An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
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Directed approximate fibrations

I. Pop

Abstract Using the notions of δ-metric space and directed metric space, we introduce and study the notion of directed approximate fibration. Some important relations of these d-maps with direct fibrations and Dold fibrations, previously introduced by the author, are established (Theorems 4.4 and 6.1). Some significant examples are given.

Keywords directed space · directed fibration · directed Dold fibration · δ-metric space · directed metric space · directed ANR (dANR) space · directed space with non-negative scalars · directed approximate homotopy lifting property (dAHLP) · directed approximate fibration · directed approximate lifting pair

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1 Introduction

Directed Algebraic Topology is a recent subject which arose from the study of some phenomena in the analysis of concurrency, traffic networks, space-time models etc. ([4], [6], [7], [8]). It was systematically developed by Marco Grandis ([10], [11], [12]). The directed spaces have privileged directions and directed paths therein do not need to be reversible. M. Grandis introduced and studied its 'non-reversible' homotopical tools corresponding to ordinary homotopies, fundamental group and fundamental n-groupoids: directed homotopies, fundamental monoids and fundamental n-categories. Also some directed homotopy constructions were considered: pushouts and pullbacks, mapping cones and homotopy fibres, suspensions and loops, cofibre and fibre sequences. As for directed fibrations, M. Grandis refers to these (more precisely to the so-called lower and upper d-fibrations) only in relation with the directed h-pullbacks. But then this theme was approached by L. Fajstrup [5], E. Goubault et al. [9], and I. Pop [14–16].

The basics of Directed Algebraic Topology which we will use are taken from the 2003 paper of Grandis [10]:

A *directed space*, or a *d-space*, is a topological space X equipped with a set $\mathrm{d}X$ of continuous maps $a : \mathbf{I} = [0,1] \longrightarrow X$, called *directed paths*, or *d-paths*, satisfying the following three axioms:

(i) (constant paths) every constant map $\mathbf{I} \longrightarrow X$ is a directed path,

(ii) (reparametrisation) $\mathrm{d}X$ is closed under composition with (weakly) increasing maps $\mathbf{I} \longrightarrow \mathbf{I}$,

(iii) (concatenation) $\mathrm{d}X$ is closed under concatenation (the product of consecutive paths, which will be denoted by $*$ or by $+$).

We use the notations $\underline{X}$ or $\uparrow X$ if X is the underlying topological space; if $\underline{X}$ (or $\uparrow X$) is given, then the set of directed paths is denoted by $\mathrm{d}\underline{X}$ (resp. $\mathrm{d}\uparrow X$) and the underlying space by $|\underline{X}|$) (resp. $|\uparrow X|$).
The standard d-interval with the directed paths given by increasing (weakly) maps $\mathbf{I} \longrightarrow \mathbf{I}$ is denoted by $\uparrow\mathbf{I} = \uparrow[0,1]$.

A *directed map*, or *d-map* $f : \underline{X} \longrightarrow \underline{Y}$, is a continuous mapping between d-spaces which preserves the directed paths: if $a \in \mathrm{d}\underline{X}$ then $f \circ a \in \mathrm{d}\underline{Y}$. The category of directed spaces and directed maps is denoted by $\mathrm{d}\mathbf{Top}$ (or $\uparrow\mathbf{Top}$). A directed path $a \in \mathrm{d}\underline{X}$ defines a directed map $a : \uparrow\mathbf{I} \longrightarrow \underline{X}$ which is also a *path* of $\underline{X}$.

For two points $x, x' \in \underline{X}$ we write $x \preceq x'$ if there exists a directed path from x to x'. The equivalence relation $\simeq$ spanned by $\preceq$ yields the partition of a d-space $\underline{X}$ in its directed path components and a functor $\uparrow\Pi_0 : \mathrm{d}\mathbf{Top} \longrightarrow \mathbf{Set}$, $\uparrow\Pi_0(\underline{X}) = |\underline{X}|/\simeq$. A non-empty d-space $\underline{X}$ is *directed path connected* if $\uparrow\Pi_0(\underline{X})$ contains only one element.

The *directed cylinder* of a d-space $\underline{X}$ is the d-space $\uparrow(\underline{X} \times \mathbf{I})$, denoted by $\underline{X}\times \uparrow \mathbf{I}$ or $\uparrow \mathbf{I}\underline{X}$, for which a path $\mathbf{I} \longrightarrow |\underline{X}| \times \mathbf{I}$ is directed if and only if its components $\mathbf{I} \longrightarrow |\underline{X}|$, $\mathbf{I} \longrightarrow \mathbf{I}$ are directed. The directed maps $\partial^\alpha : \underline{X} \longrightarrow \underline{X}\times \uparrow\mathbf{I}$, $\alpha = 0, 1$, defined by $\partial^\alpha(x) = (x, \alpha)$, are called the *faces* of the cylinder.

If $f, g : \underline{X} \longrightarrow \underline{Y}$ are directed maps, a *directed homotopy* φ from f to g, denoted by $\varphi : f \longrightarrow g$, or $\varphi : f \preceq g$, is a d-map $\varphi : \underline{X}\times \uparrow\mathbf{I} \longrightarrow \underline{Y}$ such that $\partial^0 \circ \varphi = f$ and $\partial^1 \circ \varphi = g$. The equivalence relation defined by the d-homotopy preorder $\preceq$ is denoted by $f \simeq_d g$ or simply $f \simeq g$. This means that there exists a finite sequence $f \preceq f_1 \succeq f_2 \preceq f_3 \succeq ...g$.

2 Directed fibrations

In this section we resume sections 2 and 3 from [14] including definitions and results that are necessary or interesting for this paper.

Definition 2.1 *Let $p : \underline{E} \to \underline{B}$,$f : \underline{X} \to \underline{B}$ be directed maps. A d-map $f' : \underline{X} \to \underline{E}$ is called a directed lift of f with respect to p if $p \circ f' = f$.*

Definition 2.2 *A directed map $p : \underline{E} \to \underline{B}$ is said to have the directed homotopy lifting property with respect to a d-space $\underline{X}$ if, given d-maps $f' : \underline{X} \to \underline{E}$ and $\varphi : \underline{X}\times \uparrow \mathbf{I} \to \underline{B}$, and $\alpha \in \{0, 1\}$, such that $\varphi \circ \partial^\alpha = p \circ f'$,*

there is a directed lift of φ, $\varphi' : \underline{X}\times \uparrow \mathbf{I} \to \underline{E}$, *with respect to* p, $p \circ \varphi' = \varphi$, *such that* $\varphi' \circ \partial^\alpha = f'$.

$$\begin{array}{ccc} \underline{E} & \xrightarrow{\quad p \quad} & \underline{B} \\ {\scriptstyle f'}\uparrow & \overset{=}{\nearrow_{\varphi'}} & \overset{=}{\uparrow}{\scriptstyle \varphi} \\ \underline{X} & \xrightarrow[\quad \partial^\alpha \quad]{} & \underline{X}\times \uparrow \boldsymbol{I} \end{array} \tag{2.1}$$

A directed map $p : \underline{E} \to \underline{B}$ *is called directed (Hurewicz) fibration if* p *has the directed homotopy lifting property with respect to all directed spaces (dHLP). This property is also called the directed covering homotopy property (dCHP).*

Properties of directed fibrations

P1. If $p : \underline{E} \to \underline{B}$ has the directed homotopy lifting property with respect to $\underline{X}$ and $f_0, f_1 : \underline{X} \to \underline{B}$ are directed homotopic, $f_0 \simeq_d f_1$, then f_0 has a directed lift with respect to p iff f_1 has this property.

