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On the structure of le-semigroups

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Abstract In this paper, $(0, m)$-ideal elements and 0-minimal $(0, m)$-ideal elements in *poe*-semigroups are investigated. Then, for any positive integers $m,\ n$, relations $\mathcal{I}_n, {}_m\mathcal{I}, \mathcal{H}_m^n$, $\mathcal{B}_m^n$ and $\mathcal{Q}_m^n$ on le-semigroups are introduced. We show that, in any le-semigroup, $\mathcal{B}_m^n \subseteq \mathcal{Q}_m^n \subseteq \mathcal{H}_m^n$ and provide some sufficient conditions to get the equality. We, then, prove that either a $\mathcal{Q}_m^n$-class (resp. $\mathcal{H}_m^n$-class) is $(m, 0)$ and $(0, n)$-regular or none of its element is $(m, 0)$ and $(0, n)$-regular respectively. Next we show that either a $\mathcal{B}_m^n$-class (resp. $\mathcal{Q}_m^n$-class, $\mathcal{H}_m^n$-class) is (m, n)-right weakly regular or none of its element is (m, n)-right weakly regular. Finally, we define a strong (m, n)-quasi-ideal element in an le-semigroup and provide some sufficient conditions on an (m, n)-quasi-ideal element to be a strong (m, n)-quasi-ideal element.

Keywords $\vee e(le)$-semigroup · (m, n)-ideal element · (m, n)-quasi-ideal element

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1 Introduction

In [14], S. Lajos introduced the concept of an (m, n)-ideal in a semigroup. Prompted by this, several authors studied (m, n)-ideals in various algebraic structures such as rings, semirings and ordered semigroups etc; for example, Akram et al. [1], Bussaban and Changphas [3], Changphas [4], Tilidetzke [18], Yaqoob and Chinram [19] and many others. In [2], Bhuniya and Kumbhakar studied bands and normal bands in le-semigroups and obtained several characterizations of regular and intra-regular le-semigroups. Kehayopulu [5] studied (m, n)-ideal elements and (m, n)-quasi-ideal elements in *poe*-semigroups and le-semigroups. In [6, 7, 10], Kehayopulu characterized intra-regular $\vee e$-semigroups and regular le-semigroups in terms of left (resp. right)-ideal elements, ideal-elements, bi-ideal elements and quasi-ideal elements of respective semigroups. In [8, 9], Kehayopulu studied left (resp. right) duo regular, left-regular and left duo *poe*-semigroups. In [13], Kehayopulu characterized idempotent ideal elements of le-semigroups in terms of semisimple elements while intra-regular le-semigroup in terms of prime and semiprime ideal elements. The *Green-Kehayopulu* relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ have been introduced and investigated by Kehayopulu in [11,12]. In [17], Petro and Pasku studied the property of *Green-Kehayopulu* relation $\mathcal{H}$ and obtained some properties of *Green-Kehayopulu* relation $\mathcal{H}$ on le-semigroup which differ from *Green-relation* $\mathcal{H}_{plain}$ on plain semigroups. They also proved that an $\mathcal{H}$-class H of an le-semigroup satisfies *Green's condition* ($a, b \in H$ implies $ab \in H$) if and only if it contains an idempotent element and provided several conditions under which an $\mathcal{H}$-class of an le-semigroup forms a subsemigroup. In [15], Pasku and Petro showed that, in an le-semigroup, $\mathcal{B} \subseteq \mathcal{H}$ and provided some necessary and sufficient conditions for a $\mathcal{B}$-class to be regular or intra-regular. They further investigated several conditions under which a $\mathcal{B}$-class formed a subsemigroup.

In this paper, we first characterize $(0, m)$-ideal elements and 0-minimal $(0, m)$-ideal elements in poe-semigroups. Then, we investigate (m, n)-regular le-semigroups and prove that an le-semigroup S is (m, n)-regular if and only if $a \wedge q = q^m a q^n$ for each (m, n)-quasi-ideal element q and for each ideal-element a of S besides some more results on (m, n)-regular le-semigroups. We, then, have defined the relations ${}_m\mathcal{I}, \mathcal{I}_n, \mathcal{B}_m^n, \mathcal{Q}_m^n$ and $\mathcal{H}_m^n$ on le-semigroups and provided some sufficient conditions under which these relations are equal. We also study the $(m, 0)$-regularity [$(0, n)$-regularity] and (m, n)-right weakly regularity of $\mathcal{B}_m^n$-classes, $\mathcal{Q}_m^n$-classes and $\mathcal{H}_m^n$-classes respectively in le-semigroups. Finally, after introducing the concept of strong (m, n)-quasi-ideal elements in le-semigroups, we prove that, on le-semigroups, the relations $\mathcal{H}_m^n$ and $\mathcal{Q}_m^n$ coincide if and only if each (m, n)-quasi-ideal element is a strong (m, n)-quasi-ideal element and provide some sufficient conditions so that (m, n)-quasi-ideal elements are strong (m, n)-quasi-ideal elements.

Definition 1.1 *Let S be a non-empty set. The triplet $(S, \cdot, \leq)$ is called an ordered semigroup (or po-semigroup) if $(S, \cdot)$ is a semigroup and $(S, \leq)$ is a partially ordered set such that*

$$a \leq b \Rightarrow ac \leq bc \text{ and } ca \leq cb$$

for all $a, b, c \in S$. A po-semigroup with a greatest element "e" (i.e., for each $a \in S, a \leq e$) is said to be a poe-semigroup.

Let S be a poe-semigroup and let $a \in S$. The element a is called a *subsemigroup element* if $a^2 \leq a$ and a *left (resp. right) ideal* element of S if $ea \leq a$ (resp. $ae \leq a$). It is called an *ideal element* of S if it is both a left and a right-ideal element of S and a *bi-ideal element* of S if $aea \leq a$. Further a is called an *idempotent element* if $a = a^2$; and a *quasi-ideal element* if $ae \wedge ea$ exists and $ae \wedge ea \leq a$. An element $z \in S$ is called a *zero element* of S if $za = az = z$ and $z \leq a$ for each $a \in S$. The zero element, if exists, is always unique. We shall denote it, in whatever follows, by the symbol 0. A poe-semigroup S is called *regular (left-regular, right-regular)* if

$a \leq aea$ ($a \leq ea^2, a \leq a^2e$) for each $a \in S$ and *commutative* if $ab = ba$ for all $a, b \in S$.

Definition 1.2 *A poe-semigroup S is said to be a $\vee e$-semigroup if it is an upper semilattice under $\vee$ and*

$$c(a \vee b) = ca \vee cb \text{ and } (a \vee b)c = ac \vee bc$$

for all $a, b, c \in S$. A $\vee e$-semigroup which is a lattice is said to be an le-semigroup.

It is well known that in an *le*-semigroup S,

$$a \leq b \Leftrightarrow a \wedge b = a \text{ and } a \vee b = b$$

for all $a, b \in S$.

For simplicity, throughout the paper, for any element a of an ordered semigroup S, we shall write a^n for $aa \cdots a$ ($n-$copies of a). Also the integers m, n will stand for positive integers throughout the paper until and unless otherwise specified.

Definition 1.3 *[5] Let S be a poe-semigroup and m, n be non-negative integers. An element a of S is called an (m, n)-ideal element of S if $a^m ea^n \leq a$.*

Remark 1.1 Throughout the paper, we shall use the convention $a^0 b = ba^0 = b$, for all $a, b \in S$. In particular, for $m = 0$, $n = 1$ (resp. $m = 1$, $n = 0$ and $m = 1 = n$), each (m, n)-ideal element a of S is a left-ideal element (resp. right-ideal element and bi-ideal element). Clearly each left-ideal element (resp. each right-ideal element and each bi-ideal element) is a $(0, n)$-ideal element for each positive integer n (resp. $(m, 0)$-ideal element for each positive integer m and (m, n)-ideal element for each positive integers m, n). Therefore the concept of a $(0, n)$-ideal element (resp. $(m, 0)$-ideal element and (m, n)-ideal element) is a generalization of the concept of a left-ideal element (resp. a right-ideal element and a bi-ideal element).

Remark 1.2 For each positive integers m and n, any $(m, 0)$-ideal element (resp. $(0, n)$-ideal element) is an (m, n)-ideal element as $a^m ea^n \leq a^m e \leq a$ ($a^m ea^n \leq ea^n \leq a$).

