An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXVII, 2021, f. 2
https://doi.org/10.47743/anstim.2021.00015

Slant submanifolds in an almost paracontact metric manifold

Sunil Kumar Chanyal

Abstract In this paper, slant submanifolds of a pseudo-Riemannian manifold equipped with an almost paracontact structure are defined and studied. Some characterization theorems for slant submanifolds are obtained. We also present some examples of slant submanifolds when the ambient space is an almost paracontact metric manifold.

Keywords almost paracontact structure · slant submanifold · pseudo-Riemannian manifold

Mathematics Subject Classification (2010) 53C40 · 53C42 · 53C50

1 Introduction

Sato [21] introduced an almost paracontact structure (φ, ξ, η) satisfying $\varphi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on a differentiable manifold which is analogus to almost contact structure [5,20]. The paracontact structure is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure) and every differentiable manifold endowed with paracontact structure has a positive definite Riemannian metric. S. Kaneyuki and M. Kozai [14,15] defined an almost paracontact structure on an odd dimensional pseudo-Riemannian manifold and studied its properties. S. Zamkovoy [25] started the systematic study of almost paracontact metric manifold and proved that any almost paracontact structure admits a pseudo-Riemannian metric of signature $(n+1, n)$, which in recent years have been studied by many geometers (see [9,17,18,24], and references therein).

The notion of slant submanifolds was defined by B.-Y. Chen in 1990 as a natural generalization of both complex and totally real submanifolds in a Hermitian manifold [11]. Many geometers have shown interest in the study of slant submanifolds. A. Lotta [16] translated the notion of slant submanifolds to the almost contact metric manyfolds and J. L. Cabrerizo et al. studied slant submanifolds in Sasakian Manifolds [6–8].

The study of slant submanifolds in a pseudo-Riemannian (also called semi-Riemannian) manifold was also initiated by Chen in 2009 [12, 13]. Arslan et al. [4], S. Uddin et al. [23], P. Alegre [1], Carriazo [10] studied slant submanifolds in different structures; neutral Kaehler manifold, LP cosymplectic manifold, Lorentzian Sasakian and para Sasakian manifolds, neutral almost contact pseudo-metric manifolds. Recently P. Alegre and A. Carriazo [2, 3] studied slant submanifolds in a para Hermitian manifold.

The aim of this paper is to study slant submanifolds in almost paracontact metric manifolds. The paper is organised as follows. In Section 2 definition and examples of metric manifolds are given. Some basic results for submanifolds of pseudo-Riemannian manifolds are also given in this section. Section 3 contains the study of slant submanifolds in an almost paracontact metric manifold. In section 4 we consider an almost paracontact metric manifold whose structure is related to another pseudo-Riemannian manifold endowed with para Hermitian structure and study the relation between their slant submanifolds. Some examples of slant submanifolds are constructed in section 5.

2 Preliminaries

A $(2n+1)-$dimensional smooth manifold $\overline{M}$ is said to be equipped with an almost paracontact structure (φ, ξ, η) if it admits a tensor field φ of type (1, 1), a vector field ξ and a $1-form$ η satisfying the following conditions:

$$\text{(i)} \quad \varphi(\xi) = 0,\, \eta \circ \varphi = 0 \qquad \text{(ii)} \quad \eta(\xi) = 1,\, \varphi^2 = I - \eta \otimes \xi \tag{2.1}$$

and the restriction of φ on $2n-$dimensional distribution $D := ker\,\eta$, is an almost paracomplex structure, i.e., the eigenbundles D^+, D^- corresponding to the eigenvalues 1, -1 of φ respectively, have equal dimension n.

Let g be a compatible pseudo-Riemannian metric on $\overline{M}$ i.e.,

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

for any $X,\, Y \in T\overline{M}$ then $\overline{M}$ is called an almost paracontact metric manifold equipped with almost paracontact metric structure $(\varphi,\, \xi,\, \eta,\, g)$. On an almost paracontact metric manifold, we have

$$\eta(X) = g(X,\, \xi), \qquad g(X,\, \varphi Y) = -g(\varphi X,\, Y). \tag{2.3}$$

Note that g is necessarily of signature $(n+1,\, n)$. The $2-$form Φ on $\overline{M}$ defined as $\Phi(X,\, Y) = g(X,\, \varphi Y)$ is called fundamental $2-$form on $\overline{M}$. An almost paracontact manifold is called a paracontact manifold if $\Phi(X,\, Y) = d\eta(X,\, Y)$. In this case $(\overline{M}, \varphi, \xi, \eta, g)$ is called a paracontact manifold, the $2n-$dimensional distribution $D := ker\,\eta$, is the contact distribution and η is a contact form.

Remark 2.1 Some authors [19,22] call $\overline{M}$ an almost paracontact metric structure manifold if it admits a Riemannian metric g satisfying $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$. In this paper $\overline{M}$ is a pseudo-Riemannian manifold and the metric g satifies (2.2).