P2. Let $p : \underline{E} \to \underline{B}$ be a directed fibration and $a \in d\underline{B}$ with $a(\alpha) = p(e_\alpha), e_\alpha \in |\underline{E}|$, and $\alpha \in \{0, 1\}$. Then there exists a directed path $a_\alpha \in d\underline{E}$ which is a lift of a, $p \circ a_\alpha = a$, with the α-endpoint e_α, $a_\alpha(\alpha) = e_\alpha$.

Examples of directed fibrations

E1. Let $\underline{F}$ and $\underline{B}$ be arbitrary directed spaces and let $p : \underline{B} \times \underline{F} \to \underline{B}$ be the projection. Then p is a directed fibration.

E2. Let $p : E \to |\underline{B}|$ be a Hurewicz fibration. For the space E consider the maximal d-structure compatible with $d\underline{B}$ and p, i.e., $d(\uparrow E) = \{a \in E^I / p \circ a \in d\underline{B}\}$. Then $p :\uparrow E \to \underline{B}$ is a directed fibration.

E3. If $p : \underline{E} \to \underline{B}$ is a directed fibration, then the opposite map $p : \underline{E}^{op} \to \underline{B}^{op}$ is also a directed fibration.

For an intrinsic characterization of the dHLP we need some notations.

Given a d-map $p : \underline{E} \to \underline{B}$ and $\alpha \in \{0, 1\}$, we consider the following d-subspace of the product (in d**Top**) $\underline{E} \times \underline{B}^{\uparrow \mathbf{I}}$

$$\underline{B}_\alpha = \{(e, \omega) \in \underline{E} \times \underline{B}^{\uparrow \mathbf{I}} | \omega(\alpha) = p(e)\} \tag{2.2}$$

(The d-structure of $\underline{B}^{\uparrow \mathbf{I}}$ is given by the exponential law, $d\mathbf{Top}(\uparrow \mathbf{I}, d\mathbf{Top}(\uparrow \mathbf{I}, \underline{B})) \approx d\mathbf{Top}(\uparrow \mathbf{I}\times \uparrow \mathbf{I}, \underline{B})$,[10]).

Definition 2.3 *A directed lifting pair for a directed map* $p : \underline{E} \to \underline{B}$ *is a pair of d-maps*

$$\lambda_\alpha : \underline{B}_\alpha \to \underline{E}^{\uparrow \mathbf{I}}, \alpha = 0, 1, \tag{2.3}$$

satisfying the following conditions:

$$\lambda_\alpha(e, \omega)(\alpha) = e, \tag{2.4}$$

$$p \circ \lambda_\alpha(e, \omega) = \omega, \tag{2.5}$$

for each $(e, \omega) \in \underline{B}_\alpha$.

Theorem 2.4 *(i) A directed map $p : \underline{E} \to \underline{B}$ is a directed fibration if and only if there exists a directed lifting pair for p.*

(ii) If $p : \underline{E} \to \underline{B}$ is a directed fibration, the d-spaces $\underline{B}_0$ and $\underline{B}_1$ are d-homotopy equivalent.

3 Directed metric spaces. Directed ANR

In this paper we use directed spaces and maps defined by so called δ- metric spaces, which is why we recall the definition and some of their properties after the book [12].

A *δ-metric space* is a set X equipped with a *δ-metric* $\delta : X \times X \to [0, \infty]$, satisfying the axioms

$$\delta(x, x) = 0, \quad \delta(x, y) + \delta(y, z) \geq \delta(x, z). \tag{3.1}$$

Examples:

1) The *standard δ-line* $\delta\mathbf{R}$ has the δ-metric $\delta(x, y) = y - x$ if $x \leq y$, and $\delta(x, y) = \infty$, otherwise.

2) The *standard δ-interval* $\delta\mathbf{I} = \delta[0, 1]$ has the subspace structure of δ-line.

3) *A product* $\prod_i X_i$ of δ-metric spaces has a δ-metric given by $\delta(\mathbf{x}, \mathbf{y}) = sup \quad \delta_i(x_i, y_i)$, for $\mathbf{x} = (x_i), \mathbf{y} = (y_i)$. In particular $\delta\mathbf{R}^n$ is a δ-space with $\delta(\mathbf{x}, \mathbf{y}) = sup_i(y_i - x_i), \mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n)$, if $x_i \leq y_i, \forall i = 1, ...n$, and $\delta(\mathbf{x}, \mathbf{y}) = \infty$, otherwise. As a δ-subspace of $\delta\mathbf{R}^n$ we have $\delta\mathbf{I}^n$.

4) *The tensor product* $X \otimes Y$ of two δ-spaces is the cartesian product with the δ-metric given by $\delta((x, y), (x', y')) = \delta(x, x') + \delta(y, y')$. In particular, between the standard models are also the $\delta\mathbf{R}^{\otimes n}$ and $\delta\mathbf{I}^{\otimes n}$.

5) *n-dimensional* $(n > 0)$, *δ-sphere* is the quotient $\delta\mathbf{S}^n = (\delta\mathbf{I}^{\otimes n})/(\partial\mathbf{I}^n)$.

6) *The opposite δ-metric space* $R(X) = X^{op}$ has the opposite δ-metric, $\delta^{op}(x, y) = \delta(y, x)$. A δ-metric is symmetric if $\delta = \delta^{op}$.

A δ-metric space $X{=}(X, \delta)$ has an associated bitopological space (X, τ^{-1}, τ^{+}). At the point $x_0 \in X$, the *past topology* τ^{-} (resp. the *future topology* τ^{+}) has a canonical system of fundamental neighborhoods consisting of the *past discs* D^{-} (resp. *future discs* D^{+}) centred in x_0

$D^{-}(x_0, \varepsilon) = \{x \in X | \delta(x, x_0) < \varepsilon\}$, $D^{+}(x_0, \varepsilon) = \{x \in X | \delta(x_0, x) < \varepsilon\}, (\varepsilon > 0)$.

If X is a δ- metric space, then its *symmetrised metric space* $!(X, \delta) = (X, !\delta)$ has the δ -metric $!\delta$ defined by

$$!\delta(x, x') = inf_{\mathbf{x}} \left(\sum_j (\delta(x_{j-1}, x_j) \wedge \delta(x_j, x_{j-1})) \right), (\mathbf{x} = (x_0, ..., x_p), x_0 = x, x_p = x'). \tag{3.2}$$

The topology defined by the metric $!\delta$ is called the *symmetric topology* of the δ-metric space X.

If X is a δ- metric space, for a map of sets $a : \mathbf{I} = [0, 1] \to X$ and $\mathbf{t}$ a finitely strictly increasing sequence $(0 = t_0 < t_1 < ... < t_p = 1)$, define

$L_{\mathbf{t}}(a) = \sum_j \delta(a(t_{j-1}), a(t_j))$, and then the length of a by $L(a) = sup_{\mathbf{t}} L_{\mathbf{t}}(a)$. If $L(a)$ is finite, a is called *L-feasible*.

Now if X is a δ-metric space, consider X equipped with the symmetric topology, defined by symmetric δ-metric $!\delta$, and with the d-structure for which a (continuous) path $a : \mathbf{I} \to X$ is distinguished if and only if it is L-feasible. We will name the directed space obtained in this way a *directed metric space*.

