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The holomorph of an extra-special p-group

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Abstract Let G be a group and $S(G)$ be a group of permutations on the set G. Define the holomorph of G to be the normalizer of the image in $S(G)$ of the right regular representation, $\mathrm{Hol}(G) = \mathrm{N}_{S(G)}(\rho(G)) = \mathrm{Aut}(G)\rho(G)$. If N is a regular subgroup of $S(G)$, then $\mathrm{N}_{S(G)}(N)$ is isomorphic to the holomorph of G. Miller has shown that the group $T(G) = \mathrm{N}_{S(G)}(\mathrm{Hol}(G))/\mathrm{Hol}(G)$ acts regularly on the set of the regular subgroups N of $S(G)$ which are isomorphic to G, and have the same holomorph as G, in the sense that $\mathrm{N}_{S(G)}(N) = \mathrm{Hol}(G)$. In this paper we study $T(G)$, when G is an extra-special p-group.

Keywords holomorph · multiple holomorph · regular subgroup · extra-special p-group

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1 Introduction

Let G be a group and $S(G)$ be the group of permutations on the set G. The image $\rho(G)$ of the right regular representation of G, is a regular subgroup of $S(G)$, and its normalizer is the semidirect product of $\rho(G)$ by the automorphism group $\mathrm{Aut}(G)$ of G,

$$\mathrm{N}_{S(G)}(\rho(G)) = \mathrm{Aut}(G)\rho(G).$$

We will refer to this group as the holomorph $\mathrm{Hol}(G)$ of G. More generally, if N is a regular subgroup of $S(G)$, then $\mathrm{N}_{S(G)}(N)$ is isomorphic to the holomorph of N. Let us thus consider the set

$$\mathcal{H}(G) = \{N \leq S(G) : \ N \textit{ is regular}, \ N \cong G \textit{ and } \mathrm{N}_{S(G)}(N) = \mathrm{Hol}(G)\}$$

of the regular subgroups of $S(G)$ which are isomorphic to G, and have in some sense the same holomorph as G.
G. A. Miller has shown [6] that the so-called multiple holomorph of G

$$\mathrm{NHol}(G) = \mathrm{N}_{S(G)}(\mathrm{Hol}(G)) = \mathrm{N}_{S(G)}(\mathrm{N}_{S(G)}(\rho(G)))$$

acts transitively on $\mathcal{H}(G)$, and thus the group

$$T(G) = \mathrm{NHol}(G)/\mathrm{Hol}(G)$$

acts regularly on $\mathcal{H}(G)$.
Let G be a non-abelian group. The inversion map $\mathrm{inv} : g \mapsto g^{-1}$ on G, which conjugates $\rho(G)$ to the image $\lambda(G)$ of the left regular representation λ, induces an involution in $T(G)$. One expects $T(G)$ to contain further involutions. More generally, let N be a non-abelian, regular subgroup of G that has the same holomorph as G (that is, such that $N_{S(G)}(N) = \mathrm{Hol}(G) = N_{S(G)}(\rho(G))$), and consider the group $S(N)$. The inversion map inv_N on N yields an involution in $T(N)$. This corresponds to an involution in $T(G)$ under the standard isomorphism $S(N) \to S(G)$ induced by the bijection $N \to G$ that maps $n \in N$ to $1^n \in G$. Under this isomorphism that maps $\rho(N)$ to N, inv_N maps onto an involution in $T(G)$. So each such N yields an involution in $T(G)$, although different N need not induce different involutions in $T(G)$.
Recently T. Kohl has described [5] the set $\mathcal{H}(G)$ and the group $T(G)$ for G dihedral or generalized quaternion. In [2] Caranti and Dalla Volta redo, via a commutative ring connection, the work of Mills [7], which determined $\mathcal{H}(G)$ and $T(G)$ for G a finitely generated abelian group. In [1] Caranti and Dalla Volta have studied the case of finite perfect groups. In all these cases, $T(G)$ turns out to be an elementary abelian 2-group.
T. Kohl mentions in [5] an example which $T(G)$ is a non-abelian 2-group. Let p be a prime. The finite p-group G is called extra-special if the center, commutator subgroup and Frattini subgroup of G all coincide and are cyclic of order p. This paper aims to find $T(G)$ for an extra-special p-group G with exponent $p > 2$. The present paper consists of three sections. In Section 2, we give some results about the automorphism of G, where G is an extra-special p-group. In Section 3, we prove that if G is an extra-special p-group and $\exp(G) = p > 2$, then $T(G)$ is cyclic of order $p - 1$.