Definition 1.4 *[5] An element q of a poe-semigroup S is called an (m, n)-quasi-ideal element of S if $q^m e \wedge eq^n$ exists and $q^m e \wedge eq^n \leq q$. Clearly every quasi-ideal element is an (m, n)-quasi-ideal element for each positive integers m and n such that $q^m e \wedge eq^n$ exists, but the converse is not true in general.*

Example 1.1 Let $S = \{a, b, c, d\}$. Define the binary operation $'\cdot'$ as follows:

$\cdot$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	a	a	b	a

Then the set $\mathcal{P}(S)$ of all non-empty subsets of S is a *poe*-semigroup under the binary operation of set product induced by the binary operation defined by the above table and partially ordered by set inclusion. The set $\{a, d\}$ is an (m, n)-ideal element and an (m, n) quasi-ideal element for each integers $m, n \geq 2$, but $\{a, d\}$ is neither a bi-ideal element nor a quasi-ideal element of the *poe*-semigroup $\mathcal{P}(S)$.

We denote by (a), $< a >_{<m,n>}$ and $(a)_{<m,n>}$, the ideal-element, (m, n)-ideal element and (m, n)-quasi-ideal element of S generated by the element a of S i.e., the least ideal-element, the least (m, n)-ideal element and the least (m, n) quasi-ideal element of S greater than the element a and had been described by [5,13] as follows:

$$\begin{aligned}(a) &= a \vee ea \vee ae \vee eae;\\ < a >_{<m,n>} &= a \vee a^m e a^n;\\ (a)_{<m,n>} &= a \vee (a^m e \wedge e a^n).\end{aligned}$$

Thus an element $a \in S$ is an ideal (resp. (m, n)-ideal, (m, n) quasi-ideal) element if and only if $(a) = a$ (resp. $< a >_{<m,n>} = a$, $(a)_{<m,n>} = a$).

2 Characterization of $(0, m)$-ideal elements

Lemma 2.1 *[5] Let S be a $\vee e$-semigroup, $a \in S$ and $m, n, k \geq 0$ are integers. Then*
(1) $(a \vee a^m e a^k)^m e = a^m e$;
(2) $e(a \vee a^k e a^n)^n = e a^n$;
(3) $< a >_{<m,n>}$ exists and $< a >_{<m,n>} = a \vee a^m e a^n$.

Lemma 2.2 *Let S be an le-semigroup, $m \in \mathbb{Z}^+$ and a be a subsemigroup element of S. Then a is an $(1, m)$-ideal element of S if and only if there exist a $(0, m)$-ideal element c and a right-ideal element b of S such that $bc^m \leq a \leq b \wedge c$.*

Proof. Let a be any $(1, m)$-ideal element of S. Then $a \vee ea^m$ and $a \vee ae$ are $(0, m)$-ideal element and right-ideal element of S respectively. Let $b = a \vee ae$ and $c = a \vee ea^m$. Then

$$\begin{aligned}bc^m &= (a \vee ae)(a \vee ea^m)^m\\ &= a(a \vee ea^m)^m \vee ae(a \vee ea^m)^m\\ &= (a^2 \vee aea^m)(a \vee ea^m)^{m-1} \vee aea^m && \text{(by Lemma 2.1)}\\ &\leq (a^2 \vee a)(a \vee ea^m)^{m-1} \vee a && (\text{as } aea^m \leq a)\\ &= a(a \vee ea^m)^{m-1} \vee a && (\text{as } a^2 \leq a)\\ &= (a^2 \vee aea^m)(a \vee ea^m)^{m-2} \vee a\\ &\leq (a^2 \vee a)(a \vee ea^m)^{m-2} \vee a && (\text{as } aea^m \leq a)\end{aligned}$$

$$
\begin{aligned}
&= a(a \vee ea^m)^{m-2} \vee a && (\text{as } a^2 \leq a) \\
&\vdots \\
&= a.
\end{aligned}
$$

As $a \leq b \wedge c$, we have $bc^m \leq a \leq b \wedge c$, as required.

Conversely, assume that b is a right-ideal element and c is a $(0,m)$-ideal element of S such that $bc^m \leq a \leq b \wedge c$. As $a \leq b \wedge c$, b is a right-ideal element and $bc^m \leq a$, we have $aea^m \leq (b \wedge c)e(b \wedge c)^m \leq bec^m \leq bc^m \leq a$. Therefore a is an $(1,m)$ ideal element of S.

□

Definition 2.3 *Let S be a poe-semigroup and a be any left-ideal (right-ideal) element of S. Then a is said to be a minimal left-ideal (right-ideal) element of S if for every left-ideal (right-ideal) element b of S, $b \leq a$ implies $b = a$. Further any non-zero left-ideal (right-ideal) element a of a poe-semigroup S with 0 is said to be a 0-minimal if for each left-ideal (right-ideal) element b of S, $b \leq a$ implies either $b = 0$ or $b = a$.*

Similarly we may define a minimal and a 0-minimal (m,n)-ideal (resp. $(m,0)$-ideal, $(0,n)$-ideal) element for each positive integers m and n.

Lemma 2.4 *Let S be a poe-semigroup with zero, $m \in \mathbb{Z}^+$ and let a be any 0-minimal left-ideal element of S. Then a subsemigroup element b of S is a $(0,m)$-ideal element of S smaller than a if and only if either $b^m = 0$ or $b = a$.*

Proof. Let S be a *poe*-semigroup with 0 and b be a subsemigroup element as well as a $(0,m)$-ideal element of S smaller than the 0-minimal left-ideal element a of S. As eb^m is a left-ideal element of S and $eb^m \leq b \leq a$, so by minimality of the left-ideal element a of S, $eb^m = 0$ or $eb^m = a$. If $eb^m = a$, then $a = eb^m \leq b$. Therefore $b = a$. In the other case when $eb^m = 0$, as $eb^m = 0 \leq b^m$, b^m is a left-ideal element of S smaller than a (as b is a subsemigroup element and, so, $b^m \leq b$). Now, by 0-minimality of the left-ideal element a of S, we have $b^m = 0$ or $b^m = a$. Since b is a subsemigroup element, we have $b^m \leq b$. Therefore $a = b^m \leq b$. Hence in both of the cases, either $b^m = 0$ or $b = a$, as required.
The converse is obvious.

□

Lemma 2.5 *Let S be a poe-semigroup with zero, $m \in \mathbb{Z}^+$ and let a be any subsemigroup element of S. If a is a 0-minimal $(0,m)$-ideal element of S, then $a^m = 0$ or a is a 0-minimal left-ideal element of S.*

Proof. Let S be a *poe*-semigroup with 0 and a be any subsemigroup 0-minimal $(0,m)$-ideal element of S. Therefore $a^m \leq a$ and $e(a^m)^m = e\underbrace{a^m a^m \ldots a^m}_{m\text{-times}} \leq \underbrace{aa \ldots a}_{m\text{-times}} = a^m$, it follows that a^m is $(0,m)$-ideal element of S. As $a^m \leq a$, by minimality of a 0-minimal $(0,m)$-ideal element a of S, $a^m = 0$ or $a^m = a$. Suppose $a^m = a$. Now $ea = ea^m \leq a$ implies a is left-ideal element of S. It remains to show that a is a 0-minimal left-ideal element of S. So, let b be any left-ideal element of S such that $b \leq a$. As b is a $(0,m)$-ideal element of S, a is a 0-minimal $(0,m)$-ideal element of S and $b \leq a$, we have either $b = 0$ or $b = a$. Hence a is a 0-minimal left-ideal element of S.

□

Lemma 2.6 *Let S be an le-semigroup, $m \in \mathbb{Z}^+$ and a be any subsemigroup element of S. Then a is a $(0,m)$-ideal element of S if and only if $ba \leq a$ for some $(0,m{-}1)$-ideal element b $(a \leq b)$ of S.*

Proof. Suppose a is a $(0,m)$-ideal element of S. By Lemma 2.1, $e(a \vee ea^{m-1})^{m-1} = ea^{m-1}$. So it follows that $e(a \vee ea^{m-1})^{m-1} = ea^{m-1} \leq a \vee ea^{m-1}$ i.e., $a \vee ea^{m-1}$ is a $(0,m{-}1)$-ideal element of S. Let $b = a \vee ea^{m-1}$. Then, as a is a subsemigroup and $(0,m)$-ideal element of S, $ba = (a \vee ea^{m-1})a = a^2 \vee ea^m \leq a^2 \vee a = a$, as required.