Let $\uparrow \mathcal{M}tr$ the category of directed metric spaces and directed maps.

Definition 3.1 *Let $f, g : \underline{X} \to \underline{Y}$ be d-maps between directed metric spaces and $\mathcal{A} = \{U_\lambda | \lambda \in \Lambda\}$ a covering of $\underline{Y}$. Then if for each $x \in |\underline{X}|$ there exists an index $\lambda \in \Lambda$ such that $f(x), g(x) \in U_\lambda$, f and g are said to be $\mathcal{A}$-near (or $\mathcal{A}$-close). Particularly, if for $\varepsilon > 0$ and for each $x \in |\underline{X}|$, $\delta(f(x), g(x)) < \varepsilon$, f and g are said to be ε-near (or ε-close).*

A d-homotopy $\varphi : \underline{X} \times \uparrow I \to \underline{Y}$, with $\underline{X}$ and $\underline{Y}$ directed metric spaces, is called an $\mathcal{A}$-d-homotopy (ε-d-homotopy) if for arbitrary $x \in |\underline{X}|$, there exists an index $\lambda \in \Lambda$ such that $h(\{x\} \times [0,1]) \subset U_\lambda$ (resp. $\varphi(\{x\} \times [0,1]) \subset D^-(x,\varepsilon) \cap D^+(x,\varepsilon)$).

If the d-maps $f, g : \underline{X} \to \underline{Y}$ between directed metric spaces are d-homotopic by a sequence of $\mathcal{A}$-d-homotopies (ε-d-homotopies), we say that f and g are $\mathcal{A}$- d-homotopic (ε-d-homotopic).

Definition 3.2 *A directed absolute neighborhood retract for the class $\uparrow \mathcal{M}tr$ (abbreviated dANR for the class $\uparrow \mathcal{M}tr$) is a directed metric space $\underline{Y}$ such that every directed homeomorphic image of $\underline{Y}$ as a closed directed subspace $\uparrow Y'$ of a directed metric space $\underline{Z}$ in $\uparrow \mathcal{M}tr$ is necessarily a directed neighborhood retract of $\underline{Z}$, i.e., $\uparrow Y'$ is a directed retract of an open d-subspace $\uparrow U$ of $\underline{Z}$.*

In order to prove an important proposition in the next section, we introduce a notion that is an adaptation in the context of the directed algebraic topology of the notion of topological linear space. This notion is inspired by the paper [2].

Definition 3.3 *A directed space with scalars ≥ 0 is a directed space $\underline{L}$ endowed with two directed map $+ : \underline{L} \times \underline{L} \to \underline{L}$, $. : \uparrow \mathbf{R}_{\geq 0} \times \underline{L} \to \underline{L}$, with $\uparrow \mathbf{R}_{\geq 0}$ the d-subspace of the directed real line $\uparrow \mathbf{R}$ of the nonnegative real numbers, such that the addition is an operation of commutative group and the scalar multiplication satisfies the usual axioms of a linear space.*

Example 3.1 1)$\uparrow \mathbf{R}$ with the usual addition, and with the multiplication $a.x$ with $a \geq 0$, 2) A product of d-spaces with scalars ≥ 0 ,particularly $\uparrow \mathbf{R}^n$, 3) If $\underline{X}$ is a directed space then the d-space $D(\underline{X})$ of the bounded directed real functions on $\underline{X}$ is a d-space with scalars ≥ 0.

Definition 3.4 *A subset in a d-space with scalars ≥ 0 $\underline{L}$ is said to be d-convex if for every finite numbers of points $x_0, ..., x_n \in \underline{L}$, we have $\alpha_0 x_0 + ... + \alpha_n x_n \in \underline{L}$, for arbitrary non-negative real numbers $\alpha_0, ..., \alpha_n$ satisfying*

the relation $\alpha_0 + ... + \alpha_n = 1$. For an arbitray set S in $|\underline{L}|$ the intersection of all d-convex sets of $\underline{L}$ is the smallest d-convex set of $\underline{L}$ which contains S, and it is called the d-convex hull of S in $\underline{L}$.

Definition 3.5 *A set S in a d-space with scalars ≥ 0 $\underline{L}$ is said to be bounded if for every neighborhood U of the zero element θ in $\underline{L}$ there exists a real number $\alpha > 0$ such that $S \subset \alpha U := \{\alpha.x | x \in U\}$.*

The following proposition is analogous to a classical result (e.g., [13], Theorem 2.1, Ch.III), and whose demonstration can be adapted for our directed context.

Proposition 3.6 *If $\underline{L}$ is a bounded directed space with scalars ≥ 0 and $\chi : \underline{L} \to D(\underline{L})$ is the canonical imbedding , then $\chi(\underline{L})$ is a closed subset of the d-convex hall of $\chi(\underline{L})$ in $D(\underline{L})$.*

Proposition 3.7 *If $\underline{Y}$ is a dANR and $\mathcal{A} = \{U_\lambda | \lambda \in \Lambda\}$ a given open covering of $\underline{Y}$, then there exists an open covering $\mathcal{B}$ of $\underline{Y}$, which is a refinement of $\mathcal{A}$, such that any two $\mathcal{B}$-near directed maps $f, g : \underline{X} \to \underline{Y}$ defined on arbitrary d-space $\underline{X}$ are $\mathcal{A}$-homotopic.*

Proof. By Proposition 3.6, we may consider $\underline{Y}$ as closed subset of a d-convex set Z in the δ-Banach space $D(\underline{Y})$ and with the structure of d-metric space. Since $\underline{Y}$ is a dANR, there exists and open neighborhood $\uparrow W$ of $\underline{Y}$ in $\underline{Z}$ together with a d-retraction $r :\uparrow W \supset \underline{Y}$. From the given covering $\mathcal{A}$ of $\underline{Y}$, we construct $\mathcal{B}$ as follows. At first we use the fact that $r^{-1}(\mathcal{A}) = \{r^{-1}(U_\lambda) | \lambda \in \Lambda\}$ is an open covering of $\uparrow W$. Since $|\uparrow W|$ is an open set of a d-convex set $\uparrow Z$ in the locally d-convex space $D(\underline{Y})$, $r^{-1}(\mathcal{A})$ has an open refinement $\mathcal{G} = \{W_\mu | \mu \in M\}$ such that $\uparrow W_\mu$ is d-convex for every $\mu \in M$. For each $\mu \in M$, let $V_\mu = W_\mu \cap |\underline{Y}|$. Then $\mathcal{B} = \{V_\mu | \mu \in M\}$ is an open refinement of the given covering $\mathcal{A}$.

Now let $f, g : \underline{X} \to \underline{Y}$ be two $\mathcal{B}$-near d-maps. Since $\uparrow Z$ is d-convex, we can define a d-homotopy $k : \underline{X} \times \uparrow \mathbf{I} \to \uparrow Z$, by taking $k(x,t) = (1-t)f(x) + tg(x)$, for every $x \in \underline{X}$ and every $t \in \uparrow \mathbf{I}$.

Next let us prove that $k(x,t) \in W$ for every $x \in \underline{X}$ and every $t \in \uparrow \mathbf{I}$. Since f and g are $\mathcal{B}$-near, for an arbitrary point $x \in \underline{X}$ there exists an index $\mu \in M$ such that $f(x)$ and $g(x)$ are contained in $V_\mu \subset W_\mu$. Since W_μ is convex, it follows that $k(x,t) \in W_\mu \subset W$ for every $t \in \uparrow \mathbf{I}$. In fact, we have proved that $k : \underline{X} \times \uparrow \mathbf{I} \to \uparrow W$ is a directed $\mathcal{G}$-homotopy .

