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On a new generalization of bihyperbolic Pell numbers

Dorota Bród · Anetta Szynal-Liana · Iwona Włoch

Abstract In this paper we introduce a new one-parameter generalization of bihyperbolic Pell numbers. We give some of their properties, among others the Binet formula, convolution identity and the generating function.

Keywords Pell numbers · hyperbolic numbers · bihyperbolic numbers · recurrence relations

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1 Introduction

Hyperbolic numbers are two dimensional number system. A hyperbolic number is defined as $z = x+\mathbf{h}y$, where $x, y \in \mathbb{R}$ and $\mathbf{h}$ is a unipotent (hyperbolic) imaginary unit such that $\mathbf{h}^2 = 1$ and $\mathbf{h} \neq \pm 1$. Hence the set of hyperbolic numbers is defined as

$$\mathbb{H} = \{z \colon z = x + \mathbf{h}y, x, y \in \mathbb{R}, \mathbf{h}^2 = 1\}.$$

The hyperbolic numbers were introduced by J. Cockle, see [5]-[8]. Many algebraic properties of hyperbolic numbers are presented in [12] and [13].

The set of bihyperbolic numbers is defined as follows

$$\mathbb{H}_2 = \{\zeta \colon \zeta = x_0 + j_1 x_1 + j_2 x_2 + j_3 x_3\},$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $j_1, j_2, j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1 \tag{1.1}$$

and

$$j_1 j_2 = j_2 j_1 = j_3, j_1 j_3 = j_3 j_1 = j_2, j_2 j_3 = j_3 j_2 = j_1. \tag{1.2}$$

The multiplication of bihyperbolic numbers can be made analogously as multiplication of algebraic expressions. The addition and multiplication on $\mathbb{H}_2$ are commutative and associative. Moreover, the multiplication is distributive over addition. Hence $(\mathbb{H}_2, +, \cdot)$ is a commutative ring. Hyperbolic numbers and bihyperbolic numbers are well-studied in the literature, see [1, 12].

Consider the special cases of sequences $\{a_n\}$ which are defined recurrently as a linear combination of the preceding k terms

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_k a_{n-k} \quad \text{for } n \geq k, \tag{1.3}$$

where $k \geq 2$, b_i are integers, $i = 1, 2, \ldots, k$ and $a_0, a_1, \ldots, a_{k-1}$ are given numbers. By recurrence (1.3) for $k = 2$ we get the well-known sequences e.g. Fibonacci, Lucas, Pell, Jacobsthal sequence. In the literature sequences defined by (1.3) for $k = 2$ are called sequences of the Fibonacci type.

Using the concept of Pell numbers we will introduce bihyperbolic Pell numbers and bihyperbolic r-Pell numbers. Recall some properties of Pell numbers.

The Pell sequence $\{P_n\}$ is defined by the following recurrence $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with initial conditions $P_0 = 0$, $P_1 = 1$. The Binet formula of the Pell numbers has the form

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$$

Moreover, the Pell numbers are defined by the following formula

$$P_n = \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} 2^k.$$

The matrix generator of the sequence $\{P_n\}$ is $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. It is known that

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Hence we get the well-known formula (Cassini's identity) $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$. Another interesting properties of the Pell numbers are given in [9].

Let $n \geq 0$ be an integer. Define bihyperbolic Pell sequence $\{BhP_n\}$ by the following recurrence relation

$$BhP_n = P_n + j_1 P_{n+1} + j_2 P_{n+2} + j_3 P_{n+3}, \tag{1.4}$$

where P_n is nth Pell number.

In this paper we study one-parameter generalization of the bihyperbolic Pell numbers.

2 The r-Pell numbers

In the literature there are some generalizations of the Pell numbers, see [3,4,10,11]. In [2] a one-parameter generalization of the Pell numbers was investigated. We recall this generalization.

Let $n \geq 0$, $r \geq 1$ be integers, the *r-Pell sequence* $\{P(r,n)\}$ is defined by the following recurrence relation

$$P(r,n) = 2^r P(r,n-1) + 2^{r-1} P(r,n-2) \text{ for } n \geq 2 \tag{2.1}$$

with initial conditions $P(r,0) = 2$, $P(r,1) = 1 + 2^{r+1}$.
It is easily seen that $P(1,n) = P_{n+2}$.