Conversely assume that b is a $(0,m{-}1)$-ideal element of S and a be an element of S such that $ba \leq a$ with $a \leq b$. Now $ea^m \leq eb^{m-1}a \leq ba \leq a$. Therefore a is a $(0,m)$-ideal element of S.

□

Lemma 2.7 *Let S be a poe-semigroup and let $a \in S$. Then a is a minimal $(m,m{-}1)$-ideal element $(m \in \mathbb{Z}^+, m \geq 2)$ if and only if a is a minimal bi-ideal element of S.*

Proof. Let S be a *poe*-semigroup and a be a minimal $(m,m{-}1)$-ideal element of S. Then, by definition, $a^m e a^{m-1} \leq a$, we have $(a^m e a^{m-1})^m e(a^m e a^{m-1})^{m-1} \leq a^m e a^{m-1}$. Therefore $a^m e a^{m-1}$ is a $(m,m{-}1)$-ideal element of S such that $a^m e a^{m-1} \leq a$. So, by minimality of a, we have $a^m e a^{m-1} = a$. Now

$$aa = (a^m e a^{m-1})(a^m e a^{m-1}) \leq a^m e a^{m-1} = a$$

and

$$aea = a^m e a^{m-1} e a^m e a^{m-1} \leq a^m e a^{m-1} = a.$$

This implies that a is a bi-ideal element of S. Now we show that a is a minimal bi-ideal element of S. So take any bi-ideal element b of S such that $b \leq a$. As $b^m e b^{m-1} = b(b^{m-1} e b^{m-2})b \leq beb \leq b$, b is an $(m,m{-}1)$-ideal element of S. Since a is a minimal $(m,m{-}1)$-ideal element of S, $b = a$. Hence a is a minimal bi-ideal element of S.

Conversely assume that a is a minimal bi-ideal element of S. As $a^m e a^{m-1} = a(a^{m-1} e a^{m-2})a \leq aea \leq a$, a is an $(m,m{-}1)$-ideal element of S. To show that a is a minimal $(m,m{-}1)$-ideal element of S, take any $(m,m{-}1)$-ideal element b of S such that $b \leq a$. As $(b^m e b^{m-1})(b^m e b^{m-1}) = b^m(e b^{m-1} b^m e)b^{m-1} \leq b^m e b^{m-1}$ and $(b^m e b^{m-1})e(b^m e b^{m-1}) = b^m(e b^{m-1} e b^m e)b^{m-1} \leq b^m e b^{m-1}$, it follows that $b^m e b^{m-1}$ is a bi-ideal element of S. Since a is a minimal bi-ideal element of S and $b^m e b^{m-1} \leq a$, we have $b^m e b^{m-1} = a$. As $b^m e b^{m-1} \leq b$, $a \leq b$. Now, as $b \leq a$, we have $b = a$. Hence a is a minimal $(m, m-1)$-ideal element of S.

□

Lemma 2.8 *Let S be a commutative poe-semigroup such that $e^2 = e$ and let m, n be any non-negative integers. Then*
(1) *$I_{<m,0>}$, the set of all $(m, 0)$-ideal elements of S, is a subsemigroup of S.*
(2) *$I_{<0,n>}$, the set of all $(0, n)$-ideal elements of S, is a subsemigroup of S.*
(3) *$I_{<m,n>}$, the set of all (m, n)-ideal elements of S, is a subsemigroup of S.*

Proof. Straightforward.

□

3 (m, n)-regular le-semigroups

Definition 3.1 *Let S be a poe-semigroup and let m, n be non-negative integers. An element $a \in S$ is said to be an (m, n)-regular element of S if $a \leq a^m e a^n$. Further S is said to be (m, n)-regular if every element of S is (m, n)-regular. A $(1, 1)$-regular poe-semigroup is said to be regular.*

It is clear from Definition 3.1 that, for each non-negative integers m and n, every (m, n)-regular *poe*-semigroup is an (r, s)-regular *poe*-semigroup ($r \leq m, s \leq n$ are non-negative integers). In particular, for each $m, n \in \mathbb{Z}^+$, (m, n)-regular *poe*-semigroup is regular. On the other hand, for each $m \in \mathbb{Z}^+$, $(m, 0)$-regular *poe*-semigroup need not be a regular *poe*-semigroup.

Example 3.1 Let $S = \{x, y, z\}$. Define a binary operation $'\cdot'$ and an order $'\leq'$ as follows:

$\cdot$	x	y	z
x	z	x	z
y	x	y	z
z	z	z	z

$$\leq := \{(x, x), (y, y), (z, z), (z, x), (x, y), (z, y)\}.$$

Clearly S is a *poe*-semigroup with greatest element $e = y$. As $(x, xyx) \notin \leq$, S is not a regular *poe*-semigroup, but S is $(1, 0)$-regular as $a \leq ae$ for each $a \in S$.

Lemma 3.2 *[5] Let S be an le-semigroup, $a \in S$ and m, n non-negative integers. Then the followings hold:*
(1) $(a \vee (a^m e \wedge e a^n))^m e \leq a^m e$.
(2) $e(a \vee (a^m e \wedge e a^n))^n \leq e a^n$.
(3) $(a)_{<m,n>}$ *exists and* $(a)_{<m,n>} = a \vee (a^m e \wedge e a^n)$.

Theorem 3.3 *Let S be an le-semigroup and m, n be positive integers. Then S is (m,n)-regular if and only if $a \wedge q = q^m a q^n$ for each (m,n)-quasi-ideal element q and for each ideal-element a of S.*

Proof. Let S be an (m,n)-regular le-semigroup, q be an (m,n)-quasi-ideal element and a be an ideal-element of S. Since $q^m a q^n \leq q^m e$ and $q^m a q^n \leq e q^n$, we have $q^m a q^n \leq q^m e \wedge e q^n$. As q is an (m,n)-quasi-ideal element of S, we get $q^m e \wedge e q^n \leq q$. Thus $q^m a q^n \leq q$. Since a is an ideal-element of S, we have $q^m a q^n \leq eae \leq ae \leq a$. Therefore $q^m a q^n \leq q \wedge a$. As S is (m,n)-regular, we have

$$
\begin{aligned}
&(a \wedge q) \\
&\leq (a \wedge q)^m e (a \wedge q)^n \\
&\leq (a \wedge q)^m e ((a \wedge q)^n)^m e ((a \wedge q)^n)^n \\
&\leq (a \wedge q)^m e ((a \wedge q)^m)^n e ((a \wedge q)^n)^n \\
&= (a \wedge q)^m e \underbrace{(a \wedge q)^m (a \wedge q)^m \ldots (a \wedge q)^m}_{n\text{-times}} e \underbrace{(a \wedge q)^n (a \wedge q)^n \ldots (a \wedge q)^n}_{n\text{-times}} \\
&\leq q^m e \underbrace{a^m a^m \ldots a^m}_{n\text{-times}} e \underbrace{a^n a^n \ldots a^n}_{n-1\text{-times}} q^n \\
&= q^m e a (a^{m-1} \underbrace{a^m a^m \ldots a^m}_{n-1\text{-times}} \underbrace{a^n a^n \ldots a^n}_{n-1\text{-times}}) q^n \\
&\leq q^m e a e q^n \\
&\leq q^m a q^n.
\end{aligned}
$$

Hence $a \wedge q = q^m a q^n$.

Conversely assume that $a \wedge q = q^m a q^n$ for each (m,n)-quasi-ideal element q and for each ideal-element a of S. Take any $b \in S$. As $(b)_{<m,n>}$ and (b) are (m,n)-quasi-ideal element and ideal-elements of S respectively, we have

$$(b) \wedge (b)_{<m,n>} = (b)^m_{<m,n>} (b) (b)^n_{<m,n>} \leq (b)^m_{<m,n>} e (b)^n_{<m,n>} \leq b^m e b^n.$$

As $b \leq (b) \wedge (b)_{(m,n)}$, we have $b \leq b^m e b^n$. Hence S is (m,n)-regular.

□

Theorem 3.4 *[5] Let S be an le-semigroup and m, n non-negative integers. Then, the following conditions are equivalent:*
(1)S is $< m, n >$-regular.
(2) $a^m e a^n = a$ for each $a \in I_{<m,n>}$.
(3) $q^m e q^n = a$ for each $q \in Q_{<m,n>}$.
(4) $(< a >_{<m,n>})^m e (< a >_{<m,n>})^n =< a >_{<m,n>}$ for each $a \in S$.
(5) $((a)_{<m,n>})^m e ((a)_{<m,n>})^n = (a)_{<m,n>}$ for each $a \in S$.