We recall some standard material from [1] and [3] and complement it with a couple of Lemmas that will be useful in the rest of the paper.
Let G be a group. Denote by $S(G)$ the group of permutations of the set G, under left-to-right composition. Let

$$\begin{cases} \rho : G \to S(G) \\ \quad g \mapsto (h \mapsto hg), \end{cases}$$

be the right regular representation of G.

Definition 1.1 *The holomorph of G is*

$$\mathrm{Hol}(G) = \mathrm{N}_{S(G)}(\rho(G)) = \mathrm{Aut}(G)\rho(G).$$

Theorem 1.2 *(see [3], Theorem 1.2) Let G be a finite group. The following data are equivalent.*

i) A regular normal subgroup N of $\mathrm{Hol}(G)$.
ii) A map $\gamma : G \to \mathrm{Aut}(G)$ such that for $g, h \in G$ and $\beta \in \mathrm{Aut}(G)$

$$\begin{cases} \gamma(gh) = \gamma(h)\gamma(g), \\ \gamma(g^\beta) = \gamma(g)^\beta. \end{cases}$$

iii) A group operation $\circ$ on G such that for $g, h, k \in G$

$$\begin{cases} (gh) \circ k = \ (g \circ k)k^{-1}(h \circ k), \\ \mathrm{Aut}(G) \leq \mathrm{Aut}(G, \circ). \end{cases}$$

The data of (1)-(3) are related as follows.

1) $g \circ h = g^{\gamma(h)}$ for $g, h \in G$.
2) Each element of N can be written uniquely in the form $\nu(h) = \gamma(h)\rho(h)$, for some $h \in G$.
3) For $g, h \in G$ one has $g^{\nu(h)} = g \circ h$.
4) The map

$$\begin{cases} \nu : (G, \circ) \to N \\ \qquad h \quad \mapsto \gamma(h)\rho(h) \end{cases}$$

is an isomorphism.

An element of $S(G)$ conjugating $\rho(G)$ to N can be modified by a translation, and assumed to fix 1. We have the following Lemma

Lemma 1.3 *(see [1], Lemma 4.2) Suppose $N \in \mathcal{H}(G)$, and let $\vartheta \in \mathrm{NHol}(G)$ such that $\rho(G)^\vartheta = N$ and $1^\vartheta = 1$. Then*

$$\vartheta : G \to (G, \circ)$$

is an isomorphism such that

$$\rho(g)^\vartheta = \nu(g^\vartheta) = \gamma(g^\vartheta)\rho(g^\vartheta),$$

for $g \in G$.
Conversely, an isomorphism $\vartheta : G \to (G, \circ)$ conjugates $\rho(G)$ to N.

The next Lemma shows that an isomorphism ϑ as in Lemma 1.3 determines γ uniquely.

Lemma 1.4 *(see [3], Lemma 1.5) Let $N \in \mathcal{H}(G)$. If*

$$\vartheta : G \to (G, \circ),$$

is an isomorphism, then

$$(gh)^\vartheta = (g^\vartheta)^{\gamma(h^\vartheta)}h^\vartheta,$$

for $g, h \in G$, It follows that γ associated to N is given by

$$g^{\gamma(h)} = (g^{\vartheta^{-1}} h^{\vartheta^{-1}})^{\vartheta} h^{-1}.$$

In the next Lemma we note that γ is well defined for the elements of $T(G)$.

Lemma 1.5 *(see [3], Lemma 1.6) Let $N \in \mathcal{H}(G)$, and let $\vartheta : G \to (G, \circ)$ be an isomorphism.*

If $\alpha \in \mathrm{Aut}(G)$, then $\alpha\vartheta$ represents the same element of $T(G)$, and has the same associated γ.