By (2.1) we obtain

$$\begin{array}{l} P(r,0) = 2 \\ P(r,1) = 1 + 2^{r+1} \\ P(r,2) = 2^{r+1} + 2 \cdot 4^r \\ P(r,3) = 2^{r-1} + 3 \cdot 4^r + 2 \cdot 8^r \\ P(r,4) = \frac{3}{2} \cdot 4^r + 4 \cdot 8^r + 2 \cdot 16^r \\ P(r,5) = \frac{1}{4} \cdot 4^r + 3 \cdot 8^r + 5 \cdot 16^r + 2 \cdot 32^r. \end{array} \tag{2.2}$$

In [2] it was proved that the r-Pell numbers can be used for counting of independent sets of special classes of graphs. We will recall some properties of the r-Pell numbers, they will be useful in the next section.

Theorem 2.1 [2] (*Binet formula*) *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$P(r,n) = \left(1 + \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}\right) r_1^n + \left(1 - \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}\right) r_2^n,$$

where

$$r_1 = \frac{1}{2}\big(2^r + \sqrt{4^r + 2^{r+1}}\big),\ r_2 = \frac{1}{2}\big(2^r - \sqrt{4^r + 2^{r+1}}\big). \tag{2.3}$$

Proposition 2.2 [2] *Let $n \geq 4$, $r \geq 1$ be integers. Then*

$$P(r,n) = (8^r + 4^r)P(r,n-3) + (2^{3r-1} + 2^{2r-2})P(r,n-4).$$

Theorem 2.3 [2] *Let $n \geq 1$, $r \geq 1$ be integers. Then*

$$\sum_{l=0}^{n-1} P(r,l) = \frac{P(r,n) + 2^{r-1}P(r,n-1) - 3}{3 \cdot 2^{r-1} - 1}. \tag{2.4}$$

Theorem 2.4 [2] (*convolution identity*) *Let n, m, r be integers $m \geq 2, n \geq 1$, $r \geq 1$. Then*

$$P(r,m+n) = 2^{r-1}P(r,m-1)P(r,n) + 2^{2r-2}P(r,m-2)P(r,n-1). \tag{2.5}$$

Theorem 2.5 [2] *The generating function of the sequence $\{P(r,n)\}$ has the following form*

$$f(x) = \frac{2 + x}{1 - 2^r x - 2^{r-1}x^2}.$$

3 The bihyperbolic r-Pell numbers

Let $r \geq 1$, $n \geq 0$ be integers, the nth bihyperbolic r-Pell number BhP_n^r we define as

$$BhP_n^r = P(r,n) + j_1P(r,n+1) + j_2P(r,n+2) + j_3P(r,n+3), \quad (3.1)$$

where $P(r,n)$ is nth r-Pell number.

By (2.2) and (3.1) we obtain

$$\begin{aligned} BhP_0^r &= 2 + j_1(1+2^{r+1}) + j_2(2^{r+1}+2\cdot 4^r) + j_3(2^{r-1}+3\cdot 4^r + 2\cdot 8^r) \\ BhP_1^r &= 1 + 2^{r+1} + j_1(2^{r+1}+2\cdot 4^r) + j_2(2^{r-1}+3\cdot 4^r + 2\cdot 8^r) \\ &\quad + j_3(\tfrac{3}{2}\cdot 4^r + 4\cdot 8^r + 2\cdot 16^r) \\ BhP_2^r &= 2^{r+1} + 2\cdot 4^r + j_1(2^{r-1}+3\cdot 4^r + 2\cdot 8^r) \\ &\quad + j_2(\tfrac{3}{2}\cdot 4^r + 4\cdot 8^r + 2\cdot 16^r) + j_3(\tfrac{1}{4}\cdot 4^r + 3\cdot 8^r + 5\cdot 16^r + 2\cdot 32^r). \end{aligned} \quad (3.2)$$

By the definition of bihyperbolic r-Pell numbers we get the following recurrence relations.

Theorem 3.1 *Let $n \geq 2$, $r \geq 1$ be integers. Then*

$$BhP_n^r = 2^r BhP_{n-1}^r + 2^{r-1} BhP_{n-2}^r, \quad (3.3)$$

where BhP_0^r, BhP_1^r are given by (3.2).