Theorem 3.5 *Let S be an le-semigroup and let m, n be non-negative integers. Then S is (m,n)-regular if and only if $a \wedge b = a^m b^n$ for each $(m,0)$-ideal element a and for each $(0,n)$-ideal element b of S.*

Proof. The statement is trivially true for $m = 0 = n$. If $m = 0$ and $n \neq 0$ or $m \neq 0$ and $n = 0$, then the result follows by Theorem 3.4. So, let $m \neq 0$, $n \neq 0$, a be any $(m,0)$-ideal element and b be any $(0,n)$-ideal element of S. Then $a^m b^n \leq a^m e \leq a$ and $a^m b^n \leq e b^n \leq b$. Therefore $a^m b^n \leq a \wedge b$. As

S is (m,n)-regular, we have

$$\begin{aligned}
&(a \wedge b)\\
&\leq (a \wedge b)^m e (a \wedge b)^n\\
&\leq a^m e b^n\\
&\leq a^m e b^{n-1}(b^m e b^n)\\
&\leq a^m e b^{n-1} b^{m-1}(b^m e b^n) e b^n\\
&\leq a^m e b^{n-1} b^{m-1} b^{m-1}(b^m e b^n) e b^n e b^n\\
&\vdots\\
&\leq a^m e b^{n-1} \underbrace{b^{m-1} b^{m-1} \dots b^{m-1}}_{n-1\text{-times}} (b^m e b^n) \underbrace{e b^n e b^n \dots e b^n}_{n-1\text{-times}}\\
&= a^m e b^{n-1} (b^{m-1})^{n-1} b^m \underbrace{e b^n e b^n \dots e b^n}_{n\text{-times}}\\
&= a^m e b^{mn} \underbrace{e b^n e b^n \dots e b^n}_{n\text{-times}}\\
&\leq a^m (e b^n)^n\\
&\leq a^m b^n
\end{aligned}$$

Therefore $a \wedge b = a^m b^n$.

Conversely assume that $a \wedge b = a^m b^n$ for each $(m,0)$-ideal element a and for each $(0,n)$-ideal element b of S. For any $a \in S$, as $< a >_{<m,0>}$ is a $(m,0)$-ideal element and e is a $(0,n)$-ideal element of S, we have

$$< a >_{<m,0>} = < a >_{<m,0>} \wedge e = (< a >_{<m,0>})^m e^n \leq a^m e.$$

Similarly $< a >_{<0,n>} \leq e a^n$. As $a^m e$ is an $(m,0)$-ideal element and $e a^n$ is a $(0,n)$-ideal element of S, by hypothesis, we have

$$a \leq < a >_{<m,0>} \wedge < a >_{<0,n>} \leq a^m e \wedge e a^n = (a^m e)^m (e a^n)^n \leq a^m e a^n.$$

Hence S is (m,n)-regular.

□

Theorem 3.6 *Let S be an le-semigroup and let m, n be positive integers (either $m \geq 2$ or $n \geq 2$). Then the following are equivalent:*
(1) *Each (m,n)-ideal element of S is idempotent.*
(2) *For each (m,n)-ideal elements a, b of S, $a \wedge b \leq a^m b^n$.*
(3) $< a >_{<m,n>} \wedge < b >_{<m,n>} \leq (< a >_{<m,n>})^m (< b >_{<m,n>})^n \ \forall a, b \in S$.
(4) $< a >_{<m,n>} \leq (< a >_{<m,n>})^m (< a >_{<m,n>})^n \ \forall a \in S$.
(5) *S is (m,n)-regular.*

Proof. (1) ⇒ (2) Assume that each (m,n)-ideal element of S is idempotent. Let a and b be any (m,n)-ideal elements of S. As $a \wedge b$ is an (m,n)-ideal element of S, we have

$$a \wedge b = (a \wedge b)^2 = (a \wedge b)^3 = \cdots = (a \wedge b)^m (a \wedge b)^n \leq a^m b^n.$$

(2) ⇒ (3) and (3) ⇒ (4). Obvious.

$(4) \Rightarrow (5)$. Take any $a \in S$. Then, by (4), we have

$$\begin{aligned}
&< a >_{<m,n>} \\
&\leq (< a >_{<m,n>})^m (< a >_{<m,n>})^n \\
&\leq (< a >_{<m,n>})^m (< a >_{<m,n>})^{n-1} (< a >_{<m,n>})^m (< a >_{<m,n>})^n \\
&\leq (< a >_{<m,n>})^m e (< a >_{<m,n>})^n \\
&= a^m e a^n \qquad \text{(by Lemma 2.1).}
\end{aligned}$$

As $a \leq < a >_{<m,n>}$, we have $a \leq a^m e a^n$. Hence S is (m, n)-regular.
$(5) \Rightarrow (1)$. Take any (m, n)-ideal element a of S. As S is (m, n)-regular and a is an (m, n)-ideal element, we have $a = a^m e a^n$. Now

$$a^2 = (a^m e a^n)(a^m e a^n) \leq (a^m e a^n) = a$$

and

$$a = a^m e a^n = (a^m e a^n)^m e a^n \leq (a^m e a^n)(a^m e a^n) = aa = a^2.$$

Therefore $a = a^2$. Hence each (m, n)-ideal element of S is an idempotent.

□

Lemma 3.7 *Let S be an le-semigroup and let m, n be non-negative integers. Then S is $< m, 0 >$-regular ($(0, n)$-regular) if and only if $I_{<m,0>}$ ($I_{<0,n>}$), the set of all $(m, 0)$-ideal elements of S ($(0, n)$-ideal elements of S), is $(m, 0)$-regular ($(0, n)$-regular).*

Proof. When $m = 0$, the statement holds trivially because e is the only $(0, 0)$-ideal element of S. So, let $m \neq 0$ and $a \in I_{<m,0>}$. Therefore $a^m e \leq a$. As S is $(m, 0)$-regular, we have $a \leq a^m e$. Thus $a = a^m e$. Since $e \in I_{<m,0>}$, so a is a $(m, 0)$-regular element of $I_{<m,0>}$. Hence $I_{<m,0>}$ is $(m, 0)$-regular.

Conversely assume that $I_{<m,0>}$ is $(m, 0)$-regular. Take any $a \in S$. As $< a >_{<m,0>}$ is in $I_{(m,0)}$ and $I_{<m,0>}$ is $(m, 0)$-regular, there exists $b \in I_{(m,0)}$ such that $< a >_{<m,0>} = (< a >_{<m,0>})^m b \leq (< a >_{<m,0>})^m e$. By Lemma 2.1, $(< a >_{<m,0>})^m e = a^m e$. As $a \leq < a >_{<m,0>}$, we have $a \leq a^m e$. Hence S is $(m, 0)$-regular.

□

Lemma 3.8 *Let S be an le-semigroup and let m, n be non-negative integers. Then S is (m, n)-regular if and only if $I_{<m,n>}$, the set of all (m, n)-ideal elements of S, is (m, n)-regular.*

Proof. On the lines similar to the proof of Lemma 3.7.

□

Lemma 3.9 *Let S be an le-semigroup and let m, n be non-negative integers. Then S is (m, n)-regular if and only if $Q_{<m,n>}$, the set of all (m, n)-quasi-ideal elements of S, is (m, n)-regular.*

Proof. If $m = n = 0$, then the result trivially holds as $Q_{<0,0>} = \{e\}$. When $m \neq 0$ and $n = 0$ or $m = 0$ and $n \neq 0$, then the result follows by Lemma 3.7, as the set of all $(m,0)$ $((0,n))$-quasi-ideal elements $Q_{<m,0>}$ $(Q_{<0,n>})$ coincides with the set of all $(m,0)$ $((m,0))$-ideal elements $I_{<m,0>}$ $(I_{<0,n>})$. So, let $m \neq 0$ and $n \neq 0$ and $a \in Q_{<m,n>}$. Therefore $a^m ea^n \leq a^m e \wedge ea^n \leq a$. As S is (m,n)-regular, we have $a \leq a^m ea^n$. Thus $a = a^m ea^n$. Since $e \in Q_{<m,n>}$, it follows that a is an (m,n)-regular element of $Q_{<m,n>}$. Hence $Q_{<m,n>}$ is (m,n)-regular.

