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Symmetric functions of binary products of bivariate complex Lucas polynomials and some special numbers and polynomials

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Abstract In this paper, we introduce an operator in order to derive some new symmetric properties of bivariate complex Lucas polynomials, then we give some new generating functions of the products of bivariate complex Lucas polynomials with Chebyshev polynomials of the first, second, third and fourth kinds, k-Fibonacci numbers, k-Lucas numbers, k-Pell numbers, k-Pell Lucas numbers, k-Jacobsthal numbers and k-Jacobsthal Lucas numbers. By making use of the operator defined in this paper, we give some new generating functions of the products of bivariate complex Lucas polynomials with Fibonacci polynomials, Pell polynomials and Jacobsthal polynomials.

Keywords symmetric functions · generating functions · bivariate complex Lucas polynomials · k-Fibonacci numbers · third and fourth kinds Chebyshev polynomials.

Mathematics Subject Classification (2010) 05E05 · 11B39

1 Introduction

Falcón and Plaza in [15] introduced the k-Fibonacci numbers $\{F_{k,n}\}_{n\in N}$ which is defined by

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2} \text{ for } k \geq 1 \text{ and } n \geq 2,$$

with the initial values $F_{k,0} = 1$, $F_{k,1} = k$. After that, Falcón in [16] showed some results of the k-Lucas numbers $\{L_{k,n}\}_{n\in N}$ which is defined as

$$L_{k,n} = kL_{k,n-1} + L_{k,n-2} \text{ for } k \geq 1 \text{ and } n \geq 2,$$

with initial conditions $L_{k,0} = 2$, $L_{k,1} = k$. The Binet's formulas for k-Fibonacci and k-Lucas numbers are given by

$$F_{k,n} = \frac{x_1^{n+1} - x_2^{n+1}}{x_1 - x_2} \text{ and } L_{k,n} = x_1^n + x_2^n,$$

respectively, where $x_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $x_2 = \frac{k-\sqrt{k^2+4}}{2}$ are roots of the characteristic equation $x^2 - kx - 1 = 0$. We note that

$$x_1 + x_2 = k,\ x_1\ x_2 = -1 \text{ and } x_1 - x_2 = \sqrt{k^2+4}.$$

M. Asci and E. Gurel defined and studied the bivariate complex Fibonacci and bivariate complex Lucas polynomials in [3]. They gave generating function, Binet's formula, explicit formula and partial derivation of these polynomials.

The bivariate complex Lucas polynomials $\{L_n(x,y)\}_{n\in N}$ is defined by

$$\begin{cases} L_{n+1}(x,y) = ixL_n(x,y) + yL_{n-1}(x,y), \text{ for } n \geq 1 \\ L_0(x,y) = 2,\ L_1(x,y) = ix \end{cases}.$$

The first few terms of bivariate complex Lucas polynomials are listed in the Table below:

n	0	1	2	3	4	5
$L_n(x,y)$	2	ix	$-x^2+2y$	$-x^3i+3xyi$	$x^4-4x^2y+2y^2$	$x^5i-5x^3yi+5xy^2i$

Table 1. The bivariate complex Lucas polynomials for $0 \leq n \leq 5$.

The explicit formula of bivariate complex Lucas polynomials is given by

$$L_n(x,y) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-j} \binom{n-j}{j} (ix)^{n-2j}\, y^j, \text{ (see [3]).}$$

For any integer $n \geq 2$, the Chebyshev polynomials of the first, second, third and fourth kinds $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$ are respectively defined as follows

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),$$

with initial conditions $T_0(x) = 1$ and $T_1(x) = x$, and

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. For more information, please see the paper [4].

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x),$$

with the initial values $V_0(x) = 1$ and $V_1(x) = 2x - 1$, and

$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x),$$

with the initial values $W_0(x) = 1$ and $W_1(x) = 2x + 1$, (see [14]).

Catarino in [11] studied the k-Pell sequence $\{P_{k,n}\}_{n\in N}$ which is defined by

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \text{ for } k \geq 1 \text{ and } n \geq 2,$$

with the initial terms $P_{k,0} = 0$, $P_{k,1} = 1$. After that Catarino and Vasco in [12] found some results of the k-Pell Lucas numbers $\{Q_{k,n}\}_{n\in N}$ which is defined as

$$Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2}, \text{ for } k \geq 1 \text{ and } n \geq 2,$$

with the initial terms $Q_{k,0} = 2$, $Q_{k,1} = 2$.

In this part, we define k-Jacobsthal and k-Jacobsthal Lucas numbers, Fibonacci, Pell and Jacobsthal polynomials.