Proof. Using (2.1) and (3.1), we have

$$\begin{aligned} & 2^r BhP_{n-1}^r + 2^{r-1} BhP_{n-2}^r \\ &= 2^r(P(r,n-1) + j_1P(r,n) + j_2P(r,n+1) + j_3P(r,n+2)) \\ &\quad + 2^{r-1}(P(r,n-2) + j_1P(r,n-1) + j_2P(r,n) + j_3P(r,n+1)) \\ &= P(r,n) + j_1P(r,n+1) + j_2P(r,n+2) + j_3P(r,n+3) = BhP_n^r. \end{aligned}$$

□

Proposition 3.2 *Let $n \geq 4$, $r \geq 0$ be integers. Then*

$$BhP_n^r = (8^r + 4^r)BhP_{n-3}^r + (2^{3r-1} + 2^{2r-2})BhP_{n-4}^r.$$

Proof. By Proposition 2.2 we obtain

$$\begin{aligned} BhP_n^r &= P(r,n) + j_1P(r,n+1) + j_2P(r,n+2) + j_3P(r,n+3) \\ &= (8^r+4^r)P(r,n-3) + (2^{3r-1}+2^{2r-2})P(r,n-4) \\ &\quad + j_1((8^r+4^r)P(r,n-2) + (2^{3r-1}+2^{2r-2})P(r,n-3)) \\ &\quad + j_2((8^r+4^r)P(r,n-1) + (2^{3r-1}+2^{2r-2})P(r,n-2)) \\ &\quad + j_3((8^r+4^r)P(r,n) + (2^{3r-1}+2^{2r-2})P(r,n-1)) \\ &= (8^r+4^r)(P(r,n-3) + j_1P(r,n-2) \\ &\quad + j_2P(r,n-1) + j_3P(r,n)) \\ &\quad + (2^{3r-1}+2^{2r-2})(P(r,n-4) + j_1P(r,n-3) \\ &\quad + j_2P(r,n-2) + j_3P(r,n-1)). \end{aligned}$$

Hence we have

$$BhP_n^r = (8^r + 4^r)BhP_{n-3}^r + (2^{3r-1} + 2^{2r-2})BhP_{n-4}^r,$$

which ends the proof.

□

Theorem 3.3 *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$\begin{aligned} &BhP_n^r - j_1 BhP_{n+1}^r - j_2 BhP_{n+2}^r - j_3 BhP_{n+3}^r \\ &= P(r,n) - P(r,n+2) - P(r,n+4) + P(r,n+6) - 2j_3 BhP_{n+3}^r. \end{aligned}$$

Proof. Using multiplication rules (1.1) and (1.2), we obtain

$$\begin{aligned} &BhP_n^r - j_1 BhP_{n+1}^r - j_2 BhP_{n+2}^r - j_3 BhP_{n+3}^r \\ &= P(r,n) + j_1 P(r,n+1) + j_2 P(r,n+2) + j_3 P(r,n+3) \\ &\quad -j_1(P(r,n+1) + j_1 P(r,n+2) + j_2 P(r,n+3) + j_3 P(r,n+4)) \\ &\quad -j_2(P(r,n+2) + j_1 P(r,n+3) + j_2 P(r,n+4) + j_3 P(r,n+5)) \\ &\quad -j_3(P(r,n+3) + j_1 P(r,n+4) + j_2 P(r,n+5) + j_3 P(r,n+6)) \\ &= P(r,n) + j_1 P(r,n+1) + j_2 P(r,n+2) + j_3 P(r,n+3) \\ &\quad -j_1 P(r,n+1) - P(r,n+2) - j_3 P(r,n+3) - j_2 P(r,n+4) \\ &\quad -j_2 P(r,n+2) - j_3 P(r,n+3) - P(r,n+4) - j_1 P(r,n+5) \\ &\quad -j_3 P(r,n+3) - j_2 P(r,n+4) - j_1 P(r,n+5) - P(r,n+6) \\ &= P(r,n) - P(r,n+2) - P(r,n+4) - P(r,n+6) \\ &\quad -2\big(j_1 P(r,n+5) + j_2 P(r,n+4) + j_3 P(r,n+3)\big) \\ &= P(r,n) - P(r,n+2) - P(r,n+4) + P(r,n+6) \\ &\quad -2j_3(P(r,n+3) + j_1 P(r,n+4) + j_2 P(r,n+5) + j_3 P(r,n+6)) \\ &= P(r,n) - P(r,n+2) - P(r,n+4) + P(r,n+6) - 2j_3 BhP_{n+3}^r. \end{aligned}$$

□

Now we will give the Binet formula for the bihyperbolic r-Pell numbers.