Finally, we define a d-homotopy $h : \underline{X} \times \uparrow \mathbf{I} \to \underline{Y}$, by taking $h(x,t) = r(k(x,t))$, for every $x \in \underline{X}$ and every $t \in \uparrow \mathbf{I}$. Now since k is $\mathcal{G}$-homotopy and $\mathcal{G}$ is a refinement of $r^{-1}(\mathcal{A})$, it follows that h is an $\mathcal{A}$-homotopy. On the other hand, since r is a d-retraction, we have that $f \simeq_d g$ by a directed $\mathcal{A}$-homotopy.

□

4 Directed approximate homotopy lifting property (dAHLP). Directed approximate fibrations.

Definition 4.1 *A directed map $p : \underline{E} \to \underline{B}$ between directed metric spaces has the directed approximate homotopy liting property (dAHLP) with respect to a d-space $\underline{X}$, provided that, given $\varepsilon > 0$, d-maps $f' : \underline{X} \to \underline{E}$, $\varphi : \underline{X} \times \uparrow \mathbf{I} \to \underline{B}$, and $\alpha \in \{0,1\}$, such that $\varphi \circ \partial^\alpha = p \circ f'$, there exists a directed homotopy $\varphi' : \underline{X} \times \uparrow \mathbf{I} \to \underline{E}$ such that $\varphi' \circ \partial^\alpha = f'$, and $\delta(p \circ \varphi', \varphi) < \varepsilon$, that is $\delta(p\varphi'(x,t), \varphi(x,t)) < \varepsilon$, for all $(x,t) \in |\underline{X}| \times I$.*

$$\begin{array}{ccc} \underline{E} & \xrightarrow{\quad p \quad} & \underline{B} \\ {\scriptstyle f'}\uparrow & \overset{=}{\swarrow}_{\varphi'} \quad {<\varepsilon} & \uparrow{\scriptstyle \varphi} \\ \underset{\partial^\alpha}{\underline{X}} & \xrightarrow{\qquad} & \underline{X} \times \uparrow \boldsymbol{I} \end{array} \tag{4.1}$$

Remark 4.1 Definition 4.1 can be reformulated in the terms of coverings as follows: A directed map $p : \underline{E} \to \underline{B}$ between directed metric spaces has the directed approximate homotopy liting property with respect to a d-space $\underline{X}$, provided that, given a cover $\mathcal{E}$ of $\underline{B}$, d-maps $f' : \underline{X} \to \underline{E}$, $\varphi : \underline{X} \times \uparrow \mathbf{I} \to \underline{B}$, and $\alpha \in \{0,1\}$, such that $\varphi \circ \partial^\alpha = p \circ f'$, there exists a directed homotopy $\varphi' : \underline{X} \times \uparrow \mathbf{I} \to \underline{E}$ such that $\varphi' \circ \partial^\alpha = f'$, and $p \circ \varphi'$ and φ, are $\mathcal{E}$-near.

Theorem 4.2 *A d-map $p : \underline{E} \to \underline{B}$ between directed metric spaces has the dAHLP with respect to all directed topological spaces if and only if and only if for every $\varepsilon > 0$ there is a pair of d-maps $\lambda^\varepsilon_\alpha : \underline{B}_\alpha \to \underline{E}^{\uparrow \mathbf{I}}$, such that $\lambda^\varepsilon_\alpha(e,\omega)(\alpha) = e$ and $\delta(p \circ \lambda^\varepsilon_\alpha(e,\omega), \omega) < \varepsilon$.*

Such a pair $(\lambda^\varepsilon_\alpha)_{\alpha \in \{0,1\}}$ will be called a directed approximate lifting pair for the directed map p.

Proof. If p has the dAHLP with respect to all directed topological spaces , $\alpha \in \{0,1\}$ and $\varepsilon > 0$, let $f'_\alpha : \underline{B}_\alpha \to \underline{E}$ and $F_\alpha : \underline{B}_\alpha \times \uparrow \mathbf{I} \to \underline{B}$ be the maps defined by $f'_\alpha(e,\omega) = e$, and $F_\alpha((e,\omega),t) = \omega(t)$. It is obvious that f'_α is continuous and a directed map as a projection. Regarding F_α, this is continuous since if D is an open set in $\underline{B}$, then $F^{-1}_\alpha(D) = \{((e,\omega),t) \in \underline{B}_\alpha \times \uparrow \mathbf{I} | \omega(t) \in D\}$, which is an open subset of $\underline{B}_\alpha \times \uparrow \mathbf{I}$. Then if $c : \uparrow \mathbf{I} \to \underline{B}_\alpha \times \uparrow \mathbf{I}$ is a directed path, write $c(t') = ((e(t'), \omega(t')), t(t'))$, with some directed paths $e : \uparrow \mathbf{I} \to \underline{E}, \omega : \uparrow \mathbf{I} \to \underline{B}_\alpha{}^{\uparrow \mathbf{I}}$, and $t : \uparrow \mathbf{I} \to \uparrow \mathbf{I}$. By the exponential d-structure of $\underline{B}^{\uparrow \mathbf{I}}$ ([10],p.291), the map $\widetilde{\omega} : \uparrow \mathbf{I} \times \uparrow \mathbf{I} \to \underline{B}$ defined by $\widetilde{\omega}(t',t'') = \omega(t')(t'')$ is directed. This implies that $(F_\alpha \circ c)(t') = \omega(t')(t(t')) = \widetilde{\omega}(t', t(t'))$ is directed. Hence F_α is directed. Then by the equality $F_\alpha((e,\omega),\alpha) = \omega(\alpha) = p(e) = (p \circ f'_\alpha)(e,\omega)$, i.e., $F_\alpha \circ \partial^\alpha = p \circ f'_\alpha$, by the dAHLP, for ε, there exists $F'_\alpha : \underline{B}_\alpha \times \uparrow \mathbf{I} \to \underline{E}$ such that $F'_\alpha((e,\omega),\alpha)) = f'_\alpha(e,\omega) = e$, and $\delta(p \circ F'_\alpha, F_\alpha) < \varepsilon$. Hence we can define $\lambda^\varepsilon_\alpha : \underline{B}_\alpha \to \underline{E}^{\uparrow \mathbf{I}}$ by $\lambda^\varepsilon_\alpha(e,\omega)(t) = F'_\alpha((e,\omega),t)$. By the same exponential structure, $\lambda^\varepsilon_\alpha$ is a directed map. Then $\lambda^\varepsilon_\alpha(e,\omega)(\alpha) = e$, and $\delta(p\lambda^\varepsilon_\alpha(e,\omega)(t), \omega(t)) = \delta(pF'_\alpha((e,\omega),t), \omega(t)) =$

$\delta(p(F'_\alpha(e,\omega),t),F_\alpha((e,\omega),t)<\varepsilon$, i.e., $\delta(p\circ\lambda^\varepsilon_\alpha(e,\omega),\omega)<\varepsilon$. Therefore, we have obtained a directed approximate lifting pair for p.