Conversely assume that $Q_{<m,n>}$ is (m,n)-regular and a be any element of S. Then $< a >_{<m,n>} \in Q_{<m,n>}$. Therefore there exists $b \in Q_{<m,n>}$ such that

$$(a)_{<m,n>} = ((a)_{<m,n>})^m b((a)_{<m,n>})^n.$$

Now $((a)_{<m,n>})^m b((a)_{<m,n>})^n \leq ((a)_{<m,n>})^m e((a)_{<m,n>})^n$ and, by Lemma 3.2, $((a)_{<m,n>})^m e((a)_{<m,n>})^n \leq a^m ea^n$. As $a \leq (a)_{<m,n>}$, we have $a \leq a^m ea^n$. Therefore a is (m,n)-regular and, hence, S is (m,n)-regular.

□

4 Characterization of relations $\mathcal{I}_n, {}_m\mathcal{I}, \mathcal{B}_m^n, \mathcal{Q}_m^n$ and $\mathcal{H}_m^n$

Lemma 4.1 *Let S be an le-semigroup, $a \in S$ and let m, n be non-negative integers. Then*
(1) $<< a >_{<m,n>} >_{<m,n>} = < a >_{<m,n>}$;
(2) $((a)_{<m,n>})_{<m,n>} = (a)_{<m,n>}$.

Proof. (1). By definition, we have

$$\begin{aligned} << a >_{<m,n>} >_{<m,n>} &= < a \vee a^m ea^n >_{<m,n>} \\ &= (a \vee a^m ea^n) \vee (a \vee a^m ea^n)^m e(a \vee a^m ea^n)^n \\ &= (a \vee a^m ea^n) \vee (a^m ea^n) \quad \text{(by Lemma 2.1)} \\ &= < a >_{<m,n>} . \end{aligned}$$

(2). Now

$$\begin{aligned} &((a)_{<m,n>})_{<m,n>} \\ &= (a \vee (a^m e \wedge ea^n))_{<m,n>} \\ &= (a \vee (a^m e \wedge ea^n)) \vee ((a \vee (a^m e \wedge ea^n)^m e \wedge e(a \vee (a^m e \wedge ea^n)^n). \end{aligned}$$

Now, by Lemma 3.2, $(a \vee (a^m e \wedge ea^n))^m e \leq a^m e$ and $e((a \vee (a^m e \wedge ea^n))^n \leq ea^n$. So $(a \vee (a^m \wedge ea^n))^m e \wedge e(a \vee (a^m \wedge ea^n))^n \leq a^m e \wedge ea^n$. Therefore

$$\begin{aligned} &((a)_{<m,n>})_{<m,n>} \\ &= (a \vee (a^m e \wedge ea^n)) \vee ((a \vee (a^m e \wedge ea^n))^m e \wedge e(a \vee (a^m e \wedge ea^n))^n) \\ &\leq a \vee (a^m e \wedge ea^n) \\ &= (a)_{<m,n>}. \end{aligned}$$

Also

$$\begin{aligned}&(a)_{<m,n>}\\&= a \vee (a^m e \wedge ea^n)\\&\leq (a \vee (a^m e \wedge ea^n)) \vee ((a \vee (a^m e \wedge ea^n))^m e \wedge e(a \vee (a^m e \wedge ea^n))^n)\\&= ((a)_{<m,n>})_{<m,n>}.\end{aligned}$$

Hence $((a)_{<m,n>})_{<m,n>} = (a)_{<m,n>}$, as required.

□

Corollary 4.2 *Let S be an le-semigroup, $a \in S$ and let $m,\ n$ be non-negative integers. Then*
(1) $<< a >_{<m,0>}>_{<m,0>} = < a >_{<m,0>}$;
(2) $<< a >_{<0,n>}>_{<0,n>} = < a >_{<0,n>}$.

Remark 4.1 Let S be an le-semigroup and let $m,\ n$ be non-negative integers. If $a, b \in S$ such that $a \leq b$, then
(1) $< a >_{<m,0>} \leq < b >_{<m,0>}$;
(2) $< a >_{<0,n>} \leq < b >_{<0,n>}$;
(3) $< a >_{<m,n>} \leq < b >_{<m,n>}$;
(4) $(a)_{<m,n>} \leq (b)_{<m,n>}$.

Lemma 4.3 *Let S be an le-semigroup, $a \in S$ and let $m,\ n$ be non-negative integers. Then*
(1) $<< a >_{<m,0>} \wedge < a >_{<0,n>}>_{<m,0>} = < a >_{<m,0>}$;
(2) $<< a >_{<m,0>} \wedge < a >_{<0,n>}>_{<0,n>} = < a >_{<0,n>}$.

Proof. Straightforward.

□

Lemma 4.4 *Let S be an le-semigroup, $a \in S$ and let $m,\ n$ be non-negative integers. Then*
(1) $< (a)_{<m,n>} >_{<m,0>} = < a >_{<m,0>}$;
(2) $< (a)_{<m,n>} >_{<0,n>} = < a >_{<0,n>}$.

Proof. On the lines similar to the proof of Lemma 4.1.

□

Definition 4.5 *Let S be an le-semigroup and let $m,\ n$ be positive integers. We define the relations $\mathcal{I}_n,\ {}_m\mathcal{I},\ \mathcal{H}_m^n,\ \mathcal{B}_m^n$, and $\mathcal{Q}_m^n$ as follows:*

$$\begin{aligned}\mathcal{I}_n &= \{(a,b) \in S \times S \mid\ < a >_{<0,n>} = < b >_{<0,n>}\};\\ {}_m\mathcal{I} &= \{(a,b) \in S \times S \mid\ < a >_{<m,0>} = < b >_{<m,0>}\};\\ \mathcal{H}_m^n &= {}_m \mathcal{I} \cap \mathcal{I}_n;\\ \mathcal{B}_m^n &= \{(a,b) \in S \times S \mid\ < a >_{<m,n>} = < b >_{<m,n>}\};\\ \mathcal{Q}_m^n &= \{(a,b) \in S \times S \mid (a)_{<m,n>} = (b)_{<m,n>}\}.\end{aligned}$$

Clearly all the relations defined above are equivalence relations on S.

Lemma 4.6 *Let S be an le-semigroup. If $a, b \in S$ are ${}_m\mathcal{I}$-related (resp. $\mathcal{I}_n$-related), then $a^m e = b^m e$ (resp. $ea^n = eb^n$).*

Proof. Suppose $(a, b) \in {}_m\mathcal{I}$. Then, by definition, $< a >_{<m,0>} = < b >_{<m,0>}$ i.e. $a \vee a^m e = b \vee b^m e$. Therefore $a \leq b \vee b^m e$ and $b \leq a \vee a^m e$. Thus $a^m e \leq (b \vee b^m e)^m e \leq b^m e$ (by Lemma 2.1). Similarly, from $b \leq a \vee a^m e$, we have $b^m e \leq a^m e$. Hence $a^m e = b^m e$.
Dually it may be shown that if $(a, b) \in \mathcal{I}_n$, then $ea^n = eb^n$.

□

Lemma 4.7 *Let S be an le-semigroup. If $a, b \in S$ are $\mathcal{H}_m^n$-related, then $a^m e = b^m e$, $ea^n = eb^n$ and $a^m ea^n = b^m eb^n$.*

Proof. Suppose $(a, b) \in \mathcal{H}_m^n$. Then, by definition, $(a, b) \in {}_m\mathcal{I}$ and $(a, b) \in \mathcal{I}_n$. Now, by Lemma 4.6, $a^m e = b^m e$ and $ea^n = eb^n$. Also, as $a^m e = b^m e$ and $ea^n = eb^n$, we have $a^m ea^n = b^m ea^n = b^m eb^n$, as required.

□

Lemma 4.8 *In any le-semigroup S, $\mathcal{B}_m^n \subseteq \mathcal{H}_m^n$.*

Proof. Let $(a, b) \in \mathcal{B}_m^n$. Then $< a >_{<m,n>} = < b >_{<m,n>}$ i.e., $a \vee a^m ea^n = b \vee b^m eb^n$. So $a \leq b \vee b^m eb^n$ and $b \leq a \vee a^m ea^n$. Now $a^m e \leq (b \vee b^m eb^n)^m e$ and $b^m e \leq (a \vee a^m ea^n)^m e$. Thus, by lemma 2.1, $a^m e \leq b^m e$ and $b^m e \leq a^m e$. Now

$$< a >_{<m,0>} = a \vee a^m e \leq b \vee b^m eb^n \vee a^m e \leq b \vee b^m eb^n \vee b^m e = < b >_{<m,0>}$$

and

$$< b >_{<m,0>} = b \vee b^m e \leq a \vee a^m ea^n \vee b^m e \leq a \vee a^m ea^n \vee a^m e = < a >_{<m,0>}.$$

Therefore $< a >_{<m,0>} = < b >_{<m,0>}$. Similarly we may show that $< a >_{<0,n>} = < b >_{<0,n>}$. Thus $(a, b) \in \mathcal{H}_m^n$. Hence $\mathcal{B}_m^n \subseteq \mathcal{H}_m^n$.