Definition 1.1 *[19] We define k-Jacobsthal numbers $\{J_{k,n}\}_{n\in N}$ by the following recurrence relation*

$$\begin{cases} J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2}, \text{ for } n \geq 2 \\ J_{k,0} = 0, \ J_{k,1} = 1 \end{cases}.$$

Definition 1.2 *[20] For $n \in N$, the k-Jacobsthal Lucas numbers, denoted by $\{j_{k,n}\}_{n\in N}$ is defined recursively by*

$$\begin{cases} j_{k,n} = kj_{k,n-1} + 2j_{k,n-2}, \text{ for } n \geq 2 \\ j_{k,0} = 2, \ j_{k,1} = k \end{cases}.$$

Definition 1.3 *[17] We define Pell polynomials $\{P_n(x)\}_{n\in N}$ by the following recurrence relation*

$$\begin{cases} P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \text{ for } n \geq 2 \\ P_0(x) = 0, \ P_1(x) = 1 \end{cases}.$$

Definition 1.4 *[13] For $n \in N$, the Fibonacci polynomials, denoted by $\{F_n(x)\}_{n\in N}$ is defined recursively by*

$$\begin{cases} F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \text{ for } n \geq 2 \\ F_0(x) = 1, \ F_1(x) = x \end{cases}.$$

Definition 1.5 *[18] We define Jacobsthal polynomials $\{J_n(x)\}_{n\in N}$ by the following recurrence relation*

$$\begin{cases} J_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x), \text{ for } n \geq 2 \\ J_0(x) = 0, \ J_1(x) = 1 \end{cases}.$$

2 Definitions and some properties

In this section, we introduce a symmetric function and give some properties of this symmetric function. We also give some more useful definitions from the literature which are used in the subsequent sections.

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet A is a function of the letters which is invariant under permutation of the letters of A. Taking an extra indeterminate z, one has two fundamental series (see [10]).

$$\lambda_z(A) = \Pi_{a\in A}(1+za),\ \sigma_z(A) = \frac{1}{\Pi_{a\in A}(1-za)}.$$

The expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete functions $S_n(A)$:

$$\lambda_z(A) = \sum_{n=0}^{+\infty} \Lambda_n(A)z^n,\ \sigma_z(A) = \sum_{n=0}^{+\infty} S_n(A)z^n.$$

Let us now start at the following definition.

Definition 2.1 *[1] Let A and B be any two alphabets, then we give $S_n(A-B)$ by the following form:*

$$\frac{\Pi_{b\in B}(1-zb)}{\Pi_{a\in A}(1-za)} = \sum_{n=0}^{+\infty} S_n(A-B)z^n = \sigma_z(A-B), \tag{2.1}$$

with the condition $S_n(A-B) = 0$ for $n < 0$.

Remark 2.1 Taking $A = \{0\}$ in (2.1) gives

$$\Pi_{b\in B}(1-zb) = \sum_{n=0}^{+\infty} S_n(-B)z^n = \lambda_z(-B). \tag{2.2}$$

Further, in the case $A = \{0\}$ or $B = \{0\}$, we have

$$\sum_{n=0}^{+\infty} S_n(A-B)z^n = \sigma_z(A) \times \lambda_z(-B). \tag{2.3}$$

Thus, (see [1])

$$S_n(A-B) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-B).$$

Definition 2.2 *[21] Let g be any function on R^n, then we consider the divided difference operator as the following form*

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1,\cdots,x_i,x_{i+1},\cdots x_n) - g^{\sigma}(x_1,\cdots,x_i,x_{i+1},\cdots x_n)}{x_i - x_{i+1}},$$

where g^σ is given by

$$g^\sigma(x_1,\cdots,x_i,x_{i+1},\cdots x_n) = g(x_1,\cdots x_{i-1},x_{i+1},x_i,x_{i+2}\cdots x_n).$$

Definition 2.3 *Let n be positive integer and $A = \{a_1, a_2\}$ are set of given variables, then, the n-th symmetric function $S_n(a_1 + a_2)$ is defined by*

$$S_n(A) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

with

$$\begin{aligned} S_0(A) &= S_0(a_1 + a_2) = 1, \\ S_1(A) &= S_1(a_1 + a_2) = a_1 + a_2, \\ S_2(A) &= S_2(a_1 + a_2) = a_1^2 + a_1 a_2 + a_2^2, \\ &\vdots \end{aligned}$$

Definition 2.4 *[7] Given an alphabet $A = \{a_1, a_2\}$, the symmetrizing operator $\delta_{a_1 a_2}^k$ is defined by*

$$\delta_{a_1 a_2}^k f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \ k \in N_0. \tag{2.4}$$

If $f(a_1) = a_1^n$, the operator (2.4) gives us

$$\begin{aligned} \delta_{a_1 a_2}^k (a_1^n) &= \frac{a_1^{k+n} - a_2^{k+n}}{a_1 - a_2} \\ &= S_{k+n-1}(a_1 + a_2). \end{aligned}$$

3 Generating functions of the products of bivariate complex Lucas polynomials with some well-known numbers and polynomials

The following propositions are key tools of the proof of our main result. They have been proved in [6] and [7].