Conversely, if for $\varepsilon>0$ the pair $(\lambda^\varepsilon_\alpha)_{\alpha=0,1}$ is given, let $f':\underline{X}\to\underline{E}$, and $\varphi:\underline{X}\times\mathbf{I}\to\underline{B}$ be such that $\varphi\circ\partial^\alpha=p\circ f'$. Consider the directed map $g:\underline{X}\to\underline{B}^{\uparrow\mathbf{I}}$, defined by $g(x)(t)=\varphi(x,t)$. This permits to define $\varphi':\underline{X}\times\uparrow\mathbf{I}\to\underline{E}$, by $\varphi'(x,t)=\lambda^\varepsilon_\alpha(f'(x),g(x))(t)$. For this we have $\varphi'\circ\partial^\alpha=f'$ and $\delta(p\circ\varphi',\varphi)<\varepsilon$, such that p has the directed approximate homotopy lifting property.

□

Corollary 4.3 *If $p:\underline{E}\to\underline{B}$ has the dAHLP for all directed metric spaces, then p has the dAHLP for all directed spaces, i.e., p is a directed approximate fibration.*

Proof. In this case , $\underline{B}_\alpha\subset\underline{E}\times\underline{B}^{\uparrow\mathbf{I}}$ is a directed metric space, and the proof of necessity in Theorem 4.2 shows that a directed approximate lifting pair for p exists, which then on the basis of sufficiency involves the dAHLP for all directed spaces.

□

Theorem 4.4 *Let $p:\underline{E}\to\underline{B}$ be a directed surjection between dANR's for the class $\uparrow\mathcal{M}tr$. Suppose for each $\eta>0$ there is a d-map $p_\eta:\underline{E}\to\underline{B}$ such that $\delta(p_\eta,p)<\eta$ and p_η has dHLP with respect to a d-metric space $\underline{X}$. Then p has the dAHLP with respect to $\underline{X}$.*

Proof. Using the hypothesis, we will verify the definition of the dAHLP property in the variant of the Remark 4.1.

Suppose given a cover $\mathcal{E}$ of $\underline{B}$, d-maps $f':\underline{X}\to\underline{E}$, $\varphi:\underline{X}\times\uparrow\mathbf{I}\to\underline{B}$, and $\alpha\in\{0,1\}$, such that $\varphi\circ\partial^\alpha=p\circ f'$. Denote by $\mathcal{F}$ a twice star refinement of $\mathcal{E}$, and using Proposition 3.7, consider a cover $\mathcal{D}$ such that any two $\mathcal{D}$-near maps into $\underline{B}$ are $\mathcal{F}$ -homotopic. Now consider the d-map $p_\mathcal{D}:\underline{E}\to\underline{B}$ given by hypothesis, so that p and $p_\mathcal{D}$ are $\mathcal{D}$-near and therefor $p_\mathcal{D}\circ f'$ and $p\circ f'$ are $\mathcal{D}$-near. This implies that these d-maps are $\mathcal{F}$-homotopic. In addition, if $\underline{\alpha=0}$ we can suppose there is given a $\mathcal{F}$-d-homotopy $\kappa:\underline{X}\times\uparrow[-1,0]\to B$ such that $\kappa(x,-1)=(p_\mathcal{D}f')(x)$ and $K(x,0)=(pf')(x)$, $\forall x\in|\underline{X}|$. Now we define

$$\Phi:\underline{X}\times\uparrow[-1,1]\to\underline{B} \tag{4.2}$$

by

$$\Phi(x,t)=\begin{cases}\kappa(x,t),\text{ if }t\in[-1,0],\\ \varphi(x,t),\text{ if }t\in[0,1].\end{cases} \tag{4.3}$$

Then since $p_\mathcal{D}$ has dHLP with respect to $\underline{X}$, there is a d-homotopy $\Psi:\underline{X}\times\uparrow[-1,1]\to\underline{E}$, with $\Psi(x,-1)=f'(x)$ and $p_\mathcal{D}\circ\Psi=\Phi$.

Now chose a d-map $q:\underline{X}\to\uparrow(0,1)$ such that $\varphi(\{x\}\times[0,q(x)])$ is contained in some member of $\mathcal{F}$ for each $x\in\underline{X}$. Now define $\varphi':\underline{X}\times\uparrow\mathbf{I}\to E$ by

$$\varphi'(x,t)=\begin{cases}\Psi(x,\frac{2t}{q(x)}-1), & \text{if }0\le t\le\frac{q(x)}{2},\\ \Psi(x,2t-q(x)),\text{ if }\frac{q(x)}{2}\le t\le q(x),\\ \Psi(x,t), & \text{if }q(x)\le t\le 1.\end{cases} \tag{4.4}$$

Then $\varphi'(x,0) = \Psi(x,-1) = f'(x)$, such that it is sufficient to prove that $p \circ \varphi'$ and φ are $\mathcal{E}$-near.

If $t \in [0, \frac{q(x)}{2}]$, then $y = \frac{2t}{q(x)} - 1 \in [-1,0]$, so that $p\varphi'(x,t) = p\Psi(x,y)$. Compare $p\Psi(x,y)$ to $p_{\mathcal{D}}\Psi(x,y)$. These are $\mathcal{D}$-near and therefore $\mathcal{F}$-near. But $p_{\mathcal{D}}\Psi(x,y) = \Phi(x,y) = \varphi(x,y)$, which by choosing q is $\mathcal{F}$-close to $\varphi(x,0)$. So $p\varphi'(x,t)$ and $\varphi(x,t)$ are $\mathcal{F}$-near. But since $\mathcal{F}$ twice star refines $\mathcal{E}$, we have that $p\varphi'(x,t)$ is $\mathcal{E}$-near to $\varphi(x,t)$. Suppose $t \in [\frac{q(x)}{2}, q(x)]$. Then $p\varphi'(x,t) = p\Psi(x, 2t - q(x))$ which is $\mathcal{E}$-near to $p_{\mathcal{D}}\Psi(x, 2t - q(x)) = \Phi(x, 2t - q(x)) = \kappa(x, 2t - q(x)) = \varphi(x, 2t - q(x))$ which is $\mathcal{F}$-near to $\varphi(x,t)$. So $p\varphi'(x,t)$ and $\varphi(x,t)$ are $\mathcal{F}$-near and therefore $\mathcal{E}$-near. Finally, if $q(x) \leq t \leq 1$, $p\varphi'(x,t) = p\Psi(x,t)$ which is $\mathcal{E}$-near to $p_{\mathcal{D}}\Psi(x,t) = \Phi(x,t) = \varphi(x,t)$.

If $\underline{\alpha = 1}$, we choose a $\mathcal{F}$-d-homotopy $\kappa' : \underline{X}\times \uparrow [0,1] \to \underline{B}$ such that $\kappa'(x,0) = (pf')(x)$, and $\kappa'(x,1) = (p_{\mathcal{D}}f)(x), \forall x \in |\underline{X}|$. By this we define $\Phi' : \underline{X}\times \uparrow [-1,1] \to \underline{B}$ by

$$\Phi'(x,t) = \begin{cases} \varphi(x,t+1), \text{ if } t \in [-1,0], \\ \kappa'(x,t), \quad \text{if } t \in [0,1]. \end{cases} \tag{4.5}$$

Then there is a d-homotopy $\Psi' : \underline{X}\times \uparrow [-1,1] \to \underline{E}$ satisfying $\Psi'(x,1) = f'(x)$ and $p_{\mathcal{D}}\Psi' = \Phi'$.