□

Remark 4.2 The converse of the Lemma 4.8 is not true in general as shown by the following example:

Example 4.1 [15] Let $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the multiplicative semigroup under multiplication *mod* 8. Then the power semigroup $P(Z_8) = (P(Z_8), \cdot, \cup, \cap)$, where $'\cdot'$ is the extension of the multiplication of Z_8 to the power set $P(Z_8)$ of Z_8, is an *le*-semigroup with the largest element $e = Z_8$. Observe that

$$\{2\} \vee \{2\}e\{2\} = \{2\} \cup \{2\}Z_8\{2\} = \{0, 2, 4\},$$
$$\{6\} \vee \{6\}e\{6\} = \{6\} \cup \{6\}Z_8\{6\} = \{0, 4, 6\},$$

which shows that $\{2\}$ is not $\mathcal{B}(= \mathcal{B}_1^1)$-related to $\{6\}$. On the other hand, as

$$\{2\} \vee \{2\}e = \{2\} \cup \{2\}Z_8 = \{0, 2, 4, 6\},$$
$$\{6\} \vee \{6\}e = \{6\} \cup \{6\}Z_8 = \{0, 2, 4, 6\},$$

we have $\{2\}\mathcal{R}\{6\}$. Since $P(Z_8)$ is commutative, the relations $\mathcal{R}(= {}_1\mathcal{I})$ and $\mathcal{H}(= \mathcal{H}_1^1)$ coincide. Thus $\{2\}\mathcal{H}_1^1\{6\}$ which implies that $\mathcal{H}_1^1 \nsubseteq \mathcal{B}_1^1$.

Theorem 4.9 *In any (m,n)-regular le-semigroup S, $\mathcal{B}_m^n = \mathcal{H}_m^n$.*

Proof. Let $(a,b) \in \mathcal{H}_m^n$. Then, by Lemma 4.7, $a^m ea^n = b^m eb^n$. As S is (m,n)-regular, we have $a \le a^m ea^n$ and $b \le b^m eb^n$. So $< a >_{<m,n>} = a \vee a^m ea^n = a^m ea^n$ and $< b >_{<m,n>} = b \vee b^m eb^n = b^m eb^n$. Thus $< a >_{<m,n>} = < b >_{<m,n>}$ i.e., $(a,b) \in \mathcal{B}_m^n$. So $\mathcal{H}_m^n \subseteq \mathcal{B}_m^n$. Hence, by Lemma 4.8, $\mathcal{B}_m^n = \mathcal{H}_m^n$.

□

Lemma 4.10 *In any le-semigroup S, $\mathcal{B}_m^n \subseteq \mathcal{Q}_m^n$.*

Proof. Let $(a,b) \in \mathcal{B}_m^n$. Then $< a >_{<m,n>} = < b >_{<m,n>}$ i.e., $a \vee a^m ea^n = b \vee b^m eb^n$. So $a \le b \vee b^m eb^n$ and $b \le a \vee a^m ea^n$. Therefore $a^m e \le (b \vee b^m eb^n)^m e \le b^m e$ and $ea^n \le e(b \vee b^m eb^n)^n \le eb^n$. Thus $a^m e \wedge ea^n \le b^m e \wedge eb^n$. Now

$$\begin{aligned}(a)_{<m,n>} = a \vee (a^m e \wedge ea^n) \le b \vee b^m eb^n \vee (a^m e \wedge ea^n) \le \\ b \vee b^m eb^n \vee (b^m e \wedge eb^n) = (b)_{<m,n>}.\end{aligned}$$

Similarly, as $b \le a \wedge a^m ea^n$, we may show that $(b)_{<m,n>} \le (a)_{<m,n>}$. Thus $(a)_{<m,n>} \le (b)_{<m,n>}$ i.e., $(a,b) \in \mathcal{Q}_m^n$. Hence $\mathcal{B}_m^n \subseteq \mathcal{Q}_m^n$, as required.

□

Remark 4.3 The converse of the Lemma 4.10 is not true in general. It is easy to verify from Example 4.1 that $\{2\}\mathcal{Q}_1^1(= \mathcal{Q})\{6\}$ while $\{2\}$ is not $\mathcal{B}_1^1(= \mathcal{B})$-related to $\{6\}$. Thus $\mathcal{Q}_1^1 \not\subseteq \mathcal{B}_1^1$.

Theorem 4.11 *Let S be an (m,n)-regular le-semigroup. Then $\mathcal{Q}_m^n = \mathcal{B}_m^n$.*

Proof. Let $(a,b) \in \mathcal{Q}_m^n$. Then $(a)_{<m,n>} = (b)_{<m,n>}$ i.e., $a \vee (a^m e \wedge ea^n) = b \vee (b^m e \wedge eb^n)$. So $a \le b \vee (b^m e \wedge eb^n)$ and $b \le a \vee (a^m e \wedge ea^n)$. As $\mathcal{Q}_m^n \subseteq \mathcal{H}_m^n$, we have $(a,b) \in \mathcal{H}_m^n$. So, by Lemma 4.7, $a^m ea^n = b^m eb^n$. Now

$$\begin{aligned}< a >_{<m,n>} &= a \vee a^m ea^n \\ &\le b \vee (b^m e \wedge eb^n) \vee a^m ea^n && (\text{because } a \le b \vee (b^m e \wedge eb^n) \\ &\le b \vee (b^m e \wedge eb^n) \vee b^m eb^n && (\text{as } a^m ea^n = b^m eb^n) \\ &= b \vee (b^m e \wedge eb^n) && (\text{because } b^m eb^n \le b^m e \wedge eb^n) \\ &= b \vee (b^m e)^m (eb^n)^n && (\text{by Theorem 3.5}) \\ &\le < b >_{<m,n>} .\end{aligned}$$

Similarly, as $b \le a \vee (a^m e \wedge ea^n)$, we have $< b >_{<m,n>} \le < a >_{<m,n>}$. Therefore $< a >_{<m,n>} = < b >_{<m,n>}$. Thus $(a,b) \in \mathcal{B}_m^n$. This implies that $\mathcal{Q}_m^n \subseteq \mathcal{B}_m^n$. Hence, by Lemma 4.10, $\mathcal{Q}_m^n = \mathcal{B}_m^n$.

□

Lemma 4.12 *In any le-semigroup, $\mathcal{Q}_m^n \subseteq \mathcal{H}_m^n$.*

Proof. Let S be any le-semigroup and let $(a,b) \in \mathcal{Q}_m^n$. Then $(a)_{<m,n>} = (b)_{<m,n>}$ i.e. $a \vee (a^m e \wedge ea^n) = b \vee (b^m e \wedge eb^n)$. So $a \le b \vee (b^m e \wedge eb^n)$ and $b \le a \vee (a^m e \wedge ea^n)$. Now

$$< a >_{<m,0>} = a \vee a^m e \leq (b \vee (b^m e \wedge eb^n)) \vee (b \vee (b^m e \wedge eb^n))^m e \leq < b >_{<m,0>}$$

and

$$< a >_{<0,n>} = a \vee ea^n \leq (b \vee (b^m e \wedge eb^n)) \vee e(b \vee (b^m e \wedge eb^n))^n \leq < b >_{<0,n>} .$$

Similarly, as $b \leq a \vee (a^m e \wedge ea^n)$, we have $< b >_{<m,0>} \leq < a >_{<m,0>}$ and $< b >_{<0,n>} \leq < a >_{<0,n>}$. Therefore $< a >_{<m,0>} = < b >_{<m,0>}$ and $< a >_{<0,n>} = < b >_{<0,n>}$ implying that $(a, b) \in \mathcal{H}_m^n$. Hence $\mathcal{Q}_m^n \subseteq \mathcal{H}_m^n$, as required.