Proposition 3.1 *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, then*

$$\sum_{n=0}^{+\infty} S_n(A) S_n(E) z^n = \frac{1 - a_1 a_2 e_1 e_2 z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_2^n z^n\right)}. \tag{3.1}$$

Based on the relationship (3.1) we get

$$\sum_{n=0}^{+\infty} S_{n-1}(A) S_{n-1}(E) z^n = \frac{z - a_1 a_2 e_1 e_2 z^3}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A) e_2^n z^n\right)}. \tag{3.2}$$

Proposition 3.2 *Given two alphabets* $E=\{e_1,e_2\}$ *and* $A=\{a_1,a_2\}$, *then*

$$\sum_{n=0}^{+\infty} S_{n-1}(A)S_n(E)z^n = \frac{(e_1+e_2)z - e_1e_2(a_1+a_2)z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_2^n z^n\right)}. \quad (3.3)$$

In this part, we now derive the new generating functions of the products of bivariate complex Lucas polynomials with k-Fibonacci numbers, k-Lucas numbers and k-Pell numbers, k-Pell Lucas numbers and k-Jacobsthal numbers, k-Jacobsthal Lucas numbers and Fibonacci polynomials, Pell polynomials and Jacobsthal polynomials. For the case $A=\{a_1,-a_2\}$, $E=\{e_1,-e_2\}$ with replacing a_2 by $(-a_2)$ and e_2 by $(-e_2)$ in (3.1), (3.2) and (3.3), we have

$$\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right)S_n\left(e_1+[-e_2]\right)z^n = \frac{1-a_1a_2e_1e_2z^2}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}, \quad (3.4)$$

$$\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right)S_{n-1}\left(e_1+[-e_2]\right)z^n = \frac{z-a_1a_2e_1e_2z^3}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}, \quad (3.5)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{(e_1-e_2)z+e_1e_2(a_1-a_2)z^2}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}, \quad (3.6)$$

respectively.

This case consists of six related parts. **Firstly**, the substitutions

$$\begin{cases} a_1-a_2=ix \\ a_1a_2=y \end{cases} \text{ and } \begin{cases} e_1-e_2=k \\ e_1e_2=1 \end{cases},$$

in (3.4), (3.5) and (3.6), we obtain

$$\sum_{n=0}^{+\infty} S_n\left(a_1+[-a_2]\right)S_n\left(e_1+[-e_2]\right)z^n = \frac{1-yz^2}{1-ikxz-(y\left(k^2+2\right)-x^2)z^2-ikxyz^3+y^2z^4}, \quad (3.7)$$

$$\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right)S_{n-1}\left(e_1+[-e_2]\right)z^n = \frac{z-yz^3}{1-ikxz-(y\left(k^2+2\right)-x^2)z^2-ikxyz^3+y^2z^4}, \quad (3.8)$$

$$\sum_{n=0}^{+\infty} S_{n-1}\left(a_1+[-a_2]\right)S_n\left(e_1+[-e_2]\right)z^n = \frac{kz+ixz^2}{1-ikxz-(y\left(k^2+2\right)-x^2)z^2-ikxyz^3+y^2z^4}, \quad (3.9)$$

respectively, we have the following theorems.

Theorem 3.3 *For* $n\in N$, *the new generating function of the product of bivariate complex Lucas polynomials with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)F_{k,n}z^n = \frac{2-ikxz+\left(x^2-2y\right)z^2}{1-ikxz-(y\left(k^2+2\right)-x^2)z^2-ikxyz^3+y^2z^4}. \quad (3.10)$$

Proof. Recall that, we have $F_{k,n} = S_n(e_1 + [-e_2])$ (see [5]). We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} L_n(x,y)F_{k,n}z^n &= \sum_{n=0}^{+\infty}\left(2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2])\right) S_n(e_1+[-e_2])z^n \\ &= 2\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\ &\quad - ix\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n.\end{aligned}$$

Using the relationships (3.7) and (3.9), we obtain

$$\begin{aligned}\sum_{n=0}^{+\infty} L_n(x,y)F_{k,n}z^n &= \frac{2\left(1-yz^2\right)}{1-ikxz-(y\left(k^2+2\right)-x^2)z^2-ikxyz^3+y^2z^4} \\ &- \frac{ix\left(kz+ixz^2\right)}{1-ikxz-(y\left(k^2+2\right)-x^2)z^2-ikxyz^3+y^2z^4} \\ &= \frac{2-ikxz+\left(x^2-2y\right)z^2}{1-ikxz-(y\left(k^2+2\right)-x^2)z^2-ikxyz^3+y^2z^4}.\end{aligned}$$

This completes the proof.

□

Theorem 3.4 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with k-Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)L_{k,n}z^n = \frac{4-3ikxz+\left(2x^2-2y\left(k^2+2\right)\right)z^2-ikxyz^3}{1-ikxz-(y\left(k^2+2\right)-x^2)z^2-ikxyz^3+y^2z^4}. \tag{3.11}$$

Proof. By [5], we have $L_{k,n} = 2S_n(e_1 + [-e_2]) - kS_{n-1}(e_1 + [-e_2])$. Then, we can see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} L_n(x,y)L_{k,n}z^n &= \sum_{n=0}^{+\infty} (2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2])) \\
&\times (2S_n(e_1 + [-e_2]) - kS_{n-1}(e_1 + [-e_2]))z^n \\
&= 4\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\
&- 2k\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\
&- 2ix\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\
&+ ikx\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n.
\end{aligned}$$