If we define a tensor field $\mathfrak{h}$ on the paracontact manifold as $\mathfrak{h} = \frac{1}{2}\mathcal{L}_\xi\varphi$, $\mathcal{L}$ denotes Lie derivative, then $\mathfrak{h}$ is a symmetric operator and satisfies the following properties:

$$\varphi \circ \mathfrak{h} = -\mathfrak{h} \circ \varphi, \quad tr(\mathfrak{h}) = 0, \quad \mathfrak{h}\xi = 0, \quad \text{and } \overline{\nabla}_X\xi = -\varphi X + \varphi\mathfrak{h}X, \tag{2.4}$$

where $\overline{\nabla}$ is the Levi-Civita connection of g. A paracontact structure for which ξ is Killing vector field is called a K-paracontact structure. An almost paracontact metric structure $(\varphi,\, \xi,\, \eta,\, g)$ is K-paracontact if and only if

$$\overline{\nabla}_X\xi = -\varphi X. \tag{2.5}$$

An almost paracontact metric structure $(\varphi,\, \xi,\, \eta,\, g)$ is called a para-Sasakian if and only if [25]

$$(\overline{\nabla}_X\varphi)Y = -g(X,\, Y)\xi + \eta(Y)X. \tag{2.6}$$

Thus a para-Sasakian manifold is a K-paracontact manifold but the converse is not true in general. It is known that a $3-\mathit{dimensional}$ K-paracontact manifold is a para-Sasakian manifold.

Example 2.1 Consider $\mathbb{R}^5$ with standard coordinates (x^1, y^1, x^2, y^2, z) and we define the following structure:

a vector field $\xi = 2\frac{\partial}{\partial z}$ and a one form $\eta = \dfrac{1}{2}\left(dz - \displaystyle\sum_{i=1}^{2} y^i dx^i\right)$, a pseudo-Riemannian metric

$$g = \eta\otimes\eta + \frac{1}{4}\left(\sum_{i=1}^{2} dx^i\otimes dx^i - \sum_{i=1}^{2} dy^i\otimes dy^i\right)$$

and a tensor Φ of type (1, 1) as

$$\Phi\left(\sum_{i=1}^{2}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}\right) + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^{2}\left(X_i\frac{\partial}{\partial y^i} + Y_i\frac{\partial}{\partial x^i} + Y_i y^i\frac{\partial}{\partial z}\right)$$

It is easy to see that $(\mathbb{R}^5,\, \Phi, \xi, \eta, g)$ is an almost paracontact metric manifold. Furthermore, it is easily seen that

$$\left\{2\frac{\partial}{\partial y^i},\, 2\left(\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z}\right),\, \xi\right\}$$

forms a $\Phi-$basis of $T\mathbb{R}^5$. Since Levi-Civita connection of g satisfies equation (2.6), it is a para-Sasakian pseudo-Riemannian manifold.
More examples are given in section 5.

Now, let M be a submanifold isometrically immersed in $\overline{M}$, the induced pseudo-Riemannian metric on M is also denoted by g. We denote by TM the Lie algebra of vector fields in M and by $T^{\perp}M$ the set of all vector fields normal to M. Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.7}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$ and ∇ is the Levi-Civita connection of M, $\nabla^{\perp}$ is connection in the normal bundle, h is second fundamental form of M and A_V denote the shape operator associated with V. The second fundamental form and shape operator are connected by $g(A_V X, Y) = g(h(X, Y), V)$.

For any $X \in TM$ and $V \in T^{\perp}M$, we write

$$\varphi X = TX + NX, \qquad \varphi V = tV + nV, \tag{2.8}$$

where TX(resp. tV) and NX(resp. nV) are respectively tangential and normal components of φX(resp. φV). From (2.3) and (2.8), we obtain

$$g(X, TY) = -g(TX, Y), \qquad \text{for any } X, Y \in TM. \tag{2.9}$$

M is called invariant (resp. anti-invariant) submanifold if N(resp. T) is identically zero i.e., $\varphi X \in TM$(resp. $\varphi X \in T^{\perp}M$). The derivatives of T and N are respectively given by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y \quad \text{and} \quad (\nabla_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y. \tag{2.10}$$

3 Slant submanifolds of an almost paracontact manifold

P. Alegre studied slant submanifolds in pseudo-Riemannian manifold equipped with para Hermitian structure [2]. He defined slant submanifolds as follows. A submanifold M of a para Hermitian manifold $(\overline{M}, J, g)$ is called slant if for every space-like or time-like vector filed $X \in TM$, the quotient $\dfrac{g(TX, TX)}{g(JX, JX)}$ is constant. He further classified slant submanifolds into three types:

Type 1 if for any space-like (time-like) vector field X, TX is time-like (space-like) and $\frac{|TX|}{|JX|} > 1$.

Type 2 if for any space-like (time-like) vector field X, TX is time-like (space-like) and $\frac{|TX|}{|JX|} < 1$.

Type 3 if for any space-like (time-like) vector field X, TX is space-like (time-like).

Now, we define slant submanifolds in almost paracontact manifolds as follows:

Definition 3.1 *Let M be a submanifold of an almost paracontact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$, it is called slant submanifold if for any $x \in M$ and any space-like or time-like vector field $X \in T_xM$ linearly independent to ξ, the quotient $\dfrac{g(TX, TX)}{g(\varphi X, \varphi X)}$ is constant i.e., independent on the choice of X and $x \in M$.*

It is clear from the definition 3.1 that if M is invariant i.e., $\varphi = T$ then this quotient is equal to 1 and for anti-invariant submanifolds this quotient is zero. A slant submanifold is called proper slant submanifold if it is neither invariant nor anti-invariant submanifold.

The following proposition shows that for any proper slant submanifold M of a paracontact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$, $\xi \in TM$.

Proposition 3.2 *Let M be a submanifold of a paracontact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$. If $\xi \in T^{\perp}M$, then M is anti-invariant manifold.*

Proof. The proof is similar to the proof of proposition 1 in [1].