Now chose a d-map $q' : \underline{X} \to \uparrow (0,1)$ such that $\varphi(\{x\} \times [q'(x),1])$ is contained in some member of $\mathcal{F}$ for each $x \in \underline{X}$. Now define $\varphi' : \underline{X}\times \uparrow \mathbf{I} \to E$ by

$$\varphi'(x,t) = \begin{cases} \Psi'(x, \frac{2t}{q'(x)} - 1), & \text{if } 0 \leq t \leq \frac{q'(x)}{2}, \\ \Psi'(x, 2t - q'(x)), & \text{if } \frac{q'(x)}{2} \leq t \leq q'(x), \\ \Psi'(x,t), & \text{if } q'(x) \leq t \leq 1. \end{cases} \tag{4.6}$$

Then $\varphi'(x,1) = \Psi'(x,1) = f'(x)$, and as above is verified that $p \circ \varphi'$ and φ are $\mathcal{E}$-near.

□

Corollary 4.5 *In the conditions of Theorem 4.4, if any d-map p_η is a directed fibration, then p is a directed approximate fibration.*

A slight change in the proof of Theorem 4.4 also gives the following proposition.

Proposition 4.6 *Let $p : \underline{E} \to \underline{B}$ be a d-map between dANR's. If p has dAHLP with respect to a d-space $\underline{X}$, then for every $\varepsilon > 0$ and $\alpha \in \{0,1\}$ there is $\eta > 0$ such that whenever $f' : \underline{X} \to \underline{E}$ and $\varphi : \underline{X}\times \uparrow \mathbf{I} \to \underline{B}$ are d-maps such that $\delta(p \circ f', \varphi \circ \partial^\alpha) < \eta$, there is a d-map $\varphi' : \underline{X}\times \uparrow \mathbf{I} \to \underline{E}$ such that $\varphi' \circ \partial^\alpha = f'$ and $\delta(p \circ \varphi', \varphi) < \varepsilon$.*

Example 4.1 This example is inspired by an example from [3]. In the directed space $\mathbf{R} \times \delta\mathbf{R}$ consider the d-subspace $d\mathbf{W}$, with $\mathbf{W}$ the"Warsaw circle" in $\mathbf{R}^2$, that is $W = W_1 \cup C$, where $W_1 = \{(x, sin\frac{1}{x})| - \frac{1}{2\pi} < x \leq$

$\frac{1}{2\pi}, x \neq 0\} \cup \{(0,y)| -1 \leq y \leq 1\}$ and C is an arc in $\mathbf{R}^2$ which meets W_1 only in its end points $(\frac{1}{2\pi}, 0)$ and $(-\frac{1}{2\pi}, 0)$. Also consider the directed subspace $d\mathbf{S}^1$ of $\mathbf{R} \times \delta\mathbf{R}$ and the product $d\mathbf{T}^2 = d\mathbf{S}^1 \times d\mathbf{S}^1$. Then there is a compactum $A \subset \mathbf{T}^2$ such that A is homeomorphic to $\mathbf{W}$ and such that for the point $x_0 \in \mathbf{S}^1$ there is a d-homeomorphism

$$h : d(\mathbf{T}^2 - A) \to d\mathbf{S}^1 \times d(\mathbf{S}^1 - \{x_0\}). \tag{4.7}$$

This is because $\mathbf{W}$ being shape equivalent to the circle $\mathbf{S}^1$ the spaces $\mathbf{R}^2 - \mathbf{W}$ and $\mathbf{R}^2 - \mathbf{S}^1$ are homeomorphic (see [1], Ch.VII, §7).

By using the d-map h we define the d-map

$$p : d\mathbf{S}^1 \times d\mathbf{S}^1 \to d\mathbf{S}^1, \tag{4.8}$$

given by

$$p(x) = \begin{cases} \pi h(x), \text{ if } x \in (\mathbf{S}^1 \times \mathbf{S}^1) - A, \\ x_0, \quad \text{if } x \in A. \end{cases} \tag{4.9}$$

This d-map has the property that $p^{-1}(x_0) = A$ and $p^{-1}(y)$ is an essential copy of $d\mathbf{S}^1$ in $d\mathbf{T}^2$ for each $y \neq x_0$

Now, using Corollary 4.5, we can prove that p is a directed approximate fibration. Let $\varepsilon > 0$ be given and U be an open interval in $\mathbf{S}^1$ such that $x_0 \in U \subset N(x_0, \frac{\varepsilon}{2})$. Then since $\uparrow p^{-1}(\mathbf{S}^1 - U)$ and $\uparrow p^{-1}(\overline{U})$ are d-homemorphic to $d\mathbf{S}^1 \times \delta\mathbf{I}$, the d-map $p| \uparrow p^{-1}(\mathbf{S}^1 - U)$ can be extended to a d-map $p_\varepsilon : d\mathbf{S}^1 \times d\mathbf{S}^1 \to d\mathbf{S}^1$ which is directed equivalent to π. Thus, p_ε is a d-fibration, and $\delta(p, p_\varepsilon) < \varepsilon$. By Corollary 4.5, p is a directed approximate fibration. On the other hand, p is not even a directed fibration. For suppose that p has the dHLP with respect to $\delta\mathbf{S}^1$.Now suppose $x_0 = (0, -1)$, and for an arbitrary point $x \in \mathbf{S}^1$ we choose a d-path ω from x to x_0 in $d\mathbf{S}^1$. Then consider the d-homotopy $\varphi : d\mathbf{S}^1 \times \uparrow \mathbf{I} \to d\mathbf{S}^1$ given by $\varphi(x,t) = \omega(t)$, and let $f' : d\mathbf{S}^1 \to \uparrow p^{-1}(x)$ be a d-homeomorphism. By assumption φ lifts to a d-homotopy $\varphi' : d\mathbf{S}^1 \times \uparrow \mathbf{I} \to d\mathbf{S}^1 \times d\mathbf{S}^1$, such that $\varphi'(x,0) = f'(x)$ and $\varphi'(x,1) \in A$ for each $x \in \mathbf{S}^1$. In other words, f' is d-homotopic with a d-map from $\delta\mathbf{S}^1$ to A which is d-null-homotopic. This is a contradiction because f' is a directed isomorphism.

5 Directed Dold fibrations

The directed Dold fibrations was defined and studied in [15]. We recall the definition and main properties of these d-applications needed here.

Definition 5.1 *A directed homotopy $\varphi : \underline{X} \times \uparrow \mathbf{I} \to \underline{Y}$ is called semistationary if either $\varphi(x,t) = \varphi(x, \frac{1}{2}), (\forall)x \in X, t \in [0, \frac{1}{2}]$ or $\varphi(x,t) = \varphi(x, \frac{1}{2}), (\forall)x \in X, t \in [\frac{1}{2}, 1]$. In the first case we say that φ is lower semistationary and in the second case we say that φ is upper semistationary.*

Definition 5.2 *A directed map $p : \underline{E} \to \underline{B}$ is called a directed Dold fibration if p has the directed covering homotopy property with respect to each d-space $\underline{X}$ and for all semistationary directed homotopies $\varphi : \underline{X} \times \uparrow \mathbf{I} \to \underline{B}$ (dCSHP).*

Given a d-map $p : \underline{E} \to \underline{B}$ and $\alpha \in \{0, 1\}$, we consider the following d-subspaces of the directed product $\underline{E} \times \underline{B}^{\uparrow \mathbf{I}}$:

$$\underline{B}^s_0 = \{(e, \omega) \in \underline{E} \times \underline{B}^{\uparrow \mathbf{I}} | \omega(t) = p(e), (\forall) t \in [0, \frac{1}{2}]\}, \tag{5.1}$$

and

$$\underline{B}^s_1 = \{(e, \omega) \in \underline{E} \times \underline{B}^{\uparrow \mathbf{I}} | \omega(t) = p(e), (\forall) t \in [\frac{1}{2}, 1]\}. \tag{5.2}$$

(The d-structure of $\underline{B}^{\uparrow \mathbf{I}}$ is given by the exponential law, $d\mathbf{Top}(\uparrow \mathbf{I}, d\mathbf{Top}(\uparrow \mathbf{I}, \underline{B})) \approx d\mathbf{Top}(\uparrow \mathbf{I} \times \uparrow \mathbf{I}, \underline{B})$,[10].)