Using the relationships (3.7) , (3.8) and (3.9), we obtain

$$\begin{aligned}
\sum_{n=0}^{+\infty} L_n(x,y)L_{k,n}z^n &= \frac{4\left(1 - yz^2\right)}{1 - ikxz - (y\left(k^2+2\right) - x^2)z^2 - ikxyz^3 + y^2z^4} \\
&- \frac{2k\left(ixz + kyz^2\right)}{1 - ikxz - (y\left(k^2+2\right) - x^2)z^2 - ikxyz^3 + y^2z^4} \\
&- \frac{2ix\left(kz + ixz^2\right)}{1 - ikxz - (y\left(k^2+2\right) - x^2)z^2 - ikxyz^3 + y^2z^4} \\
&+ \frac{ikx\left(z - yz^3\right)}{1 - ikxz - (y\left(k^2+2\right) - x^2)z^2 - ikxyz^3 + y^2z^4} \\
&= \frac{4 - 3ikxz + \left(2x^2 - 2y\left(k^2+2\right)\right)z^2 - ikxyz^3}{1 - ikxz - (y\left(k^2+2\right) - x^2)z^2 - ikxyz^3 + y^2z^4}.
\end{aligned}$$

This completes the proof.

□

– Put $k = 1$ in the relationships (3.10) and (3.11), we obtain the following corollaries.

Corollary 3.5 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Fibonacci numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)F_nz^n = \frac{2 - ixz + \left(x^2 - 2y\right)z^2}{1 - ixz - (3y - x^2)z^2 - ixyz^3 + y^2z^4}.$$

Corollary 3.6 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)L_n z^n = \frac{4-3ixz+2\left(x^2-3y\right)z^2-ixyz^3}{1-ixz-(3y-x^2)z^2-ixyz^3+y^2z^4}.$$

Secondly, the substitutions

$$\begin{cases} a_1-a_2=ix \\ a_1a_2=y \end{cases} \text{and} \begin{cases} e_1-e_2=2 \\ e_1e_2=k \end{cases},$$

in (3.4), (3.5) and (3.6), we give

$$\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{1-kyz^2}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4}, \tag{3.12}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{z-kyz^3}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4}, \tag{3.13}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{2z+ikxz^2}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4}, \tag{3.14}$$

respectively, we have the following results.

Theorem 3.7 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with k-Pell numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)P_{k,n}z^n = \frac{ixz+4yz^2+ikxyz^3}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4}. \tag{3.15}$$

Proof. By referred to [5], we have

$$P_{k,n} = S_{n-1}(e_1+[-e_2]).$$

We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} L_n(x,y)P_{k,n}z^n =& \sum_{n=0}^{+\infty}\left(2S_n(a_1+[-a_2])-ixS_{n-1}(a_1+[-a_2])\right)S_{n-1}(e_1+[-e_2])z^n \\ =& 2\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\ & -ix\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.\end{aligned}$$

then, we get the following equality:

$$\begin{aligned}\sum_{n=0}^{+\infty} L_n(x,y)P_{k,n}z^n &= \frac{2\left(ixz+2yz^2\right)}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4} \\ &- \frac{ix\left(z-kyz^3\right)}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4} \\ &= \frac{ixz+4yz^2+ikxyz^3}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4}.\end{aligned}$$

This completes the proof.

□

Theorem 3.8 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with k-Pell Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)Q_{k,n}z^n = \frac{4-6ixz+\left(2kx^2-4y\,(k+2)\right)z^2-2ikxyz^3}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4}. \tag{3.16}$$

Proof. Recall that, we have [5].

$$Q_{k,n} = 2S_n(e_1+[-e_2]) - 2S_{n-1}(e_1+[-e_2]),$$

by using the relationships (3.12) , (3.13) and (3.14), we get

$$\begin{aligned}
\sum_{n=0}^{+\infty} L_n(x,y)Q_{k,n}z^n &= \sum_{n=0}^{+\infty}(2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2])) \\
&\times (2S_n(e_1+[-e_2]) - 2S_{n-1}(e_1+[-e_2]))z^n \\
&= 4\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&- 4\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&- 2ix\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&+ 2ix\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&= \frac{4\left(1-kyz^2\right)}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4} \\
&- \frac{4\left(ixz+2yz^2\right)}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4} \\
&- \frac{2ix\left(2z+ikxz^2\right)}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4} \\
&+ \frac{2ix\left(z-kyz^3\right)}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4} \\
&= \frac{4-6ixz+\left(2kx^2-4y\,(k+2)\right)z^2-2ikxyz^3}{1-2ixz-(2y\,(k+2)-kx^2)z^2-2ikxyz^3+k^2y^2z^4}.
\end{aligned}$$

This completes the proof.

□

– Put $k = 1$ in the relationships (3.15) and (3.16), we deduce the following corollaries.

Corollary 3.9 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Pell numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)P_nz^n = \frac{ixz+4yz^2+ixyz^3}{1-2ixz-(6y-x^2)z^2-2ixyz^3+y^2z^4}.$$

Corollary 3.10 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Pell Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y) Q_n z^n = \frac{4 - 6ixz + 2\left(x^2 - 6y\right) z^2 - 2ixyz^3}{1 - 2ixz - (6y - x^2) z^2 - 2ixyz^3 + y^2 z^4}.$$

Thirdly, the substitutions

$$\begin{cases} a_1 - a_2 = ix \\ a_1 a_2 = y \end{cases} \text{ and } \begin{cases} e_1 - e_2 = k \\ e_1 e_2 = 2 \end{cases},$$

in (3.4), (3.5) and (3.6), we give

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n = \frac{1 - 2yz^2}{1 - ikxz - ((k^2+4)\, y - 2x^2) z^2 - 2ikxyz^3 + 4y^2 z^4}, \tag{3.17}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n = \frac{z - 2yz^3}{1 - ikxz - ((k^2+4)\, y - 2x^2) z^2 - 2ikxyz^3 + 4y^2 z^4}, \tag{3.18}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n = \frac{kz + 2ixz^2}{1 - ikxz - ((k^2+4)\, y - 2x^2) z^2 - 2ikxyz^3 + 4y^2 z^4}, \tag{3.19}$$

respectively, thus we get the following theorems.