Remark 1.1 (see [1], Theorem 5.2) For a goup G, denote by ι the morphism

$$\begin{cases} \iota : G \to \mathrm{Aut}(G) \\ \quad g \mapsto (h \mapsto h^g = g^{-1}hg), \end{cases}$$

that maps $g \in G$ to the inner automorphism $\iota(g)$ it induces.

We write $\mathrm{Aut}_c(G)$ for the group of central automorphisms of G, that is

$$\mathrm{Aut}_c(G) = \{\alpha \in \mathrm{Aut}(G) \mid \ [g, \alpha] = g^{-1}\alpha(g) \in Z(G), \ \ \forall g \in G\}.$$

Lemma 1.6 *(see [3], Lemma 2.1) Let G be a finite p-group of class two, and N is a regular normal subgroup of* $\mathrm{Hol}(G)$ *. The following are equivalant.*

i) $G' \leq ker(\gamma)$, *that is,* $\gamma(G') = 1$,
ii) $\gamma(G)$ *is abelian,*
iii) $[\gamma(G), G] \leq Z(G)$, *that is,* $\gamma(G) \leq \mathrm{Aut}_c(G)$,
iv) $[\gamma(G), G] \leq ker(\gamma)$.

Moreover, these conditions imply $[G', \gamma(G)] = 1$.

We now introduce a linear technique that will simplify the calculations in the main section.

Proposition 1.7 *(see [3], Proposition 2.2) Let G be a finite p-group of class two, for $p > 2$. There is a one-to-one correspondence between*

1) A regular normal subgroup N of $\mathrm{Hol}(G)$ *such that*

(a) $\gamma(G) \leq \mathrm{Aut}_c(G)$, *and*
(b) $[Z(G), \gamma(G)] = 1$,

and
2) The bilinear maps

$$\triangle : G/Z(G) \times G/G' \to Z(G)$$

such that

$$\triangle(g^\beta, h^\beta) = \triangle(g,h)^\beta$$

for $g \in G/Z(G),\ h \in G/G'$ *and* $\beta \in \mathrm{Aut}(G)$. *The correspondence is given by*

$$\triangle(gZ(G), hG') = [g, \gamma(h)], \qquad \text{for } g, h \in G.$$

Lemma 1.8 *(see [3], Lemma 2.4) Let* G *be a finite* p*-group of class two, and* N *is a regular normal subgroup of* $\mathrm{Hol}(G)$. *Suppose*

i) $\gamma(G) \leq \mathrm{Aut}_c(G)$, *and*
ii) $[Z(G), \gamma(G)] = 1$.

Then for $g, h \in G$ *we have*

$$\begin{aligned}[\nu(g), \nu(h)] &= \nu([g,h][g,\gamma(h)][h,\gamma(g)]^{-1}) \\ &= \nu([g,h] + \triangle(g,h) - \triangle(h,g)).\end{aligned}$$

Remark 1.2 (see [3], Section 3) Let G be a finite p-group of class two, for $p > 2$. The map

$$\begin{aligned}\triangle_c : G/Z(G) \times G/G' &\to G' \\ (gZ(G), hG') &\mapsto [g,h]^c,\end{aligned}$$

for integers c, satisfy the condition (2) of Proposition 1.7. To Δ_c we then associate the γ_c given by

$$g^{\gamma_c(h)} = g\Delta_c(g,h) = g[g,h]^c = g^{h^c},$$

that is,

$$\gamma_c(h) = \iota(h^c)$$

All these γ_c yields regular normal subgroups N of $\mathrm{Hol}(G)$. To see which γ_c yield subgroups $N \in \mathcal{H}(G)$, consider, for integers d coprime to p, consider the bijection $\vartheta_d \in S(G)$ given by the d-th power map, $\vartheta_d : x \longmapsto x^d$ on G, and write d' for the inverse of d modulo $\exp(G)$. we can see that the ϑ_d are all distinct as d ranges in the integers coprime to p between 1 and $\exp(G/Z(G)) - 1$. Also the ϑ_d yield a subgroup of $T(G)$ isomorphic to the units of the integers modulo $\exp(G/Z(G))$.