□

Corollary 4.13 *Let S be an (m, n)-regular le-semigroup. Then $\mathcal{B}_m^n = \mathcal{Q}_m^n = \mathcal{H}_m^n$.*

Lemma 4.14 *If B_x and B_y are two (m, n)-regular $\mathcal{B}_m^n$-classes contained in the same $\mathcal{H}_m^n$-class of an le-semigroup S, then $B_x = B_y$.*

Proof. As x and y are (m, n)-regular elements of S, $x \leq x^m ex^n$ and $y \leq y^m ey^n$. Therefore $< x >_{<m,n>} = x \vee (x^m ex^n) = x^m ex^n$ and $< y >_{<m,n>} = y \vee (y^m ey^n) = y^m ey^n$. Since x and y are contained in the same $\mathcal{H}_m^n$-class, by Lemma 4.7, $x^m ex^n = y^m ey^n$. So $< x >_{<m,n>} = < y >_{<m,n>}$. Therefore $x\mathcal{B}_m^n y$. Hence $B_x = B_y$.

□

Lemma 4.15 *If Q_x and Q_y are two (m, n)-regular $\mathcal{Q}_m^n$-classes contained in the same $\mathcal{H}_m^n$-class of an le-semigroup S, then they coincide.*

Proof. As x and y are (m, n)-regular elements of S, $x \leq x^m ex^n \leq x^m e \wedge ex^n$ and $y \leq y^m ey^n \leq y^m e \wedge ey^n$. Therefore $(x)_{<m,n>} = x \vee (x^m e \wedge ex^n) = x^m e \wedge ex^n x$ and $(y)_{<m,n>} = y \vee (y^m e \wedge ey^n) = y^m e \wedge ey^n$. Since x and y are contained in the same $\mathcal{H}_m^n$-class, by Lemma 4.7, $x^m e = y^m e$ and $ex^n = ey^n$. Therefore $x^m e \wedge ex^n = y^m e \wedge ex^n$. This implies that $(x)_{<m,n>} = (y)_{<m,n>}$. So $x\mathcal{Q}_m^n y$ implies $Q_x = Q_y$, as required.

□

5 $(m, 0)$-regularity $[(0, n)$-regularity$]$ and (m, n)-right weakly regularity of $\mathcal{B}_m^n$-classes, $\mathcal{Q}_m^n$-classes and $\mathcal{H}_m^n$-classes

Proposition 5.1 *A $\mathcal{Q}_m^n$-class Q of an le-semigroup S is (m,0)-regular [(0,n)-regular] if it contains an (m,0)-regular [(0,n)-regular] element.*

Proof. Let $a \in Q$ be an $(m, 0)$-regular element and $b \in Q$. Then $(b)_{<m,n>} = (a)_{<m,n>} = a \vee (a^m e \wedge ea^n) \leq a \vee a^m e = a^m e$ (since a is $(m, 0)$-regular, $a \leq a^m e$). By Lemma 4.10 and Lemma 4.7, $a^m e = b^m e$. So $(b)_{<m,n>} \leq b^m e$. Thus $b \leq b^m e$. Therefore b is a $(m.0)$-regular element and, hence, Q is $(m, 0)$-regular. The dual statement may be proved on similar lines.

□

Proposition 5.2 *An $\mathcal{H}^n_m$-class H of a $\vee e$-semigroup S is (m,0)-regular [(0,n)-regular] if it contains an (m,0)-regular [(0,n)-regular] element.*

Proof. Let $a \in H$ be an $(m,0)$-regular element and $c \in H$. Then $< b >_{<m,0>} =< a >_{<m,0>}= a \vee a^m e = a^m e$ (since a is $(m,0)$-regular, $a \leq a^m e$). By Lemma 4.7, $a^m e = b^m e$. This implies that $< b >_{<m,0>} \leq b^m e$. Hence $b \leq b^m e$. So b is $(m,0)$-regular element. Hence H is $(m,0)$-regular. The dual statement follows on the similar lines.

□

Corollary 5.3 *An le-semigroup S is $(m,0)$-regular [$(0,n)$-regular] if and only if each $\mathcal{Q}^n_m$-class ($\mathcal{H}^n_m$-class) of S contains an $(m,0)$-regular [$(0,n)$-regular] element.*

Definition 5.4 *Let S be a poe-semigroup and m, n be positive integers. An element a of S is said to be an (m,n)-right regular element if $a \leq a^m e a^n e$. Further S is said to be (m,n)-right weakly regular if each element of S is (m,n)-right weakly regular.*

Proposition 5.5 *A $\mathcal{B}^n_m$-class B of a $\vee e$-semigroup S is (m,n)-right weakly regular if it contains an (m,n)-right weakly regular element.*

Proof. Let $a \in B$ be an (m,n)-right weakly regular element and let $b \in B$ be any element of B. Then $a \leq a^m e a^n e$. So, it follows that $a^m e a^n \leq (a^m e a^n e)^m e (a^m e a^n e)^n \leq a^m e a^n e$. Since $a, b \in B$, we have $< b >_{<m,n>}=< a >_{<m,n>}= a \vee a^m e a^n \leq a \vee a^m e a^n e = a^m e a^n e$. By Lemmas 4.8 and 4.7, we have $a^m e a^n = b^m e b^n$. This implies that $< b >_{<m,n>}\leq b^m e b^n e$. So $b \leq b^m e b^n e$. Thus b is an (m,n)-right weakly regular element of B. Hence B is (m,n)-right weakly regular.

□

Proposition 5.6 *A $\mathcal{Q}^n_m$-class Q of an le-semigroup S is (m,n)-right weakly regular if it contains an (m,n)-right weakly regular element.*

Proof. Let $a \in Q$ be an (m,n)-right weakly regular element and let $b \in Q$ be any element of Q. Then $a \leq a^m e a^n e$. Thus $a^m e \leq (a^m e a^n e)^m e \leq a^m e a^n e$. Since $a, b \in Q$, we have $(b)_{<m,n>} = (a)_{<m,n>} = a \vee (a^m e \wedge e a^n) \leq a \vee a^m e \leq a \vee a^m e a^n e = a^m e a^n e$. As, by lemmas 4.10 and 4.7, $a^m e a^n = b^m e b^n$, we have $(b)_{<m,n>} \leq b^m e b^n e$. So $b \leq b^m e b^n e$. This implies that b is an (m,n)-right weakly regular element of Q and, hence, Q is (m,n)-right weakly regular.

□

Proposition 5.7 *An $\mathcal{H}^n_m$-class H of a $\vee e$-semigroup S is (m,n)-right weakly regular if it contains an (m,n)-right weakly regular element.*

Proof. On the lines similar to the proof of Proposition 5.6.

□

Corollary 5.8 *An le-semigroup S is (m,n)-right weakly regular if and only if each $\mathcal{B}^n_m$-class (resp. $\mathcal{Q}^n_m$-class, $\mathcal{H}^n_m$-class) of S contains an (m,n)-right weakly regular element.*

6 Strong (m, n)-quasi-ideal element

In a *poe*-semigroup S, meet of an $(m, 0)$-ideal element a and a $(0, n)$-ideal element b is an (m, n)-quasi-ideal if $(a \wedge b)^m e \wedge e(a \wedge b)^n$ exists. The converse statement does not hold in general. As an example [5], let $S = \{a, b, c, d\}$. Define a binary operation $'\cdot'$ as follows:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	d	d
c	a	c	c	a
d	a	c	b	d

Then the set $\mathcal{P}(S)$ of all non-empty subsets of S is a *poe*-semigroup under the binary operation of set products induced by the binary operation defined by the above table and partially ordered by set inclusion. The set $\{a, b\}$ is an $(1, 1)$-quasi-ideal element of S. The sets $\{a\}$, $\{a, b, c, d\}$, $\{a, c\}$ and $\{a\}$, $\{a, b, c, d\}$, $\{a, d\}$ are the $(1, 0)$ and $(0, 1)$-ideal element of S respectively, but $\{a, b\}$ is not the intersection of any $(1, 0)$ and $(0, 1)$-ideal element.

Definition 6.1 *Let S be a $\vee e$-semigroup and q be an (m, n)-quasi-ideal element of S. Then q is said to be a strong (m, n)-quasi-ideal element of S if q can be written as the meet of an $(m, 0)$-ideal element and a $(0, n)$-ideal element.*

Theorem 6.2 *Let S be a $\vee e$-semigroup and q be an (m, n)-quasi-ideal element of S. Then the following are equivalent:*
(1) *q is a strong (m, n)-quasi-ideal element of S;*
(2) $< q >_{<m,0>} \wedge < q >_{<0,n>} = q$;
(3) $< a >_{<m,0>} \wedge < b >_{<0,n>} \leq q \quad (\forall a, b \leq q)$.