Theorem 3.11 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y) J_{k,n} z^n = \frac{ixz + 2kyz^2 + 2ixyz^3}{1 - ikxz - ((k^2+4)\, y - 2x^2) z^2 - 2ikxyz^3 + 4y^2 z^4}. \tag{3.20}$$

Proof. By [5], we have $J_{k,n} = S_{n-1}(e_1 + [-e_2])$. Then, we can see that

$$\begin{aligned} \sum_{n=0}^{+\infty} L_n(x,y) J_{k,n} z^n &= \sum_{n=0}^{+\infty} (2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2])) S_{n-1}(e_1 + [-e_2]) z^n \\ &= 2\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\ &\quad - ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n, \end{aligned}$$

then, we get the following equality:

$$\begin{aligned}\sum_{n=0}^{+\infty} L_n(x,y)J_{k,n}z^n &= \frac{2\left(ixz+kyz^2\right)}{1-ikxz-\left(\left(k^2+4\right)y-2x^2\right)z^2-2ikxyz^3+4y^2z^4}\\ &- \frac{ix\left(z-2yz^3\right)}{1-ikxz-\left(\left(k^2+4\right)y-2x^2\right)z^2-2ikxyz^3+4y^2z^4}\\ &= \frac{ixz+2kyz^2+2ixyz^3}{1-ikxz-\left(\left(k^2+4\right)y-2x^2\right)z^2-2ikxyz^3+4y^2z^4}.\end{aligned}$$

This completes the proof.

□

Theorem 3.12 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with k-Jacobsthal Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)j_{k,n}z^n = \frac{4-3ikxz+\left(4x^2-2y\left(k^2+4\right)\right)z^2-2ikxyz^3}{1-ikxz-\left(\left(k^2+4\right)y-2x^2\right)z^2-2ikxyz^3+4y^2z^4}. \tag{3.21}$$

Proof. We know that

$$j_{k,n} = 2S_n(e_1+[-e_2]) - kS_{n-1}(e_1+[-e_2]),\ \text{(see [9])}.$$

We see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} L_n(x,y) j_{k,n} z^n &= \sum_{n=0}^{+\infty} \left(2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2])\right) \\
&\times \left(2S_n(e_1+[-e_2]) - kS_{n-1}(e_1+[-e_2])\right) z^n \\
&= 4\sum_{n=0}^{+\infty} S_n(a_1+[-a_2]) S_n(e_1+[-e_2]) z^n \\
&- 2k\sum_{n=0}^{+\infty} S_n(a_1+[-a_2]) S_{n-1}(e_1+[-e_2]) z^n \\
&- 2ix\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2]) S_n(e_1+[-e_2]) z^n \\
&+ ikx\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2]) S_{n-1}(e_1+[-e_2]) z^n \\
&= \frac{4\left(1-2yz^2\right)}{1-ikxz-((k^2+4)\,y-2x^2)z^2-2ikxyz^3+4y^2z^4} \\
&- \frac{2k\left(ixz+kyz^2\right)}{1-ikxz-((k^2+4)\,y-2x^2)z^2-2ikxyz^3+4y^2z^4} \\
&- \frac{2ix\left(kz+2ixz^2\right)}{1-ikxz-((k^2+4)\,y-2x^2)z^2-2ikxyz^3+4y^2z^4} \\
&+ \frac{ikx\left(z-2yz^3\right)}{1-ikxz-((k^2+4)\,y-2x^2)z^2-2ikxyz^3+4y^2z^4},
\end{aligned}$$

after a simple calculation, we have

$$\sum_{n=0}^{+\infty} L_n(x,y) j_{k,n} z^n = \frac{4-3ikxz+\left(4x^2-2y\left(k^2+4\right)\right)z^2-2ikxyz^3}{1-ikxz-((k^2+4)\,y-2x^2)z^2-2ikxyz^3+4y^2z^4}.$$

This completes the proof.

□

– Put $k=1$ in the relationships (3.20) and (3.21), we have the following corollaries.