□

Let M be a submanifold isometrically immersed in an almost paracontact manifold $(\overline{M}, \varphi, \xi, \eta, g)$ and $\xi \in TM$. We write

$$TM = \mathcal{D} \oplus < \xi >,$$

where $\mathcal{D}$ denote the orthogonal distribution to $< \xi >$. Consider M with dimension 2. This implies the distribution $\mathcal{D}$ is one dimensional. If $X \in TM - < \xi >$ is non-light like, we have $g(\varphi X, X) = 0$ as well as $g(\varphi X, \xi) = 0$. Therefore we state:

Proposition 3.3 *A surface M isometrically immersed in an almost paracontact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $\xi \in TM$ is an anti-invariant submanifold.*

Analogous to the definition given by P. Alegre, we have the following definition.

Definition 3.4 *Let M be a proper slant submanifold of an almost paracontact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $\xi \in TM$. We say that it is of*

Type 1 if for any space-like (time-like) vector field X orthogonal to ξ, TX is time-like (space-like) and $\frac{|TX|}{|\varphi X|} > 1$.

Type 2 if for any space-like (time-like) vector field X orthogonal to ξ, TX is time-like (space-like) and $\frac{|TX|}{|\varphi X|} < 1$.

Type 3 if for any space-like (time-like) vector field X orthogonal to ξ, TX is space-like (time-like).

Remark 3.1 The case $g(TX, TX) = 0$(respectively $g(TX, TX) = g(\varphi X, \varphi X)$) for any space-like or time-like X orthogonal to ξ, corresponds to the anti-invariant case (invariant case), that is $T \equiv 0(N \equiv 0$ or equivalently $T \equiv \varphi)$. Let us assume that $g(TX, TX) = 0$ for any space-like (time-like) vector field X orthogonal to ξ. Since any light-like vector can be approximated by a sequence of space-like or time-like vector fields and $T\xi = 0$. So for any vector field X we have $g(TX, TX) = 0$ and it does not depend on the causal character of X. Therefore

$$0 = g(T(X+Y),\, T(X+Y)) = 2g(TX,\, TY), \text{ for any } X,\, Y \in TM.$$

Thus the equation $g(TX,\, TY) = 0$ holds for any $Y \in TM$. But $g(TX,\, Z) = 0$ for any $Z \in T(TM)^{\perp}$, where $TM = T(TM) \oplus T(TM)^{\perp}$. Therefore $TX = 0$ for all $X \in TM$ as g is non-degenerate. Hence M is anti-invariant. Thus the case $g(TX, TX) = 0$ is excluded. Similarly the invariant case can be shown.

Now we prove the following characterization theorem for slant submanifolds of an almost paracontact metric manifold.

Theorem 3.5 *Let M be a submanifold of an almost paracontact metric manifold $(\overline{M},\, \varphi,\, \xi,\, \eta,\, g)$ such that $\xi \in TM$. Then,*

(1) M is slant of type 1 if and only if for any space-like (time-like) vector field $X \in TM - <\xi>$, TX is time-like (space-like), and there exists a constant $\lambda \in (1,\, +\infty)$ such that

$$T^2 = \lambda(I - \eta \otimes \xi). \tag{3.1}$$

We write $\lambda = \cosh^2\theta$, with $\theta > 0$.

(2) M is slant of type 2 if and only if for any space-like (time-like) vector field $X \in TM - <\xi>$, TX is time-like (space-like), and there exists a constant $\lambda \in (0,\, 1)$ such that

$$T^2 = \lambda(I - \eta \otimes \xi). \tag{3.2}$$

We write $\lambda = \cos^2\theta$, with $0 < \theta < \frac{\pi}{2}$.

(3) M is slant of type 3 if and only if for any space-like (time-like) vector field $X \in TM - <\xi>$, TX is space-like (time-like), and there exists a constant $\lambda \in (-\infty,\, 0)$ such that

$$T^2 = \lambda(I - \eta \otimes \xi). \tag{3.3}$$

We write $\lambda = -\sinh^2\theta$, with $\theta > 0$.

In each case θ is called the slant angle.

Proof. Let M be slant submanifold of type 1 i.e., for any space-like vector field X orthogonal to ξ, TX is time-like and φX is also time-like. Moreover they satisfy $|TX| > |\varphi X|$. So there exists $\theta > 0$ such that

$$\cosh\theta = \frac{|TX|}{|\varphi X|} = \frac{\sqrt{-g(TX,\, TX)}}{\sqrt{-g(\varphi X,\, \varphi X)}}. \tag{3.4}$$

So replacing X by TX, we obtain

$$\cosh\theta = \frac{|T^2X|}{|\varphi TX|} = \frac{|T^2X|}{|TX|}. \tag{3.5}$$

Also we have

$$g(T^2X,\, X) = g(\varphi TX,\, X) = -g(TX,\, TX) = |TX|^2. \tag{3.6}$$

From equations (3.4), (3.5) and (3.6), we have

$$g(T^2X,\, X) = |TX|^2 = |T^2X||X|. \tag{3.7}$$

Since both T^2X and X are space-like vector fields, (3.7) implies they are collinear i.e., $T^2X = \lambda X$ and from (3.4) we obtain $\lambda = \cosh^2\theta$. It follows that for any space-like vector field $X \in TM$ we have

$$T^2X = \lambda(X - \eta(X)\xi). \tag{3.8}$$

Now consider Y is a time-like vector field orthogonal to ξ, then TY and φY both are space-like. So we define

$$\cosh\theta = \frac{|TY|}{|\varphi Y|} = \frac{\sqrt{g(TY,\, TY)}}{\sqrt{g(\varphi Y,\, \varphi Y)}}.$$

Proceeding in similar manner we get (3.8) for any time-like vector field $X \in TM$. Since (3.8) holds for any space-like or time-like vector field X, it also holds for light-like vector fields and we have (3.1). The converse is a straight forward computation.