2 Preliminaries

In this section we state some lemmas to get an automorphism of G, where G is an extra-special p-group and $\exp(G) = p$. We also use this result in the main theorem.

Lemma 2.1 *(see [4], Corollary 1.5) Suppose that* G *is a finite group and* $g \in G$. *Let* D *be the conjugacy class of* G *containing* g. *Then* $|D| = |G : C_G(g)|$.

Lemma 2.2 *Let G be an extra-special p-group, such that $x \in G \setminus Z(G)$, then $|G : C_G(x)| = p$.*

Proof. For every element $x \in G \setminus Z(G)$, the conjugacy class $cl(x) = xG' = xZ(G)$. On the other hand by Lemma 2.1, we have $|cl(x)| = |G : C_G(x)|$, so $p = |Z(G)| = |xZ(G)| = |cl(x)|$, thus $|G : C_G(x)| = p$.

□

Lemma 2.3 *Let G be an extra-special p-group, such that $x \in G \setminus Z(G)$, $y \in G \setminus C_G(x)$ and $[x, y] \neq 1$. Then $|G : C_G(x) \cap C_G(y)| = p^2$.*

Proof. First $C_G(x) \neq C_G(y)$. Since $x \in C_G(x)$, but $y \notin C_G(x)$. Hence $C_G(x)C_G(y) = G$, as $|G : C_G(x)| = |G : C_G(y)| = p$. It follows that

$$|G : C_G(x) \cap C_G(y)| \leq |G : C_G(x)||G : C_G(y)| = p^2.$$

On the other hand, $p \mid |G : C_G(x) \cap C_G(y)|$, thus $p \leq |G : C_G(x) \cap C_G(y)| \leq p^2$, therefore $|G : C_G(x) \cap C_G(y)| = p^t$ where $1 \leq t \leq 2$. If $t = 1$, then $|G : C_G(x) \cap C_G(y)| = p$ and $|G|/|G : C_G(x) \cap C_G(y)| = p$, so $|G| = |G : C_G(x) \cap C_G(y)|p$, using Lemma 2.2, $|G| = |C_G(x)|p$. Then $|C_G(x)| = |G : C_G(x) \cap C_G(y)|$ and $C_G(x) \cap C_G(y) \subseteq C_G(x)$, we have $C_G(x) = C_G(y) \cap C_G(x)$, therefore $C_G(x) \subseteq C_G(y)$. Similarly we have $C_G(y) \subseteq C_G(x)$, which is a contradiction, and so $t = 2$.

□

Lemma 2.4 *Let G be an extra-special p-group of exponent p, and $x, y \in G$, such that $[x, y] \neq 1$. Then the subgroup $K = \langle x, y \rangle$ is an extra-special p-group of order p^3 and exponent p.*

Proof. By Lemma 2.3, for $x \in G \setminus Z(G)$, there exists $y \in G \setminus C_G(x)$, such that $[x, y] \neq 1$. As $G' = Z(G) \cong C_p$. Furthermore x^p and y^p lie in $Z(G)$, as $G/Z(G)$ is elementary abelian. Thus $K = \langle x, y, Z(G) \rangle$ has order p^3. Since $[x, y] \neq 1$, and $\exp(G) = p$, so K is non-abelian with $\exp(K) = p$.

□

Theorem 2.5 *(see [8] Theorem 5.3.8) An extra-special p-group is a central product of non-abelian subgroups of order p^3 and has order p^{2n+1}. Conversely a finite central product of non-abelian groups of order p^3 is an extra-special p-group.*

Lemma 2.6 *Let G be an extra-special p-group, such that $x \in G \setminus Z(G)$, $y \in G \setminus C_G(x)$ and $[x, y] \neq 1$. If $K = \langle x, y \rangle$ and $C = C_G(x) \cap C_G(y)$. Then $G = KC$ and $K \cap C = Z(G)$, where $[x, c] = [y, c] = 1$, for every $c \in C$.*

Proof. By Theorem 2.5, G is a central product of non-abelian subgroups of order p^3, clearly we have $G = KC$ and $Z(G) = K \cap C$, as $[x, c] = [y, c] = 1$, for every $c \in C$.

