta)\,r+l)^n}$, then

$$\frac{d\rho}{d\theta} = \frac{n\,((w\cos\theta+v\sin\theta)\,r+2l)\,r^{2n-1}}{((w\cos\theta+v\sin\theta)\,r+l)^{n+1}}\frac{dr}{d\theta} - \frac{n\,(v\cos\theta-w\sin\theta)\,r^{2n+1}}{((w\cos\theta+v\sin\theta)\,r+l)^{n+1}}$$

and the equation (2.3) is transformed into the linear equation

$$\frac{d\rho}{d\theta} = n\alpha\rho + \frac{1}{(a+c\sin 2\theta+b\cos 2\theta)^n}. \tag{2.4}$$

The general solution of linear equation (2.3) is

$$\rho\,(\theta,k) = e^{\alpha n\theta}\,(k+f\,(\theta)),$$

where $k \in \mathbb{R}$, $f(\theta)$ is the function defined in the statement of Theorem 2.1. Consequently, the general solution of (2.3) is

$$F\,(r,\theta) = \frac{r^{2n}}{(rg\,(\theta)+l)^n} - e^{\alpha n\theta}\,(k+f\,(\theta)) = 0, \tag{2.5}$$

where $k \in \mathbb{R}$, $f(\theta)$ and $g(\theta)$ are the functions defined in the statement of Theorem 2.1. Going back through the changes of variables we obtain the first integral

$$H(x,y) = (\frac{x^2+y^2}{wx+vy+l})^n e^{-\alpha n \arctan \frac{y}{x}} - \int_0^{\arctan \frac{y}{x}} \frac{e^{-\alpha n s}}{(a+c\sin 2s + b\cos 2s)^n}\, ds.$$

From solution (2.5) we can obtain a two different values of r: one of them is equal to $r_1(\theta,k) = \frac{1}{2}\left(g(\theta)\rho^{\frac{1}{n}}(\theta,k) - \sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta,k)\right)^2 + 4l\rho^{\frac{1}{n}}(\theta,k)}\right)$ and we do not consider this case (because since $-(|w|+|v|) < g(\theta) < |w|+|v|$, then $r_1(\theta)$ can not be strictly positive) and the second value is

$$r(\theta,k) = \frac{1}{2}\left(g(\theta)\rho^{\frac{1}{n}}(\theta,k) + \sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta,k)\right)^2 + 4l\rho^{\frac{1}{n}}(\theta,k)}\right),$$

where $\rho(\theta,k) = e^{\alpha n\theta}(k+f(\theta))$.

Notice that system (1.2) has a periodic orbit if and only if equation (2.3) has a strictly positive 2π periodic solution. This, moreover, is equivalent to the existence of a solution of (2.3) that satisfies $r(0,r_*) = r(2\pi,r_*)$ and $r(\theta,r_*) > 0$ for any θ in $[0,2\pi]$.

To go a step further, we remark that the solution such as $r(0,r_0) = r_0 > 0$, corresponds to the value

$$k = \frac{r^{2n}}{(l+wr)^n}.$$

The solution $r(\theta,r_0)$ of the differential equation (2.3) such that $r(0,r_0) = r_0$ is

$$r(\theta,r_0) = \frac{1}{2}\left(g(\theta)\rho^{\frac{1}{n}}(\theta,r_0) + \sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta,r_0)\right)^2 + 4l\rho^{\frac{1}{n}}(\theta,r_0)}\right),$$

where $\rho(\theta,r_0) = e^{\alpha n\theta}\left(\frac{r_0^{2n}}{(l+wr_0)^n} + f(\theta)\right)$.