Now consider second case i.e., M is a submanifold of type 2. So for any space-like vector field X orthogonal to ξ, we have $|TX| < |\varphi X|$, so there exists $\theta > 0$ such that

$$\cos\theta = \frac{|TX|}{|\varphi X|} = \frac{\sqrt{-g(TX,\, TX)}}{\sqrt{-g(\varphi X,\, \varphi X)}}.$$

As in previous case, we can prove $g(T^2X,\, X) = |T^2X||X|$. Since both T^2X and X are space-like vector fields, it follows that they are collinear i.e., $T^2X = \lambda X$ but in this case we obtain $\lambda = \cos^2\theta$ and Hence the result. The converse is a direct computation.

Lastly, if M is a submanifold of type 3, then for any space-like vector field X orthogonal to ξ, TX is also space-like and there exists $\theta > 0$ such that

$$\sinh\theta = \frac{|TX|}{|\varphi X|} = \frac{\sqrt{g(TX,\, TX)}}{\sqrt{-g(\varphi X,\, \varphi X)}}.$$

Once again we can prove $g(T^2X,\, X) = |T^2X||X|$. Since both T^2X and X are space-like vector fields, it follows that they are collinear i.e., $T^2X = \lambda X$ but in this case we obtain $\lambda = -\sinh^2\theta$ and Hence the result. The converse is a direct computation.

□

Corollary 3.6 *Let M be a slant submanifold of an almost paracontact metric manifold $(\overline{M},\, \varphi,\, \xi,\, \eta,\, g)$ such that $\xi \in TM$ and θ be the slant angle. Then, for any $X,\, Y \in TM$, we have:*
If M is of type 1, then

$$g(TX,\, TY) = -\cosh^2\theta\,(g(X,\, Y) - \eta(X)\eta(Y))\,,$$

$$g(NX,\, NY) = \sinh^2\theta\,(g(X,\, Y) - \eta(X)\eta(Y))\,.$$

If M is of type 2, then

$$g(TX,\, TY) = -\cos^2\theta\,(g(X,\, Y) - \eta(X)\eta(Y))\,,$$

$$g(NX,\, NY) = -\sin^2\theta\,(g(X,\, Y) - \eta(X)\eta(Y))\,.$$

If M is of type 3, then

$$g(TX,\, TY) = \sinh^2\theta\,(g(X,\, Y) - \eta(X)\eta(Y))\,,$$

$$g(NX,\, NY) = -\cosh^2\theta\,(g(X,\, Y) - \eta(X)\eta(Y))\,.$$

Proof. From equations (2.9) and (3.1), we get

$$g(TX,\, TY) = -g(T^2X,\, Y) = -\lambda(g(X,\, Y) - \eta(X)\eta(Y)) = \lambda g(\varphi X,\, \varphi Y).$$

Equation (2.8) yields

$$g(\varphi X,\, \varphi Y) = g(TX,\, TY) + g(NX,\, NY).$$

From last two equations, we obtain

$$g(NX,\, NY) = (1-\lambda)g(\varphi X,\, \varphi Y).$$

Hence the corollary follows from the values of λ in the theorem 3.5.

□

Remark 3.2 Since every light-like vector field can be decomposed as a sum of a space-like vector field and a time-like vector field, the conditions (3.1), (3.2) and (3.3) also hold for every light-like vector field.
Also it is necessary to ask space-like vector fields to satisfy conditions (3.1) and (3.2) for slant submanifolds of type 1 and type 2.

Theorem 3.7 *Let M be a submanifold of an almost paracontact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $\xi \in TM$. Then M is a slant submanifold of*

type 1 *if and only if* $T^2X = \cosh^2\theta(X - \eta(X)\xi)$ *for every space-like vector field X.*
type 2 *if and only if* $T^2X = \cos^2\theta(X - \eta(X)\xi)$ *for every space-like vector field X.*

Proof. Let $Y \in TM$ be a time-like vector field orthogonal to ξ. There exists a space-like vector field $X \in TM$ orthogonal to ξ such that $TX = Y$. In the first case, we obtain

$$T^2Y = T^2TX = TT^2X = \cosh^2\theta\, TX = \cosh^2\theta\, Y.$$

Hence for any time-like vector field $Y \in TM$, we have $T^2Y = \cosh^2\theta(Y - \eta(Y)\xi)$.
In similar manner, in the second case we can prove for any time-like vector field $Y \in TM$, we have $T^2Y = \cos^2\theta(Y - \eta(Y)\xi)$.

□

Here we mention that the case of slant submanifolds of type 3 is not same as in the previous theorem. Since slant submanifolds of type 3 are not always neutral submanifolds, study of these submanifolds is completely different but they can be space-like or time-like.

4 Some more results

First we consider an almost paracontact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$. Let M be an isometrically immersed submanifold in $\overline{M}$. The following theorem gives induced structure on M when it is a proper slant submanifold.

Theorem 4.1 *Let M be a proper slant submanifold of an almost paracontact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $\xi \in TM$. If M is of*

Type 1, then $\dot{\varphi} = \frac{1}{\cosh\theta}T$ defines an almost paracontact structure over M.
Type 2, then $\dot{\varphi} = \frac{1}{\cos\theta}T$ defines an almost paracontact structure over M.
Type 3, then $\dot{\varphi} = \frac{1}{\sinh\theta}T$ defines an almost contact structure over M.