Definition 5.3 *A directed semistationary lifting pair for a directed map $p : \underline{E} \to \underline{B}$ consists of a pair of d-maps*

$$\lambda^s_\alpha : \underline{B}^s_\alpha \to \underline{E}^{\uparrow \mathbf{I}}, \alpha = 0, 1, \tag{5.3}$$

satisfying the following conditions:

$$\lambda^s_\alpha(e, \omega)(\alpha) = e, \tag{5.4}$$

$$p \circ \lambda^s_\alpha(e, \omega) = \omega, \tag{5.5}$$

for each $(e, \omega) \in \underline{B}^s_\alpha$.

Theorem 5.4 *A directed map $p : \underline{E} \to \underline{B}$ is a directed Dold fibration if and only if there exists a directed semistationary lifting pair for p.*

6 Directed approximate fibrations and directed Dold fibrations

Theorem 6.1 *Let $p : \underline{E} \to \underline{B}$ be a directed map between metric spaces, with $|\underline{B}|$ a metric space. If p has the dCSHP for a d-space $\underline{X}$, then p has also dAHLP for $\underline{X}$.*

In the proof of this theorem we use the following lemma whose proof follows the steps in the proof of Lemma 3.3 of the paper [17].

Lemma 6.2 *Let $(\underline{B}, d)$ be a directed metric space and $\underline{B}^{\uparrow \mathbf{I}}$ the d-space of d-paths in $\underline{B}$. Then there exists a continuous map*

$$\delta : \underline{B}^{\uparrow \mathbf{I}} \times \uparrow (0, \infty) \to \uparrow (0, \infty), \tag{6.1}$$

directed with respect to the second argument, and such that : $t_1, t_2 \in [0, 1], |t_1 - t_2| \leq \delta(\omega, \varepsilon)$ implies $d(\omega(t_1), \omega(t_2) \leq \varepsilon$.

Proof of Theorem 6.1.
By Theorem 4.2 there exists for p a directed semistationary lifting pair

$$\lambda_\alpha^s : \underline{B}_\alpha^s \to \underline{E}^{\uparrow \mathbf{I}}, \alpha = 0, 1, \tag{6.2}$$

We shall prove that this implies that for every $\varepsilon > 0$, p admits a directed approximate lifting pair $\lambda_\alpha^\varepsilon : \underline{B}_\alpha \to \underline{E}^{\uparrow \mathbf{I}}$, and then we apply Theorem 4.2.
a) First we consider the d-map

$$s_\alpha : \underline{B}_\alpha \to \underline{B}_\alpha^s, \tag{6.3}$$

defined by $s_\alpha(e, \omega) = (e', \omega'_\alpha)$, with ω'_α given by

$$\omega'_0(t) = \begin{cases} \omega(0) = p(e), \text{ if } 0 \le t \le 1/2, \\ \omega(2t-1), \quad \text{if } 1/2 \le t \le 1. \end{cases} \tag{6.4}$$

and respectively by

$$\omega'_1(t) = \begin{cases} \omega(2t), \quad\quad\quad \text{if } 0 \le t \le 1/2, \\ \omega(1) = p(e), \text{ if } 1/2 \le t \le 1. \end{cases} \tag{6.5}$$

We can consider the spaces $\underline{B}_\alpha$ and $\underline{B}_\alpha^s$ as metric d-subspaces of the d-space $\underline{E} \times \underline{B}^{\uparrow \mathbf{I}}$ with the metric

$$d((e_1, \omega_1), (e_2, \omega_2)) = d(e_1, e_2) + d(\omega_1, \omega_2) = d(e_1, e_2) + sup\{d(\omega_1(t), \omega_2(t)) : t \in [0,1]\}. \tag{6.6}$$

Then we can see that s is a (uniformly) continuous map.
Indeed, $d(s(e_1, \omega_1), s(e_2, \omega_2)) = d((e_1, \omega'_1), (e_2, \omega'_2)) = d(e_1, e_2) + sup\{d(\omega'_1(t), \omega'_2(t)) : t \in [0,1]\} = d(e_1, e_2) + sup\{(d(\omega_1(t)), d(\omega_2(t)) : t \in [0,1])\} = d((e_1, \omega_1), (e_2, \omega_2))$, that is, s is an isometry.
b) Now let we consider the d-map $\chi : \underline{E}^{\uparrow \mathbf{I}} \times (\uparrow \mathbf{I})^{\uparrow \mathbf{I}} \to \underline{E}^{\uparrow \mathbf{I}}$, defined by

$$\chi(\theta, u)(t) = \theta(u(t)), \theta \in \underline{E}^{\uparrow \mathbf{I}}, u \in (\uparrow \mathbf{I})^{\uparrow \mathbf{I}}, t \in [0,1]. \tag{6.7}$$

Consider $\underline{E}^{\uparrow \mathbf{I}} \times (\uparrow \mathbf{I})^{\uparrow \mathbf{I}}$ with the product metric $d((\theta_1, u_1), (\theta_2, u_2)) = d(\theta_1, \theta_2) + d(u_1, u_2)$, and the d-space $\underline{E}^{\uparrow \mathbf{I}}$ with the metric of uniform convergence. Then we can see that χ is a (uniformly) continuous d-map.
Indeed, $d(\chi(\theta_1, u_1), \chi(\theta_2, u_2)) = sup\{d(\chi(\theta_1, u_1)(t), d(\chi(\theta_2, u_2)(t) : t \in [0,1]\} = sup\{d(\theta_1(u_1(t)), d(\theta_2(u_2(t)) : t \in [0,1]\} \le sup\{d(\theta_1(t), \theta_2(t)) : t \in [0,1]\} = d(\theta_1, \theta_2) \le d(\theta_1, \theta_2) + d(u_1, u_2) = d((\theta_1, u_1), (\theta_2, u_2))$.
c) Now, by Lemma 6.2, we can construct the following d-map

$$\delta' : \underline{B}_\alpha \to \uparrow (0,1), \tag{6.8}$$

$$\delta'(e, \omega) = \frac{\delta(\omega, \frac{\varepsilon}{2})}{1 + \delta(\omega, \frac{\varepsilon}{2})}, \tag{6.9}$$

for the above fixed $\varepsilon > 0$.
This d-map has the following property:

The conditions $(e,\omega)\in \underline{B}_\alpha$ and $t_1,t_2\in[0,1]$ with $|t_1-t_2|\leq \delta'(e,\omega)$ imply the inequality $d(\omega(t_1),\omega(t_2))<\frac{\varepsilon}{2}$.

d) We use the map $\delta':\underline{B}_\alpha\to\uparrow(0,1)$ to define the d-map

$$\psi:\underline{B}_\alpha\to(\uparrow \mathbf{I})^{\uparrow \mathbf{I}}, \tag{6.10}$$

by

$$\psi(e,\omega)(t)=\begin{cases}\frac{t}{2\delta'(e,\omega)}, & \text{if } 0\leq t\leq \delta'(e,\omega),\\ \frac{t+(1-\delta'(e,\omega))}{2(1-\delta'(e,\omega))}, & \text{if } \delta'(e,\omega)\leq t\leq 1.\end{cases} \tag{6.11}$$