Corollary 3.13 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Jacobsthal numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y) J_n z^n = \frac{ixz+2yz^2+2ixyz^3}{1-ixz-(5y-2x^2)z^2-2ixyz^3+4y^2z^4}.$$

Corollary 3.14 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Jacobsthal Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y) j_n z^n = \frac{4 - 3ixz + 2\left(2x^2 - 5y\right) z^2 - 2ixyz^3}{1 - ixz - (5y - 2x^2)z^2 - 2ixyz^3 + 4y^2 z^4}.$$

Fourthly, the substitutions

$$\begin{cases} a_1 - a_2 = ix \\ a_1 a_2 = y \end{cases} \text{and} \begin{cases} e_1 - e_2 = t \\ e_1 e_2 = 1 \end{cases},$$

in (3.4) and (3.6), we obtain

$$\sum_{n=0}^{+\infty} S_n\left(a_1 + [-a_2]\right) S_n\left(e_1 + [-e_2]\right) z^n = \frac{1 - yz^2}{1 - ixtz - (t^2 y + 2y - x^2)z^2 - ixytz^3 + y^2 z^4}, \tag{3.22}$$

$$\sum_{n=0}^{+\infty} S_{n-1}\left(a_1 + [-a_2]\right) S_n\left(e_1 + [-e_2]\right) z^n = \frac{tz + ixz^2}{1 - ixtz - (t^2 y + 2y - x^2)z^2 - ixytz^3 + y^2 z^4}, \tag{3.23}$$

respectively, we obtain the following theorem.

Theorem 3.15 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Fibonacci polynomials is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y) F_n(t) z^n = \frac{2 - ixtz + \left(x^2 - 2y\right) z^2}{1 - ixtz - (t^2 y + 2y - x^2)z^2 - ixytz^3 + y^2 z^4}. \tag{3.24}$$

Proof. Recall that, we have [22].

$$F_n(t) = S_n(e_1 + [-e_2]),$$

by using the relationships (3.22) and (3.23), we get

$$\begin{aligned}\sum_{n=0}^{+\infty}L_n(x,y)F_n\left(t\right)z^n&=\sum_{n=0}^{+\infty}(2S_n(a_1+[-a_2])-ixS_{n-1}(a_1+[-a_2]))\,S_n(e_1+[-e_2])z^n\\&=2\sum_{n=0}^{+\infty}S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n\\&-ix\sum_{n=0}^{+\infty}S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n\\&=\frac{2\left(1-yz^2\right)}{1-ixtz-(t^2y+2y-x^2)z^2-ixytz^3+y^2z^4}\\&-\frac{ix\left(tz+ixz^2\right)}{1-ixtz-(t^2y+2y-x^2)z^2-ixytz^3+y^2z^4}\\&=\frac{2-ixtz+\left(x^2-2y\right)z^2}{1-ixtz-(t^2y+2y-x^2)z^2-ixytz^3+y^2z^4}.\end{aligned}$$

This completes the proof.

□

Fifthly, the substitutions

$$\begin{cases}a_1-a_2=ix\\a_1a_2=y\end{cases}\text{ and }\begin{cases}e_1-e_2=2t\\e_1e_2=1\end{cases},$$

in (3.5) and (3.6), we have

$$\sum_{n=0}^{+\infty}S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n=\frac{z-yz^3}{1-2ixtz-(4yt^2-x^2+2y)z^2-2ixytz^3+y^2z^4},\tag{3.25}$$

$$\sum_{n=0}^{+\infty}S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n=\frac{2tz+ixz^2}{1-2ixtz-(4yt^2-x^2+2y)z^2-2ixytz^3+y^2z^4},\tag{3.26}$$

respectively, thus we get the following theorem.

Theorem 3.16 *For $n\in N$, the new generating function of the product of bivariate complex Lucas polynomials with Pell polynomials is given by*

$$\sum_{n=0}^{+\infty}L_n(x,y)P_n\left(t\right)z^n=\frac{ixz+4ytz^2+ixyz^3}{1-2ixtz-(4yt^2-x^2+2y)z^2-2ixytz^3+y^2z^4}.\tag{3.27}$$

Proof. We know that

$$P_n(t) = S_{n-1}(e_1 + [-e_2]), \text{ (see [22])}.$$

We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} L_n(x,y)P_n(t)z^n &= \sum_{n=0}^{+\infty}(2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2]))S_{n-1}(e_1+[-e_2])z^n \\ &= 2\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\ &\quad - ix\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\ &= \frac{2\left(ixz + 2ytz^2\right)}{1 - 2ixtz - (4yt^2 - x^2 + 2y)z^2 - 2ixytz^3 + y^2z^4} \\ &\quad - \frac{ix\left(z - yz^3\right)}{1 - 2ixtz - (4yt^2 - x^2 + 2y)z^2 - 2ixytz^3 + y^2z^4},\end{aligned}$$

after a simple calculation, we have

$$\sum_{n=0}^{+\infty} L_n(x,y)P_n(t)z^n = \frac{ixz + 4ytz^2 + ixyz^3}{1 - 2ixtz - (4yt^2 - x^2 + 2y)z^2 - 2ixytz^3 + y^2z^4}.$$

This completes the proof.

□

Sixthly, the substitutions

$$\begin{cases} a_1 - a_2 = ix \\ a_1a_2 = y \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 1 \\ e_1e_2 = 2t \end{cases},$$

in (3.5) and (3.6), we give

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n = \frac{z - 2ytz^3}{1 - ixz - (y - 2tx^2 + 4yt)z^2 - 2ixytz^3 + 4y^2t^2z^4}, \tag{3.28}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{z + 2ixtz^2}{1 - ixz - (y - 2tx^2 + 4yt)z^2 - 2ixytz^3 + 4y^2t^2z^4}, \tag{3.29}$$

respectively, we have the following theorem.