Proof. It is easy to verify that $\dot{\varphi}^2 = I - \eta \otimes \xi$ for type 1 and type 2, and $\dot{\varphi}^2 = -I + \eta \otimes \xi$ for type 3 and the induced metric is compatible with $\dot{\varphi}$ in each case.

□

Now we consider that $\overline{M}$ is a pseudo-Riemannian manifold endowed with a K-paracontact structure. The following result shows that for a proper slant submanifold M, $\nabla T^2 \neq 0$.

Theorem 4.2 *Let M be a slant submanifold of an almost paracontact metric manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $\xi \in TM$. If $\overline{M}$ is K-paracontact manifold then $\nabla T^2 = 0$ if and only if M is anti-invariant submanifold.*

Proof. Here we write Q for T^2. For any X, $Y \in TM$, we have

$$(\nabla_X Q)\, Y = \nabla_X QY - Q\,(\nabla_X Y)\,.$$

Equation (3.8) implies

$$Q\,(\nabla_X Y) = \lambda\,(\nabla_X Y - \eta(\nabla_X Y))\,,$$

for the values of λ in theorem 3.5. Again from equation (3.8) we obtain

$$Q(\nabla_X Y) = \lambda\,(\nabla_X Y - \eta(\nabla_X Y)\xi - \eta(Y)\nabla_X \xi - g(Y,\, \nabla_x \xi)\xi)\,.$$

Last three equations yield

$$\nabla_X Q = 0 \Longleftrightarrow \nabla_X \xi = 0.$$

If $\overline{M}$ is a K-paracontact pseudo-Riemannian manifold then from equations (2.5) and (2.7) we have

$$\nabla_X \xi = -TX.$$

The statement of the theorem follows from last two equations.

□

In [7], Cabrerizo et al. proved a result which was used to obtain slant submanifolds in a contact manifold from slant submanifolds of Kaehler manifold. Analogously, we state the following theorem:

Theorem 4.3 *Suppose that*

$$x(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v), f_4(u,v))$$

defines a slant surface S in $\mathbb{C}^2$ with usual para Kaehlerian structure such that $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ are non-zero and perpendicular. Then

$$y(u,v,t) = (f_1(u,v), f_2(u,v), f_3(u,v), f_4(u,v), t)$$

defines a three-dimensional slant submanifold in $\mathbb{R}^5$ with paracontact structure given in example 2.1. Moreover if we put

$$e_1 = \frac{\partial}{\partial u} + \left(2f_3 \frac{\partial f_1}{\partial u} + 2f_4 \frac{\partial f_2}{\partial u}\right) \frac{\partial}{\partial t}$$

and

$$e_2 = \frac{\partial}{\partial v} + \left(2f_3 \frac{\partial f_1}{\partial v} + 2f_4 \frac{\partial f_2}{\partial v}\right) \frac{\partial}{\partial t},$$

then $\{e_1, e_2, \xi\}$ is an orthogonal basis of the tangent bundle of the submanifold.

Proof. The proof is similar to the proof of theorem 3.5 in [7].

□

Using theorem 4.3 and examples given in [2] we obtain slant submanifolds in a paracontact manifold such that $\xi \in TM$ as shown in proposition 3.2. But when the ambient space is almost paracontact manifold, there exists proper slant submanifolds satisfying $\xi \in T^{\perp}M$.

Consider a pseudo-Riemannian manifold $(\overline{M}, g)$ endowed with a para Hermitian structure J. We recall that the almost paracontact structure (φ, ξ, η, g) on the product manifold $\overline{M} \times \mathbb{R}$ is given by

$$\varphi\left(X, s\frac{d}{dt}\right) = (JX, 0), \qquad \xi = \left(0, \frac{d}{dt}\right), \qquad \eta = dt,$$

where t is coordinate on $\mathbb{R}$ and we denote by the same symbol g the product metric of g with usual metric on $\mathbb{R}$. If (M, f) is an immersed submanifold of $\overline{M}$ i.e., M is a submanifold of $\overline{M}$ with immersion f, we denote by M_0 and M_1 respectively the immersed submanifolds (M, f_0) and $(M \times \mathbb{R}, f_1)$ of $\overline{M} \times \mathbb{R}$, where f_0 and f_1 are the natural immersions given by

$$f_0(x) = (f(x), 0), \quad f_1(x, t) = (f(x), t).$$

Notice that

$$\forall x \in M_0, \qquad T_xM_0 = T_xM \times \{0\}, \quad T_xM_0^{\perp} = T_xM^{\perp} \times \mathbb{R} \tag{4.1}$$

$$\forall (x, t) \in M_1, \; T_{(x,t)}M_1 = T_xM \times \mathbb{R}, \quad T_{(x,t)}M_1^{\perp} = T_xM^{\perp} \times \{0\}. \tag{4.2}$$

For any $x \in M$ and $X \in T_xM$, we put as usual

$$\begin{aligned} JX &= TX + NX, \\ \varphi\,(X, 0) &= T_0\,(X, 0) + N_0\,(X, 0)\,, \\ \varphi\left(X, s\tfrac{d}{dt}\right) &= T_1\left(X, s\tfrac{d}{dt}\right) + N_1\left(X, s\tfrac{d}{dt}\right), \end{aligned}$$

where $TX \in T_xM$, $NX \in T_xM^{\perp}$, $T_0\,(X, 0) \in T_xM_0$, $N_0\,(X, 0) \in T_xM_0^{\perp}$, $T_1\left(X, s\frac{d}{dt}\right) \in T_{(x,t)}M_1$, and $N_1\left(X, s\frac{d}{dt}\right) \in T_{(x,t)}M_1^{\perp}$.