A periodic solution of system (1.2) must satisfy the condition $r(2\pi,r_0) = r(0,r_0)$. Since $g(0) = g(2\pi) = w$, then $r(2\pi,r_0) = r(0,r_0)$ if and only if $\rho(2\pi,r_0) = \rho(0,r_0)$ and there are two different values with the property $r(2\pi,r_0) = r_0$, so one of them is equal to

$$-\frac{1}{2}\left(\sqrt{\left(we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right)^2 + 4le^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}} - we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right)$$

and we do not consider this case because since $a > |c| + |b|$ then $\frac{e^{-\alpha n s}}{(a+c\sin 2\theta+b\cos 2\theta)^n} > 0$ for all $\theta \in \mathbb{R}$ thus $f(\theta) = \int_0^\theta \frac{e^{-\alpha n s}}{(a+c\sin 2s+b\cos 2s)^n}\, ds > 0$ for all $\theta \in \mathbb{R}$. Since $\alpha < 0, l > 0, w \geq 0$, it follows that $\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}} > 0$ and

$$\sqrt{\left(we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right)^2 + 4le^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}} > we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}},$$

thus

$$-\frac{1}{2}\left(\sqrt{\left(we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right)^2 + 4le^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}} - we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right) < 0.$$

We only take into consideration the following value r_* which satisfies $r(2\pi, r_*) = r_* > 0$

$$r_* = \frac{1}{2}\left(\sqrt{\left(we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right)^2 + 4le^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}} + we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right).$$

After the substitution of these value r_* into $r(\theta, r_0)$ we obtain

$$r(\theta, r_*) = \frac{1}{2}\left(g(\theta)\rho^{\frac{1}{n}}(\theta, r_*) + \sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta, r_*)\right)^2 + 4l\rho^{\frac{1}{n}}(\theta, r_*)}\right), \quad (2.6)$$

where $\rho(\theta, r_*)$ and $g(\theta)$ are the functions defined in the statement of Theorem 2.1.

Clearly the curve $(r(\theta)\cos\theta, r(\theta)\sin\theta)$ in the (x, y) plane with

$$\frac{r^{2n}}{(g(\theta)r + l)^n} - e^{\alpha n\theta}\left(\frac{r_*^{2n}}{(l + wr_*)^n} + f(\theta)\right) = 0 \quad (2.7)$$

is not algebraic, due to the expression $e^{\alpha n\theta}\frac{r_*^{2n}}{(l+wr_*)^n}$.

In what follows we prove that the straight line (2.1) does not intersect the orbit (2.7). This straight line in polar coordinates becomes $rg(\theta) + 2l = 0$ where $g(\theta)$ is the function defined in the statement of Theorem 2.1. To show this, we have to show that the system

$$\begin{cases} \left(\dfrac{r^2}{g(\theta)r + l}\right)^n - e^{\alpha n\theta}\left(\frac{r_*^{2n}}{(l+wr_*)^n} + f(\theta)\right) = 0, \\ g(\theta)r + 2l = 0, \end{cases} \quad (2.8)$$

has no solutions. Indeed the system (2.8) can be written as

$$\left(\frac{r^2}{-l}\right)^n - e^{\alpha n\theta}\left(\frac{r_*^{2n}}{(l+wr_*)^n} + f(\theta)\right) = 0,$$

which leads

$$r^2 = -le^{\alpha\theta}\left(\frac{r_*^{2n}}{(l+wr_*)^n} + f(\theta)\right)^{\frac{1}{n}}. \tag{2.9}$$

Since $r_* > 0, w \geq 0, l > 0$ and $f(\theta) > 0$ for all $\theta \in \mathbb{R}$, then

$$\rho(\theta, r_*) = e^{\alpha n\theta}\left(\frac{r_*^{2n}}{(l+wr_*)^n} + f(\theta)\right) > 0 \tag{2.10}$$

for all $\theta \in \mathbb{R}$.
By (2.10) and taking into $l > 0$ singular points satisfying (2.9) do not exist. Hence, (2.8) has no solution.

To show that $r(\theta, r_*)$ is a periodic solution, we have to show that:
i) the function $\theta \to r(\theta, r_*)$ is 2π -periodic.
ii) $r(\theta, r_*) > 0$, for all $\theta \in [0, 2\pi[$.
This last condition ensures that $r(\theta, r_*)$ is well defined for all $\theta \in [0, 2\pi[$ and the periodic solution do not pass through the equilibrium point $(0, 0)$ of system (1.2).