Theorem 3.17 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Jacobsthal polynomials is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)J_n(t)z^n = \frac{ixz + 2yz^2 + 2ixytz^3}{1 - ixz - (y - 2tx^2 + 4yt)z^2 - 2ixytz^3 + 4y^2t^2z^4}. \tag{3.30}$$

Proof. By [22], we have $J_n(t) = S_{n-1}(e_1 + [-e_2])$. Then, we can see that

$$\begin{aligned}\sum_{n=0}^{+\infty} L_n(x,y) J_n(t) z^n &= \sum_{n=0}^{+\infty} (2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2]))S_{n-1}(e_1+[-e_2])z^n \\ &= 2\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\ &\quad -ix\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n,\end{aligned}$$

then, we get the following equality:

$$\begin{aligned}\sum_{n=0}^{+\infty} L_n(x,y) J_n(t) z^n &= \frac{2\left(ixz+yz^2\right)}{1-ixz-(y-2tx^2+4yt)z^2-2ixytz^3+4y^2t^2z^4} \\ &\quad - \frac{ix\left(z-2ytz^3\right)}{1-ixz-(y-2tx^2+4yt)z^2-2ixytz^3+4y^2t^2z^4} \\ &= \frac{ixz+2yz^2+2ixytz^3}{1-ixz-(y-2tx^2+4yt)z^2-2ixytz^3+4y^2t^2z^4}.\end{aligned}$$

This completes the proof. □

4 A new class of ordinary generating functions of binary products of bivariate complex Lucas polynomials with Chebyshev polynomials

In this section, we now derive the new generating functions of the products of bivariate complex Lucas polynomials with Chebyshev polynomials of the first, second, third and fourth kinds.

For the case $A = \{a_1, -a_2\}$, $E = \{2e_1, -2e_2\}$ with replacing a_2 by $(-a_2)$, e_1 by $2e_1$ and e_2 by $(-2e_2)$ in (3.1), (3.2) and (3.3), we have

$$\sum_{n=0}^{+\infty} S_n(a_1+[-a_2]) S_n(2e_1+[-2e_2]) z^n = \frac{1-4a_1a_2e_1e_2z^2}{(1-2a_1e_1z)(1+2a_2e_1z)(1+2a_1e_2z)(1-2a_2e_2z)}, \tag{4.1}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2]) S_{n-1}(2e_1+[-2e_2]) z^n = \frac{z-4a_1a_2e_1e_2z^3}{(1-2a_1e_1z)(1+2a_2e_1z)(1+2a_1e_2z)(1-2a_2e_2z)}, \tag{4.2}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2]) S_n(2e_1+[-2e_2]) z^n = \frac{2(e_1-e_2)z+4e_1e_2(a_1-a_2)z^2}{(1-2a_1e_1z)(1+2a_2e_1z)(1+2a_1e_2z)(1-2a_2e_2z)}, \tag{4.3}$$

respectively, the substitutions

$$\begin{cases} a_1 - a_2 = ix \\ a_1 a_2 = y \end{cases} \text{and} \begin{cases} e_1 - e_2 = t \\ 4e_1 e_2 = -1 \end{cases},$$

in (4.1), (4.2) and (4.3), we obtain

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{1 + yz^2}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2 z^4}, \tag{4.4}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n = \frac{z + yz^3}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2 z^4}, \tag{4.5}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{2tz - ixz^2}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2 z^4}, \tag{4.6}$$

respectively, we get the following theorems.

Theorem 4.1 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Chebyshev polynomials of the first kind is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y) T_n(t) z^n = \frac{2 - 3ixtz + \left(2y - 4t^2 y - x^2\right) z^2 + ixytz^3}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2 z^4}. \tag{4.7}$$

Proof. Recall that, we have [8]

$$T_n(t) = S_n(2e_1 + [-2e_2]) - tS_{n-1}(2e_1 + [-2e_2]).$$

Then, we can see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} L_n(x,y) T_n(t) z^n &= \sum_{n=0}^{+\infty} \left(2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2])\right) \\
&\quad \times \left(S_n(2e_1 + [-2e_2]) - tS_{n-1}(2e_1 + [-2e_2])\right) z^n \\
&= 2\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad - 2t \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
&\quad - ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad + ixt \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n.
\end{aligned}$$

By using the relationships (4.4) , (4.5) and (4.6), we get

$$\begin{aligned}
\sum_{n=0}^{+\infty} L_n(x,y)T_n(t)z^n &= \frac{2\left(1+yz^2\right)}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4} \\
&\quad -\frac{2t\left(ixz+2ytz^2\right)}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4} \\
&\quad -\frac{ix\left(2tz-ixz^2\right)}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4} \\
&\quad +\frac{ixt\left(z+yz^3\right)}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4} \\
&= \frac{2-3ixtz+\left(2y-4t^2y-x^2\right)z^2+ixytz^3}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4}.
\end{aligned}$$

This completes the proof.