Now consider (M, f) is an immersed slant submanifold with slant angle θ of an almost para Hermitian manifold $(\overline{M}, J, g)$.

Suppose M is slant submanifold of type 1 i.e., for any space-like (time-like) $X \in TM$, TX is time-like (space-like) and there exists a $\lambda \in (1, \infty)$ such that $T^2 = \lambda I$, where $\lambda = \cosh^2\theta$. By definition of T_0 and the product structure φ we have

$$T_0^2\,(X, 0) = (T^2X, 0) = \lambda\,(X, 0)\,.$$

Since ξ is orthogonal to M_0, we have

$$T_0^2 = \lambda(I - \eta \otimes \xi).$$

Thus M_0 is a slant submanifold of type 1 of $\overline{M} \times \mathbb{R}$ and the converse is also true. Similarly if M is slant submanifold of type 2 (resp. type 3), it can be proved that M_0 is also slant submanifolds of type 2 (resp. type 3) of $\overline{M} \times \mathbb{R}$ and vice-versa.
Again if M is slant submanifold of type 1 i.e., for any space-like (time-like) $X \in TM$, TX is time-like (space-like) and there exists a $\lambda \in (1, \infty)$ such that $T^2 = \lambda I$, where $\lambda = \cosh^2\theta$. By definition of T_1 and the product structure φ we have

$$T_1^2\left(X,\, s\frac{d}{dt}\right) = (T^2X,\, 0) = \lambda\left(X,\, s\frac{d}{dt}\right).$$

But this time ξ is tangent to M_1. Let $\mathbb{H}$ denotes the distribution orthogonal to ξ i.e.,

$$TM_1 = \mathbb{H} \oplus < \xi > .$$

Thus for any $\left(X,\, s\frac{d}{dt}\right) \in \mathbb{H}$ we have

$$T_1^2 = \lambda I.$$

Hence we have

$$T_1^2 = \lambda(I - \eta \otimes \xi).$$

Therefore M_1 is also a slant submanifold of type 1 of $\overline{M} \times \mathbb{R}$ and the converse can be proved easily. Similarly if M is slant submanifold of type 2 (resp. type 3), it can be proved that M_1 is also slant submanifolds of type 2 (resp. type 3) of $\overline{M} \times \mathbb{R}$ and vice-versa.
Let us summarize:

Theorem 4.4 *Let $(M,\, f)$ is an immersed slant submanifold with slant angle θ of an almost para Hermitian manifold $(\overline{M},\, J,\, g)$ and let M_0 and M_1 be the immersed submanifolds of $\overline{M} \times \mathbb{R}$ defined as above. Then*

(1) The characteristic vector field ξ of $\overline{M} \times \mathbb{R}$ is orthogonal to M_0 but tangent to M_1.
(2) If $\theta \in [0,\, \frac{\pi}{2}]$, the following statements are equivalent:
(a) M is slant in $\overline{M}$ with slant angle θ.
(b) M_0 is slant in $\overline{M} \times \mathbb{R}$ with slant angle θ.
(c) M_1 is slant in $\overline{M} \times \mathbb{R}$ with slant angle θ.
Furthermore, all the slant submanifolds namely M in $\overline{M}$; M_0 and M_1 in $\overline{M} \times \mathbb{R}$ are of same type.

5 Examples

In this section we present some examples of proper slant submanifolds of an almost paracontact metric manifold. These examples are inspired on examples given in [2]. We begin with an example of proper slant submanifold of a paracontact metric manifold.

Example 5.1 Let us consider a 3−dimensional submanifold M of $\mathbb{R}^5$ equipped with paracontact structure given in example 2.1 defined by

$$(u,\, v,\, t) \mapsto 2\,(u\, cos\,\theta,\, v,\, u\, sin\,\theta,\, 0,\, t)\,,$$

where $\theta \in [0,\, \pi/2]$. It is easy to see that M is a slant submanifold of type 2 with slant angle θ.

Example 5.2 Consider $\mathbb{R}^5$ endowed with the structure $(\Phi_1,\, \xi,\, \eta,\, g_1)$, where Φ_1 is the tensor of type $(1,\, 1)$ defined by

$$\Phi_1\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad \Phi_1\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial x^i}, \quad \Phi_1\left(\frac{\partial}{\partial z}\right) = 0,$$

$\xi = \frac{\partial}{\partial z}, \quad \eta = dz$ and g_1 is the pseudo-Riemannian metric given by

$$g_1\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right) = 1, \quad g_1\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i}\right) = -1, \quad g_1\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = 1,$$

where $(x^1,\, y^1,\, x^2,\, y^2,\, z)$ are standard coordinates on $\mathbb{R}^5$. It is easy to see that $(\Phi_1,\, \xi,\, \eta,\, g_1)$ is an almost paracontact pseudo-Riemannian structure on $\mathbb{R}^5$. For any $a,\, b \in \mathbb{R}$ satisfying $(a^2 + b^2 \neq 1)$, let M_1 be submanifold of the almost paracontact metric manifold $(\mathbb{R}^5,\, \Phi_1,\, \xi,\, \eta,\, g_1)$ defined by

$$(u,\, v,\, t) \mapsto (au,\, v,\, bu,\, u,\, t)\,. \tag{5.1}$$

It is easy to verify that M_1 is a slant submanifold with

$$T^2X \quad = \frac{a^2}{a^2 + b^2 - 1}\,(X - \eta(X)\xi)\,.$$

Moreover we have the following cases:

1. M_1 is slant submanifold of type 1 if $a^2 + b^2 > 1$ and $b^2 < 1$.
2. M_1 is slant submanifold of type 2 if $a^2 + b^2 > 1$ and $b^2 > 1$.
3. M_1 is slant submanifold of type 3 if $a^2 + b^2 < 1$.