Periodicity. Since the function $g(\theta)$ is 2π -periodic, it follows that $r(\theta, r_*)$ is 2π -periodic if and only if $\rho(\theta, r_*)$ is 2π -periodic and we have

$$\begin{aligned}\rho(\theta + 2\pi, r_*) = e^{\alpha n(\theta+2\pi)}\left(\frac{r_*^{2n}}{(l+wr_*)^n}\right) + e^{\alpha n(\theta+2\pi)} f(\theta + 2\pi) \qquad (2.11)\\ e^{\alpha n(\theta+2\pi)}\left(-\frac{e^{2\pi n\alpha}}{e^{2\pi n\alpha} - 1} f(2\pi)\right) + e^{\alpha n(\theta+2\pi)} f(\theta + 2\pi),\end{aligned}$$

we have

$$\begin{aligned} f(\theta + 2\pi) &= \int_0^{\theta+2\pi} \frac{e^{-\alpha ns}}{(a + c\sin 2\theta + b\cos 2\theta)^n} ds \\ &= f(2\pi) + \int_{2\pi}^{\theta+2\pi} \frac{e^{-\alpha ns}}{(a + c\sin 2\theta + b\cos 2\theta)^n} ds, \end{aligned}$$

we make the change of variable $u = s - 2\pi$ in the integral $\int_{2\pi}^{\theta+2\pi} \frac{e^{-\alpha ns}}{(a+c\sin 2\theta+b\cos 2\theta)^n} ds$, we get

$$\begin{aligned} f(\theta + 2\pi) &= f(2\pi) + \int_0^{\theta} \frac{e^{-\alpha n(u+2\pi)}}{(a + c\sin 2(u + 2\pi) + b\cos 2(u + 2\pi))^n} du \\ &= f(2\pi) + e^{-2\alpha n\pi} f(\theta). \end{aligned}$$

We replace $f(\theta+2\pi)$ by $f(2\pi)+e^{-2\alpha n\pi}f(\theta)$ in (2.11), and after some calculations we obtain that

$$\rho(\theta+2\pi,r_*)=\rho(\theta,r_*).$$

Since $g(\theta)$ and $\rho(\theta,r_*)$ are a $2\pi-$periodic functions, then $r(\theta,r_*)$ is also $2\pi-$periodic function.

Strict positivity of $r(\theta,r_*)$ for $\theta\in[0,2\pi[$.
Since $l>0$, and taking into (2.10), then $4l\rho^{\frac{1}{n}}(\theta,r_*)>0$, thus

$$r(\theta,r_*)=\frac{1}{2}\left(g(\theta)\rho^{\frac{1}{n}}(\theta,r_*)+\sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta,r_*)\right)^2+4l\rho^{\frac{1}{n}}(\theta,r_*)}\right)>0$$

for all $\theta\in[0,2\pi[$.

In order to prove that the periodic orbit is hyperbolic limit cycles, we consider (2.6), and introduce the Poincaré return map $\lambda\mapsto\Pi(2\pi,\lambda)=r(2\pi,\lambda)$. Therefore, a limit cycles of system (2.2) are hyperbolic if, and only if $\left.\frac{dr(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_*}\neq 1$, in

$$r_*=\frac{1}{2}(\sqrt{(we^{2\pi\alpha}(\tfrac{-f(2\pi)}{e^{2\pi n\alpha}-1})^{\frac{1}{n}})^2+4le^{2\pi\alpha}(\tfrac{-f(2\pi)}{e^{2\pi n\alpha}-1})^{\frac{1}{n}}}+we^{2\pi\alpha}(\tfrac{-f(2\pi)}{e^{2\pi n\alpha}-1})^{\frac{1}{n}}),$$

after some calculations we obtain that

$$\left.\frac{dr(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_*}=e^{2\pi\alpha}<1.$$

Consequently the limit cycle of the differential equation is hyperbolic and stable for the equation (2.3), for more details see [15]. This completes the proof of Theorem 2.1.