□

Theorem 2.7 *(see [8] Von Dyck's Theorem) Let G have a presentation*

$$G = \langle x_1, x_2, ..., x_n |\ r_j(x_1, x_2, ..., x_n), j \in J \rangle$$

so that $G = F/R$, *where* F *is the free group with basis* $\{x_1, x_2, ..., x_n\}$ *and* R *is the normal subgroup generated by* r_j. *If* M *is a group with* $M = \langle y_1, y_2, ..., y_n \rangle$ *and if* $r_j(y_1, y_2, ..., y_n) = 1$ *for all* j, *then there is a surjective homomorphism* $G \longrightarrow M$ *with* $x_i \longmapsto y_i$, *for all* i.

Now we describe automorphisms of $K = \langle x, y \rangle$. Let F be a free group with basis $\{u, v\}$, we consider the homomorphism $f : F \longrightarrow K$, where $f(u) = x^{a^{-1}}$ and $f(v) = y^a$, we have $L = \langle u^p v^p, [u, [u, v]], [v, [u, v]] \rangle \leq kerf$. By Theorem 2.7, we have $F/L \cong K$ and $\mathrm{Aut}(F/L) \cong \mathrm{Aut}(K)$.

Proposition 2.8 *Let* $\sigma \in \mathrm{Aut}(K)$ *and* $\tau \in \mathrm{Aut}(C)$, *where* $K = \langle x, y \rangle$ *and* $C = C_G(x) \cap C_G(y)$. *There exists a unique automorphism* β *of* G *such that* $\beta \mid_K = \sigma$ *and* $\beta \mid_C = \tau$, *then* $\beta \in \mathrm{Aut}(G)$ *if and only if* $\sigma \mid_{K \cap C} = \tau \mid_{K \cap C} = I$.

Proof. Suppose that $\sigma \mid_{K \cap C} = \tau \mid_{K \cap C} = I$, Lemma 2.6 implies that $G = KC$, so we can define $\beta : G \longrightarrow G$, where, $\beta(tc) = t^\sigma c^\tau$ for every $t \in K,\ c \in C$. We show that β is well defined. Let $tc = t^{'}c^{'}$, so we have $(t^{'})^{-1}t = c^{'}c^{-1} \in K \cap C = Z(G)$. Since σ and τ are identity on $Z(G)$, therefore we can write $t^\sigma = (t^{'}((t^{'})^{-1}t))^\sigma = (t^{'})^\sigma((t^{'})^{-1}t)^\sigma$. Similarly $c^\tau = (c^{'}((c^{'})^{-1}c))^\tau = (c^{'})^\tau((c^{'})^{-1}c)^\tau$. Thus $(tc)^\beta = (t^{'}c^{'})^\beta$. As K and C are normal subgroups and $\sigma \in \mathrm{Aut}(K), \tau \in \mathrm{Aut}(C)$, we can easily see that $\beta \in \mathrm{Aut}(G)$. Conversely, if $\beta \in \mathrm{Aut}(G)$, we can define $(tc)^\beta = t^\sigma c^\tau$ which $t \in K,\ c \in C$ and $z^\beta = z$, for each $z \in Z(G) = K \cap C$. Since $\beta \mid_K = \sigma$, $\beta \mid_C = \tau$ and $\beta \mid_{Z(G)} = I$, so we have $\sigma \mid_{Z(G)} = \tau \mid_{Z(G)} = I$.

□

3 Main results

Theorem 3.1 *Let* G *be an extra-special* p*-group and* $\exp(G) = p > 2$. *Then* $T(G)$ *is (cyclic) of order* $p - 1$.