Now, we have the following examples of slant submanifolds for particular values of a and b.

Example 5.3 For any $\theta > 0$,

$$(u,\, v,\, t) \mapsto \left(u\, cosh\,\theta,\, v,\, u\,\sqrt{1 - sinh^2\,\theta},\, u,\, t\right)$$

defines a slant submanifold of type 1 in the almost paracontact metric manifold $(\mathbb{R}^5,\, \Phi_1,\, \xi,\, \eta,\, g_1)$ with slant angle θ.

Example 5.4 For any $\theta \in (0,\, \pi/2)$,

$$(u,\, v,\, t) \mapsto \left(u\, cos\,\theta,\, v,\, u\,\sqrt{1 + sin^2\,\theta},\, u,\, t\right)$$

defines a slant submanifold of type 2 in the almost paracontact metric manifold $(\mathbb{R}^5,\, \Phi_1,\, \xi,\, \eta,\, g_1)$ with slant angle θ.

Example 5.5 For any $\theta > 0$ satisfying $-1 < sinh\,\theta < 1$,

$$(u,\, v,\, t) \mapsto \left(u\, tanh\,\theta,\, v,\, u\,\sqrt{1 - sinh^2\theta},\, u,\, t\right)$$

defines a slant submanifold of type 3 in the almost paracontact metric manifold $(\mathbb{R}^5,\, \Phi_1,\, \xi,\, \eta,\, g_1)$ with slant angle θ.

Example 5.6 Now consider $\mathbb{R}^5 = \{(x^1,\, y^1,\, x^2,\, y^2,\, z) : x^i, y^i, z \in \mathbb{R}\}$ endowed with different almost paracontact structure $(\Phi_2,\, \xi,\, \eta,\, g_2)$ given by

$$\Phi_2 = \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&-1&0&0\\ 0&0&0&-1&0\\ 0&0&0&0&1 \end{pmatrix},$$

$$\xi = \begin{pmatrix} 0\\0\\0\\0\\1 \end{pmatrix} \text{ and } \eta = (0\;0\;0\;0\;1)\,.$$

For any $a,\, b \in \mathbb{R}$ satisfying $(a^2 - b^2 \neq 1)$, let M_2 be submanifold of $\mathbb{R}^5$ defined by

$$(u,\, v,\, t) \mapsto (u,\, av,\, bv,\, v,\, t)\,. \tag{5.2}$$

It is clear that M_2 is a slant submanifold with

$$T^2X \quad = \frac{b^2}{-a^2 + b^2 + 1}\,(X - \eta(X)\xi)\,.$$

Moreover, we have the following cases:

1. M_2 is slant submanifold of type 1 if $a^2 - b^2 < 1$ and $a^2 > 1$.
2. M_2 is slant submanifold of type 2 if $a^2 - b^2 < 1$ and $a^2 < 1$.
3. M_2 is slant submanifold of type 3 if $a^2 - b^2 > 1$.

In the remaining part of this section, for any $a,\, b \in \mathbb{R}$, M_1 and M_2 denote submanifolds isometrically immersed in $\mathbb{R}^5$ defined by equations (5.1) and (5.2) respectively.

Example 5.7 If M_1 is considered as the submanifold of the almost paracontact metric manifold $(\mathbb{R}^5,\, \Phi_2,\, \xi,\, \eta,\, g_2)$ such that $(a^2 - b^2 \neq 1)$, it is easy to verify that M_1 is a slant submanifold with

$$T^2X \quad = \frac{1}{-a^2 + b^2 + 1}\,(X - \eta(X)\xi)\,.$$

Moreover,

1. M_1 is slant submanifold of type 1 if $a^2 - b^2 < 2$.
2. M_1 is slant submanifold of type 2 if $2 < a^2 - b^2 < 1$.
3. M_1 is slant submanifold of type 3 if $a^2 - b^2 > 1$.

Example 5.8 Consider M_2 as the submanifold of the almost paracontact metric manifold $(\mathbb{R}^5, \Phi_1, \xi, \eta, g_1)$, where $(b^2 - a^2 \neq 1)$. It is clear that M_2 is a slant submanifold with

$$T^2X \quad = \frac{a^2}{a^2 - b^2 + 1}\,(X - \eta(X)\xi)\,.$$

Moreover,

1. M_2 is slant submanifold of type 1 if $b^2 - a^2 < 1$ and $b^2 > 1$.
2. M_2 is slant submanifold of type 2 if $b^2 - a^2 < 1$ and $b^2 < 1$.
3. M_2 is slant submanifold of type 3 if $b^2 - a^2 > 1$.

Example 5.9 Now consider $\mathbb{R}^5 = \{(x^1, y^1, x^2, y^2, z) : x^i, y^i, z \in \mathbb{R}\}$ endowed with another pseudo-Riemannian metric g_3 given by

$$g_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that (Φ_1, ξ, η, g_3) and (Φ_2, ξ, η, g_3) both define almost paracontact pseudo-Riemannian metric structures on $\mathbb{R}^5$. Consider M_1 with $(b^2 - a^2 \neq 1)$. It is slant submanifold of $(\mathbb{R}^5, \Phi_1, \xi, \eta, g_3)$ with

$$T^2X \; = \frac{a^2}{a^2 - b^2 + 1}\,(X - \eta(X)\xi)$$

and slant submanifold of $(\mathbb{R}^5, \Phi_2, \xi, \eta, g_3)$ with

$$T^2X \; = \frac{-1}{b^2 - a^2 - 1}\,(X - \eta(X)\xi)\,.$$

Similarly M_2 with $(a^2 + b^2 \neq 1)$ is slant submanifold of $(\mathbb{R}^5, \Phi_1, \xi, \eta, g_3)$ as well as $(\mathbb{R}^5, \Phi_2, \xi, \eta, g_3)$.