□

3 Examples

As an example let $\alpha=v=b=-1,c=0,w=l=1,a=2$, then the system (1.2) becomes

$$\begin{aligned}\dot{x}&=x(x-y+1)^{n+1}-n\left(x^2+3y^2\right)^n\left(2x^2-y^2+xy+x+2y\right),\\ \dot{y}&=y(x-y+1)^{n+1}-n\left(x^2+3y^2\right)^n\left(3xy-x^2+y-2x\right).\end{aligned}\tag{3.1}$$

It is easy to verify that all conditions of Theorem 2.1 are satisfied. We conclude that system (3.1) has a non-algebraic limit cycle whose expression in polar coordinates (r,θ) is

$$r(\theta,r_*)=\frac{1}{2}\left(g(\theta)\rho^{\frac{1}{n}}(\theta,r_*)+\sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta,r_*)\right)^2+4\rho^{\frac{1}{n}}(\theta,r_*)}\right),$$

where $g(\theta) = \cos\theta - \sin\theta$, $\rho(\theta, r_*) = e^{-n\theta}\left(\frac{r_*^{2n}}{(1+r_*)^n} + \int_0^\theta \frac{e^{ns}}{(2-\cos 2s)^n}\, ds\right)$, $\theta \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axis is the point

$$r_* = \frac{1}{2}\left(\sqrt{\left(e^{-2\pi}\left(-\frac{f(2\pi)}{e^{-2\pi n}-1}\right)^{\frac{1}{n}}\right)^2 + 4e^{-2\pi}\left(-\frac{f(2\pi)}{e^{-2\pi n}-1}\right)^{\frac{1}{n}}} + e^{-2\pi}\left(\frac{-f(2\pi)}{e^{-2\pi n}-1}\right)^{\frac{1}{n}}\right).$$

Moreover,

$$\left.\frac{dr(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_*} = e^{-2\pi} < 1.$$

This limit cycle is a stable hyperbolic limit cycle.

For $n = 1$,
The system (3.1) is a quartic system that has a non algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = \frac{1}{2}\left(g(\theta)\,\rho(\theta, r_*) + \sqrt{(g(\theta)\,\rho(\theta, r_*))^2 + 4\rho(\theta, r_*)}\right),$$

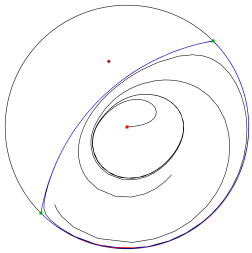

Fig. 1: Limit cycle of system (3.1) for n=1

where $g(\theta) = \cos\theta - \sin\theta$, $\rho(\theta, r_*) = e^{-\theta}\left(\frac{r_*^2}{1+r_*} + \int_0^\theta \frac{e^s}{2-\cos 2s} ds\right)$, $\theta \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axis is the point

$$r_* = 1.187\,7.$$

Moreover,

$$\left.\frac{dr(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_*} = e^{-2\pi} < 1.$$

This limit cycle is a stable hyperbolic limit cycle.
For $n = 2$,

The system is of degree 6 and that has a non algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = \frac{1}{2}\left(g(\theta)\,\rho^{\frac{1}{2}}(\theta, r_*) + \sqrt{\left(g(\theta)\,\rho^{\frac{1}{2}}(\theta, r_*)\right)^2 + 4\rho^{\frac{1}{2}}(\theta, r_*)}\right),$$

where $g(\theta) = \cos\theta - \sin\theta$, $\rho(\theta, r_*) = e^{-2\theta}\left(\frac{r_*^4}{(1+r_*)^2} + \int_0^\theta \frac{e^{2s}}{(2-\cos 2s)^2} ds\right)$ where $\theta \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axis is the point

$$r_* = 1.071\,4.$$

Moreover,

$$\left.\frac{dr(2\pi, \lambda)}{d\lambda}\right|_{\lambda = r_*} = e^{-2\pi} < 1.$$

This limit cycle is a stable hyperbolic limit cycle.

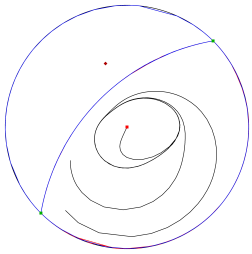

Fig. 2: Limit cycle of system (3.1) for n=2

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