□

Theorem 4.2 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Chebyshev polynomials of the second kind is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)U_n(t)z^n = \frac{2-2ixtz+\left(2y-x^2\right)z^2}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4}. \tag{4.8}$$

Proof. By [8], we have $U_n(t) = S_n(2e_1 + [-2e_2])$. Then, we can see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} L_n(x,y)U_n(t)z^n &= \sum_{n=0}^{+\infty}(2S_n(a_1+[-a_2])-ixS_{n-1}(a_1+[-a_2]))S_n(2e_1+[-2e_2])z^n \\
&= 2\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&\quad - ix\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&= \frac{2\left(1+yz^2\right)}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4} \\
&\quad - \frac{ix\left(2tz-ixz^2\right)}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4},
\end{aligned}$$

after a simple calculation, we have

$$\sum_{n=0}^{+\infty} L_n(x,y)U_n(t)z^n = \frac{2-2ixtz+\left(2y-x^2\right)z^2}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4}.$$

This completes the proof.

□

Theorem 4.3 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Chebyshev polynomials of the third kind is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)V_n(t)z^n = \frac{2 - ix\,(2t+1)\,z + \left(2y - 4yt - x^2\right)z^2 + ixyz^3}{1 - 2ixtz - (x^2+4yt^2-2y)z^2 + 2ixytz^3 + y^2z^4}. \tag{4.9}$$

Proof. By [2], we have $V_n(t) = S_n(2e_1 + [-2e_2]) - S_{n-1}(2e_1 + [-2e_2])$. Then, we can see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} L_n(x,y)V_n(t)z^n &= \sum_{n=0}^{+\infty} \left(2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2])\right) \\
&\quad \times \left(S_n(2e_1+[-2e_2]) - S_{n-1}(2e_1+[-2e_2])\right)z^n \\
&= 2\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&\quad - 2\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n \\
&\quad - ix\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&\quad + ix\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n.
\end{aligned}$$

According to relationships (4.4), (4.5) and (4.6) this gives the following equality:

$$\begin{aligned}
\sum_{n=0}^{+\infty} L_n(x,y)V_n(t)z^n &= \frac{2\left(1+yz^2\right)}{1 - 2ixtz - (x^2+4yt^2-2y)z^2 + 2ixytz^3 + y^2z^4} \\
&\quad - \frac{2\left(ixz+2ytz^2\right)}{1 - 2ixtz - (x^2+4yt^2-2y)z^2 + 2ixytz^3 + y^2z^4} \\
&\quad - \frac{ix\left(2tz - ixz^2\right)}{1 - 2ixtz - (x^2+4yt^2-2y)z^2 + 2ixytz^3 + y^2z^4} \\
&\quad + \frac{ix\left(z+yz^3\right)}{1 - 2ixtz - (x^2+4yt^2-2y)z^2 + 2ixytz^3 + y^2z^4} \\
&= \frac{2 - ix\,(2t+1)\,z + \left(2y - 4yt - x^2\right)z^2 + ixyz^3}{1 - 2ixtz - (x^2+4yt^2-2y)z^2 + 2ixytz^3 + y^2z^4}.
\end{aligned}$$

This completes the proof.

□

Theorem 4.4 *For $n \in N$, the new generating function of the product of bivariate complex Lucas polynomials with Chebyshev polynomials of the fourth kind is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)W_n(t)z^n = \frac{2 + ix\,(1-2t)\,z + \left(2y + 4yt - x^2\right) z^2 - ixyz^3}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4}. \tag{4.10}$$

Proof. By referred to [2], we have

$$W_n(t) = S_n(2e_1 + [-2e_2]) + S_{n-1}(2e_1 + [-2e_2]).$$

We see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} L_n(x,y)W_n(t)z^n &= \sum_{n=0}^{+\infty} (2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2])) \\
&\times (S_n(2e_1 + [-2e_2]) + S_{n-1}(2e_1 + [-2e_2]))z^n \\
&= 2\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \\
&+ 2\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n \\
&- ix\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \\
&- ix\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n.
\end{aligned}$$

According to relationships (4.4) , (4.5) and (4.6) this gives the following equality:

$$\begin{aligned}\sum_{n=0}^{+\infty} L_n(x,y)W_n(t)z^n &= \frac{2\left(1+yz^2\right)}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4}\\ &+ \frac{2\left(ixz+2ytz^2\right)}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4}\\ &- \frac{ix\left(2tz-ixz^2\right)}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4}\\ &- \frac{ix\left(z+yz^3\right)}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4}\\ &= \frac{2+ix\left(1-2t\right)z+\left(2y+4yt-x^2\right)z^2-ixyz^3}{1-2ixtz-(x^2+4yt^2-2y)z^2+2ixytz^3+y^2z^4}.\end{aligned}$$

This completes the proof.

□

5 Conclusion

In this paper, by making use of Eqs. (3.1) and (3.3), we have derived some new generating functions of the products of bivariate complex Lucas polynomials with k-Fibonacci numbers, k-Lucas numbers, k-Jacobsthal numbers, k-Jacobsthal Lucas numbers, k-Pell numbers and k-Pell Lucas numbers, and the products of bivariate complex Lucas polynomials with Fibonacci polynomials, Jacobsthal polynomials, Pell polynomials and Chebyshev polynomials of the first, second, third and fourth kinds. The derived theorems and corollaries are based on symmetric functions and products of these numbers and polynomials.

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