Remark 5.1 We can obtain a slant submanifold of particular type by selecting proper values of a, b in these examples.

Acknowledgements The author is grateful to the referee(s) for their valuable comments and suggestions.

References

1. Alegre, P. – *Slant submanifolds of Lorentzian Sasakian and para Sasakian manifolds*, Taiwanese. J. Math., **17** (2013), no. 3, 897-910.
2. Alegre, P.; Carriazo, A. – *Slant submanifolds of para-Hermitian manifolds*, Mediterr. J. Math., **14** (2017), no. 5, 1-14.
3. Alegre, P.; Carriazo, A. – *Bi-slant Submanifolds of para Hermitian Manifolds*, Mathematics, **618 (7)** (2019), 1-15.
4. Arslan, K.; Carriazo, A.; Chen, B.-Y.; Murathan, C. – *On slant submanifolds of neutral Kaehler manifolds*, Taiwanese J. Math., **14** (2010), no. 2, 561-584.
5. Blair, D.E. – *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, **509** (1976).
6. Cabrerizo, J.L.; Carriazo, A.; Fernández, L.M.; Fernández, M. – *Semi-slant submanifolds of a Sasakian manifold*, Geom. Dedicata, **78** (1999), no. 2, 183-199.
7. Cabrerizo, J.L.; Carriazo, A.; Fernández, L.M.; Fernández, M. – *Slant submanifolds in Sasakian manifolds*, Glasg. Math. J., **42** (2000), no. 1, 125-138.
8. Cabrerizo, J.L.; Carriazo, A.; Fernández, L.M.; Fernández, M. – *Existence and uniqueness theorem for slant immersions in Sasakian-space-forms*, Publ. Math. Debrecen, **58** (2001), no. 4, 559-574.
9. Calvaruso, G.; Martín-Molina, V. – *Paracontact metric structures on the unit tangent sphere bundle*, Ann. Mat. Pura Appl., **(4) 194** (2015), no. 5, 1359-1380.
10. Carriazo, A.; Pérez-Garcia, M.J. – *Slant submanifolds in neutral almost contact pseudo-metric manifolds*, Differential Geom. Appl., **54** (2017), part. A, 71-80.
11. Chen, B.-Y. – *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, Louvain, 1990.
12. Chen, B.-Y.; Garay, O.J. – *Classification of quasi-minimal surfaces with parallel mean curvature vector in pseudo-Euclidean 4-space E^4_2*, Results Math., **55** (2009), no. 1-2, 23-38.
13. Chen, B.-Y.; Mihai, I. – *Classification of quasi-minimal slant surfaces in Lorentzian complex space forms*, Acta Math. Hungar., **122** (2009), no. 4, 307-328.
14. Kaneyuki, S.; Williams, F.L. – *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J., **99** (1985), 173-187.
15. Kaneyuki, S.; Kozai, M. – *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math., **8** (1985), no. 1, 81-98.
16. Lotta, A. – *Slant submanifolds in contact geometry*, Bull. Math. Soc. Sc. Math. Roum., **39 (87)** (1996), no. 1-4, 183-198.
17. Martín-Molina, V. – *Paracontact metric manifolds without a contact metric counterpart*, Taiwanese J. Math., **19** (2015), no. 1, 175-191.
18. Perrone, A. – *Some results on almost paracontact metric manifolds*, Mediterr. J. Math., **13** (2016), no. 5, 3311-3326.
19. Rahman, M.S. – *A study of para-Sasakian manifolds*, IC/95/212 (Internal Report) International Centre for Theoretical Physics, Trieste, Italy, 1995.
20. Sasaki, S. – *On differentiable manifolds with certain structures which are closely related to almostcontact structure, I*, Tohoku Math. J., **(2) 12** (1960), no. 3, 459-476.
21. Satō, I. – *On a structure similar to the almost contact structure*, Tensor (N.S.), **30** (1976), no. 3, 219-224.
22. Shahid, M.H. – *Differential geometry of CR-submanifolds of a normal almost para contact manifold*, IC/92/413 (Internal Report) International Centre for Theoritical Physics, Trieste, Italy, 1992.
23. Uddin, S.; Khan, M.A.; Singh, K. – *Totally umbilical proper slant and hemislant submanifolds of an LP-cosymplectic manifold*, Math. Probl. Eng., 2011, 1-9.
24. Wełyczko, J. – *Para-CR structures on almost paracontact metric manifolds*, J. Appl. Anal., **20** (2014), no. 2, 105-117.
25. Zamkovoy, S.- *Canonical connections on paracontact manifolds*, Ann. Global Anal. Geom., **36** (2009), no. 1, 37-60.

Received: 16.III.2020 / Revised: 23.X.2020 / Accepted: 05.XI.2020

Author

Sunil Kumar Chanyal,
Department of Mathematics,
D. S. B. Campus, Kumaun University,
Nainital-263002, Uttarakhand, India,
E-mail: skchanyal.math@gmail.com, skchanyal@kunainital.ac.in