The continuity of the map ψ follows by the continuity of the map δ' and by using the"gluing lemma". And it is a d-map since $\psi(e,\omega)$ is an increasing function.

e) Finally we define the following d-maps

$$\lambda^s_\alpha:\underline{B}_\alpha\to\underline{E}^{\uparrow \mathbf{I}},\alpha=0,1, \tag{6.12}$$

as the compositions

$$\underline{B}_\alpha\xrightarrow{s'_\alpha}\underline{B}^s_\alpha\times\underline{B}_\alpha\xrightarrow{\lambda_\alpha\times\psi}\underline{E}^{\uparrow \mathbf{I}}\times(\uparrow \mathbf{I})^{\uparrow \mathbf{I}}\xrightarrow{\chi}\underline{E}^{\uparrow \mathbf{I}}, \tag{6.13}$$

for

$$s'_\alpha(e,\omega)=(s_\alpha(e,\omega),(e,\omega)). \tag{6.14}$$

Then, by a), b), c), d) we obtain that λ^s_α is a continuous d-map.

Explicitly, the d-maps λ^s_α are given as follows

$$\lambda^s_\alpha(e,\omega)(t)=\begin{cases}\lambda_\alpha(e,\omega'_\alpha)(\frac{t}{2\delta'(e,\omega)}), & \text{if } 0\leq t\leq \delta'(e,\omega),\\ \lambda_\alpha(e,\omega'_\alpha)(\frac{t+(1-2\delta'(e,\omega))}{2(1-\delta'(e,\omega))}), & \text{if } \delta'(e,\omega)\leq t\leq 1.\end{cases} \tag{6.15}$$

f) For the d-maps λ^s_α we have

$$\lambda^s_\alpha(e,\omega)(\alpha)=\lambda_\alpha(e,\omega'_\alpha)(\alpha)=e. \tag{6.16}$$

Then, if we denote, to simplify writing,

$$\delta':=\delta'(e,\omega) \tag{6.17}$$

$$\begin{aligned} d(p\lambda^s_\alpha(e,\omega)(t),\omega(t)) &= \begin{cases} d(p\lambda_\alpha(e,\omega'_\alpha)(\frac{t}{2\delta'}),\omega(t)), & \text{if } 0\leq t\leq \delta',\\ d(p\lambda(e,\omega'_\alpha)(\frac{t+(1-2\delta')}{2(1-\delta')}),\omega(t)) & \text{if } \delta'\leq t\leq 1.\end{cases}\\ &= \begin{cases} d(\omega'_\alpha(\frac{t}{2\delta'}),\omega(t)) & \text{if } 0\leq t\leq \delta',\\ d(\omega'(\frac{t+(1-2\delta')}{2(1-\delta')}),\omega(t)), & \text{if } \delta'\leq t\leq 1.\end{cases}\\ &= \begin{cases} d(\omega(\alpha),\omega(t)), & \text{if } 0\leq t\leq \delta',\\ d(\omega(\frac{1-\delta'}{1-\delta'}),\omega(t)), & \text{if } \delta'\leq t\leq 1.\end{cases}\end{aligned} \tag{6.18}$$

By c) we have:

$$\begin{array}{l} d(\omega(t),\omega(\alpha)) < \frac{\varepsilon}{2}, \text{ if } 0 \leq t \leq \delta', \text{ and if } \delta' \leq t \leq 1 \text{ then} \\ |\frac{t-\delta'}{1-\delta}' - t| = \frac{\delta'(1-t)}{1-\delta'} \leq \delta' = \delta'(e,\omega), \text{ and therefore} \\ d(\omega(\frac{t-\delta'}{1-\delta'}),\omega(t)) < \frac{\varepsilon}{2}. \end{array} \tag{6.19}$$

In such a way we obtained

$$d(p\lambda_\alpha^\varepsilon(e,\omega)(t),\omega(t)) < \frac{\varepsilon}{2}, \text{ for every } t \in [0,1], \tag{6.20}$$

and therefore

$$d(p\lambda_\alpha^\varepsilon(e,\omega),\omega) < \varepsilon. \tag{6.21}$$

Thus we have proved that p has directed approximate lifting pair. By Theorem 4.2 it follows that p has dAHLP with respect to $\underline{X}$.

Corollary 6.3 *A directed Dold fibration $p : \underline{E} \to \underline{B}$ between directed locally metric ANR's is a directed approximate fibration.*

Remark 6.1 In view of Corollary 6.3 one can ask the converse question, is every directed approximate fibration a directed Dold fibration? The answer is no, as the next example shows.

Example 6.1 Consider in $\mathbb{R}^2$ the points $O(0,0), A(2,0), B(0,2), C(1,2)$, $D(1,1), E(2,1)$ and the d-spaces $\underline{E}$ and $\underline{B}$ with $|\underline{E}| = [BC] \cup [CD] \cup [DE]$, $|\underline{B}| = [OA]$ and with the d-structures given the vectors $\overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}$ and $\overrightarrow{OA}$ respectively. Then the d-map $d : \underline{E} \to \underline{B}$ given by $p(x,y) = (x,0)$ fails to be a directed Dold fibration since, for example, the semistationary directed path $\omega :\uparrow \mathbf{I} \to \underline{B}$, given by

$$\omega(t) = \begin{cases} (0,0), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (4t-2,0), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} \tag{6.22}$$

can not be directed lifted with the initial position $(0,2) \in \underline{E}$. But we can define directed approximate lifts of this directed path. Indeed, for $\varepsilon > 0$, we define $\omega' :\uparrow \mathbf{I} \to \underline{E}$ by

$$\omega'(t) = \begin{cases} (0,2), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (4t-2,2), & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4}, \\ (1, -\frac{8}{\varepsilon}t + 2 + \frac{6}{\varepsilon}), & \text{if } \frac{3}{4} \leq t \leq \frac{3}{4} + \frac{\varepsilon}{8}, \\ (4t-2-\frac{\varepsilon}{2},1), & \text{if } \frac{3}{4} + \frac{\varepsilon}{8} \leq t \leq 1. \end{cases} \tag{6.23}$$

Now

$$(p\omega')(t) = \begin{cases} (0,0), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (4t-2,0), & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4}, \\ (1,0), & \text{if } \frac{3}{4} \leq t \leq \frac{3}{4} + \frac{\varepsilon}{8}, \\ (4t-2-\frac{\varepsilon}{2},0), & \text{if } \frac{3}{4} + \frac{\varepsilon}{8} \leq t \leq 1. \end{cases} \tag{6.24}$$

and then is immediately that

$$d(\omega(t), (p\omega')(t)) \leq \frac{\varepsilon}{2} < \varepsilon. \tag{6.25}$$

Analogously it can be defined a directed approximate lift of an arbitrary (semistationary) d-path. Moreover, if the same procedure is used to define the lifts, then for an $\varepsilon > 0$, the d-applications $\lambda_\alpha^\varepsilon$ can be obtained, such that, by Theorem 4.2, p is a regular directed approximate fibration.

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Author

I. Pop,
Faculty of Mathematics,
"Alexandru Ioan Cuza" University of Iasi,
Iasi, 700506, Romania,
E-mail: ioanpop@uaic.ro