Proof. (1) $\Rightarrow$ (2). Let q be a strong (m, n)-quasi-ideal element of S. Then $q = a \wedge b$ for some $(m, 0)$-ideal element a and $(0, n)$-ideal element b of S. Therefore $< a >_{<m,0>} = a$ and $< b >_{<0,n>} = b$. Now, as $< q >_{<m,0>} =< a \wedge b >_{<m,0>} \leq < a >_{<m,0>} = a$ and $< q >_{<0,n>} =< a \wedge b >_{<0,n>} \leq < b >_{<0,n>} = b$, it follows that $< q >_{<m,0>} \wedge < q >_{<0,n>} \leq a \wedge b = q$. Again, as $q^m e \wedge eq^n \leq q$, $q = q \vee (q^m e \wedge eq^n)$. Therefore $q \leq q \vee q^m e =< q >_{<m,0>}$ and $q \leq q \vee eq^n =< q >_{<0,n>}$. Thus $q \leq < q >_{<m,0>} \wedge < q >_{<0,n>}$. Hence $< q >_{<m,0>} \wedge < q >_{<0,n>} = q$, as required.
(2) $\Rightarrow$ (3). Obvious.
(3) $\Rightarrow$ (1). As $q \leq q$, by (3), we have $< q >_{<m,0>} \wedge < q >_{<0,n>} \leq q$. Also, as $q \leq < q >_{<m,0>} \wedge < q >_{<0,n>}$, we have $< q >_{<m,0>} \wedge < q >_{<0,n>} = q$. Since $< q >_{<m,0>}$ and $< q >_{<0,n>}$ are $(m, 0)$ and $(0, n)$-ideal elements of S respectively, by definition, q is a strong (m, n)-quasi-ideal element of S. □

Lemma 6.3 *Let S be an le-semigroup and let $a \in S$. Then the strong (m, n)-quasi-ideal element of S generated by the element a, denoted by $q_s(a)$, is equal to*

$$< a >_{<m,0>} \wedge < a >_{<0,n>} .$$

Proof. Let $a \in S$. Then $< a >_{<m,0>} \wedge < a >_{<0,n>}$ is a strong (m,n)-quasi-ideal element of S. Let q be any strong (m,n)-quasi-ideal element of S such that $a \leq q$. Then, by Theorem 6.2, we have $< a >_{<m,0>} \wedge < a >_{<0,n>} \leq q$. Therefore $q_s(a) =< a >_{<m,0>} \wedge < a >_{<0,n>}$.

□

Theorem 6.4 *Let S be an (m,n)-regular le-semigroup. Then each (m,n)-quasi-ideal element is a strong (m,n)-quasi-ideal element.*

Proof. Let S be an (m,n)-regular *le*-semigroup and q be an (m,n)-quasi-ideal element of S. As $q \leq q^m e q^n$, we have $q \leq q^m e$ and $q \leq eq^n$. Therefore $q^m e = q \vee q^m e =< q >_{<m,0>}$ and $eq^n = q \vee eq^n =< q >_{<0,n>}$. As $q \leq< q >_{<m,0>} \wedge < q >_{<0,n>}= q^m e \wedge eq^n \leq q$, we have $< q >_{<m,0>} \wedge < q >_{<0,n>}= q$. Hence, by Theorem 6.7, q is a strong (m,n)-quasi-ideal element of S.

□

Lemma 6.5 *Each (m,n)-quasi-ideal element in a distributive le-semigroup is a strong (m,n)-quasi-ideal element.*

Proof. Straightforward.

□

Lemma 6.6 *Each $\mathcal{H}_m^n$-class H of S has a strong (m,n)-quasi-ideal element which is the greatest element of H and is equal to $< a >_{<m,0>} \wedge < a >_{<0,n>}$ for $a \in H$.*

Proof. Let $a \in H$. Then, by Lemma 4.3, $<< a >_{<m,0>} \wedge < a >_{<0,n>}>_{<m,0>} =< a >_{<m,0>}$ and $<< a >_{<0,n>} \wedge < a >_{<0,n>}>_{<0,n>}=< a >_{<0,n>}$. So, we have $(< a >_{<m,0>} \wedge < a >_{<0,n>}, a) \in {}_m\mathcal{I}$ and $(< a >_{<0,n>} \wedge < a >_{<0,n>}, a) \in \mathcal{I}_n$. Now take any element $b \in H$. As $b \leq< b >_{<m,0>}=< a >_{<m,0>}$ and $b \leq< b >_{<0,n>}=< a >_{<0,n>}$, we have $b \leq< a >_{<m,0>} \wedge < a >_{<0,n>}$. Therefore $< a >_{<m,0>} \wedge < a >_{<0,n>}$ is the greatest strong (m,n)-quasi-ideal element of H.

□

Theorem 6.7 *Let S be an le-semigroup. Then $\mathcal{H}_m^n = \mathcal{Q}_m^n$ if and only if each (m,n)-quasi-ideal element is a strong (m,n)-quasi-ideal element.*

Proof. Let $(a,b) \in \mathcal{H}_m^n$. Therefore $< a >_{<m,0>}=< b >_{<m,0>}$ and $< a >_{<0,n>} =< b >_{<0,n>}$. As each (m,n) quasi-ideal element is a strong (m,n) quasi-ideal element, therefore, by Theorem 6.2, we have

$$\begin{aligned}
(a)_{<m,n>} &=< (a)_{<m,n>} >_{<m,0>} \wedge < (a)_{<m,0>} >_{<0,n>} \\
&=< a >_{<m,0>} \wedge < a >_{<0,n>} \\
&=< b >_{<m,0>} \wedge < b >_{<0,n>} \\
&=< (b)_{<m,n>} >_{<m,0>} \wedge < (b)_{<m,0>} >_{<0,n>} \\
&= (b)_{<m,n>}.
\end{aligned}$$

Thus $(a,b) \in \mathcal{Q}_m^n$. Hence, by Lemma 4.10, $\mathcal{Q}_m^n = \mathcal{H}_m^n$.

Conversely assume that q be any (m,n)-quasi-ideal element of S. Therefore $q \in \mathcal{Q}_m^n(a)$ for some $a \in S$. Let $x \in \mathcal{Q}_m^n(a)$. Then $(x)_{<m,n>} = (q)_{<m,n>} = q$ implies $x \leq q$. Thus q is the greatest element of $\mathcal{Q}_m^n(a)$. By assumption $\mathcal{Q}_m^n(a) = \mathcal{H}_m^n(a)$. Therefore, by Lemma 6.3, we have $q =< a >_{<m,n>} \wedge < a >_{<0,n>}$. Hence q is a strong (m,n)-quasi-ideal element of S.

□

Corollary 6.8 *In any distributive le-semigroup,* $\mathcal{H}_m^n = \mathcal{Q}_m^n$.

Proof. The proof follows by Lemma 4.15 and Theorem 6.7.

□

Motivation and Application: The main motivation of the present paper is to introduce the equivalence relations ${}_m\mathcal{I}, \mathcal{I}_n, \mathcal{B}_m^n, \mathcal{Q}_m^n$ and $\mathcal{H}_m^n$ on an *le*-semigroup and enhance the understanding of different classes of *le*-semigroups ((m,n)-regular, $(m,0)$-regular, $(0,n)$-regular, (m,n)-right weakly regular) by considering the structural influence of the equivalence relations ${}_m\mathcal{I}, \mathcal{I}_n, \mathcal{B}_m^n, \mathcal{Q}_m^n$ and $\mathcal{H}_m^n$. In particular, if we take $m = 1 = n$ in the equivalence relations ${}_m\mathcal{I}, \mathcal{I}_n, \mathcal{B}_m^n, \mathcal{Q}_m^n$ and $\mathcal{H}_m^n$, then we get the equivalence relations $\mathcal{L}, \mathcal{R}, \mathcal{B}, \mathcal{Q}$ and $\mathcal{H}$. Again, if we take $m = 1 = n$ in Theorem 3.5, Lemma 4.6, Lemma 4.7, Lemma 4.8, Theorem 4.9, Lemma 4.10, Theorem 4.11, Lemma 4.12, Lemma 4.14, Lemma 4.15, Corollary 6.8, then all results of the papers [15–17] are deduced as corollaries which is the main application of the results of the paper and a testimony of the genuineness of the notions introduced in the paper.

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