Proof. $\Phi(G) = Z(G) = G' \cong C_p$ is the definition of G being an extra-special p-group. By [9], we have that H/I $\cong Sp(2n, p)$, where H is normal subgroup of $\mathrm{Aut}(G)$ consisting of all elements which act trivially on $Z(G)$ and I = $\mathrm{Inn}(G)$ is the group of inner automorphisms. Since $\gamma : G \longrightarrow \mathrm{Aut}(G)$, by Theorem 1.2, we have $\gamma(G)$ is normal in $\mathrm{Aut}(G)$. As G is a p-group of class two and $Z(G) = G' \cong C_p$, we have $\mathrm{Inn}(G) = \mathrm{Aut}_c(G)$, therefore $\gamma(G) \leq \mathrm{Aut}_c(G)$. Moreover Lemma 1.6, implies

$$[Z(G), \gamma(G)] = [G', \gamma(G)] = 1.$$

We can thus use the map Δ introducted in Proposition 1.7. Here $V = G/G'$ and $W = G'$ are F_p-vector spaces, and W is isomorphic to the exterior

square $\bigwedge^2 V$ of V.
We begin with the case where $p > 3$. We can choose $a \in F_p^* \setminus \{1, -1\}$. Consider the automorphism β of V (and thus of G), such that $\beta(x) = x^{a^{-1}}$, $\beta(y) = y^a$, for $x, y \in K$ and $\beta(c) = c$, for $c \in C$. Hence, we can write

$$\Delta(x,x)^\beta = \Delta(x^\beta, x^\beta) = \Delta(x^{a^{-1}}, x^{a^{-1}}) = \Delta(x,x)^{a^{-2}}.$$

Since β is the identity on $W = G'$ and $a \neq 1$ or $a^2 \neq 1$, we obtain $\Delta(x,x) = 0$ Similarly, $\Delta(y,y)^\beta = \Delta(y^\beta, y^\beta) = \Delta(y,y) = 0$, $\Delta(x,c) = 0$ and $\Delta(y,c) = 0$. Now we consider $\Delta(x,y)$, according to the definition of the automorphism β, where we have

$$\Delta(x,y)^\beta = \Delta(x^\beta, y^\beta) = \Delta(x^{a^{-1}}, y^a) = \Delta(x,y)^{a^{-1}a} = \Delta(x,y).$$

Since β is the identity on $W = G'$ and by Proposition 1.7, we have $\Delta(x,y) = [x,y]^\lambda$ for some λ. Therefore

$$x^{\gamma(y)} = x\Delta(x,y) = x[x,y]^\lambda = x[x,y^\lambda] = xx^{-1}x^{y^\lambda} = x^{y^\lambda}.$$

That is $\gamma(y) = \iota(y^\lambda)$. For $\lambda \neq -1/2$, this is uniquely associated, as we have seen in [3] (see Section 3), to $x^{\vartheta_d} = x^d$, where $d = (1+2\lambda)'$. When $\lambda = -1/2$, there is no such ϑ_d. In this case, Lemma 1.8 implies that

$$\begin{aligned}[\nu(x), \nu(y)] &= \nu([x,y] + \Delta(x,y) - \Delta(y,x)) \\ &= \nu([x,y] - [x,y]^{-1/2} + [y,x]^{1/2}) \\ &= \nu(0) = 1,\end{aligned}$$

that is , $N \cong (G, \circ)$ is abelian, thus N is not isomorphic to G. When $\lambda \neq -1/2$, we choose $d = (1+2\lambda)' \in F_p^*$, so that $G \cong (G, \circ)$. Since we have $p-1$ choices for ϑ_d, according to Remark 1.2, it follows that $T(G)$ is cyclic of order $p-1$.
The case $p = 3$. We choose $a = -1$, since $\Delta(x,x) \in G'$ and β is identity on $W = G'$, so we have $\Delta(x,x) = 0$. Similarly $\Delta(x,c) = 0$, $\Delta(y,c) = 0$ and $\Delta(y,y) = 0$. We also have $\Delta(x,y)^\beta = \Delta(x^\beta, y^\beta) = \Delta(x,y)$.

□

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