Theorem 3.4 (*Binet formula*) *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$BhP_n^r = C_1 r_1^n (1 + j_1 r_1 + j_2 r_1^2 + j_3 r_1^3) + C_2 r_2^n (1 + j_1 r_2 + j_2 r_2^2 + j_3 r_2^3),$$

where r_1, r_2 are given by (2.3) *and*

$$C_1 = 1 + \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}},\ C_2 = 1 - \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}.$$

Proof. By Theorem 2.1 we get

$$\begin{aligned} BhP_n^r &= P(r,n) + j_1 P(r,n+1) + j_2 P(r,n+2) + j_3 P(r,n+3) \\ &= C_1 r_1^n + C_2 r_2^n + j_1(C_1 r_1^{n+1} + C_2 r_2^{n+1}) \\ &\quad + j_2(C_1 r_1^{n+2} + C_2 r_2^{n+2}) + j_3(C_1 r_1^{n+3} + C_2 r_2^{n+3}) \\ &= C_1 r_1^n \left(1 + j_1 r_1 + j_2 r_1^2 + j_3 r_1^3\right) + C_2 r_2^n \left(1 + j_1 r_2 + j_2 r_2^2 + j_3 r_2^3\right). \end{aligned}$$

□

By Theorem 3.4 we get the Binet formula for the bihyperbolic Pell numbers.

Corollary 3.5 *Let $n \geq 0$ be an integer. Then*

$$BhP_n = \frac{1}{2\sqrt{2}}\big((1+\sqrt{2})^n A - (1-\sqrt{2})^n B\big),$$

where

$$A = 1 + (1+\sqrt{2})j_1 + (3+2\sqrt{2})j_2 + (7+5\sqrt{2})j_3,$$

$$B = 1 + (1-\sqrt{2})j_1 + (3-2\sqrt{2})j_2 + (7-5\sqrt{2})j_3.$$

Proof. By Theorem 3.4 and formula (2.3) we have $C_1 = \frac{2\sqrt{2}+3}{2\sqrt{2}}$, $C_2 = \frac{2\sqrt{2}-3}{2\sqrt{2}}$, $r_1 = 1+\sqrt{2}$, $r_2 = 1-\sqrt{2}$. Hence

$$BhP_n^1 = C_1(1+\sqrt{2})^n A + C_2(1-\sqrt{2})^n B = BhP_{n+2}.$$

Thus

$$\begin{aligned} BhP_n &= \tfrac{2\sqrt{2}+3}{2\sqrt{2}}(1+\sqrt{2})^{n-2}A + \tfrac{2\sqrt{2}-3}{2\sqrt{2}}(1-\sqrt{2})^{n-2}B \\ &= \tfrac{1}{2\sqrt{2}}\big((1+\sqrt{2})^n A - (1-\sqrt{2})^n B\big). \end{aligned}$$

□

In the next theorem we give a summation formula for the bihyperbolic r-Pell numbers.

Theorem 3.6 *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$\begin{aligned} \sum_{l=0}^{n} BhP_l^r = {} & \tfrac{BhP_{n+1}^r + 2^{r-1}BhP_n^r - 3(1+j_1+j_2+j_3)}{3\cdot 2^{r-1}-1} \\ & - \left(2j_1 + (3+2^{r+1})j_2 + (3+2\cdot 4^r + 2^{r+2})j_3\right). \end{aligned}$$

Proof. By the definition of the bihyperbolic r-Pell numbers we get

$$\begin{aligned} \sum_{l=0}^{n} BhP_l^r = {} & P(r,0) + j_1P(r,1) + j_2P(r,2) + j_3P(r,3) \\ & + P(r,1) + j_1P(r,2) + j_2P(r,3) + j_3P(r,4) + \cdots \\ & + P(r,n) + j_1P(r,n+1) + j_2P(r,n+2) + j_3P(r,n+3) \\ = {} & P(r,0) + P(r,1) + \cdots + P(r,n) \\ & + j_1\left(P(r,1) + P(r,2) + \cdots + P(r,n+1) + P(r,0) - P(r,0)\right) \\ & + j_2(P(r,2) + P(r,3) + \cdots + P(r,n+2) + P(r,0) + P(r,1) \\ & - P(r,0) - P(r,1)) + j_3(P(r,3) + P(r,4) \\ & + \cdots + P(r,n+3) + P(r,0) + P(r,1) + P(r,2) \\ & - P(r,0) - P(r,1) - P(r,2)). \end{aligned}$$

By formula (2.4) we obtain

$$\begin{aligned}\sum_{l=0}^{n} BhP_l^r = \tfrac{1}{3\cdot 2^{r-1}-1}[&P(r,n+1)+2^{r-1}P(r,n)-3\\ &+j_1(P(r,n+2)+2^{r-1}P(r,n+1)-3)\\ &+j_2(P(r,n+3)+2^{r-1}P(r,n+2)-3)\\ &+j_3(P(r,n+4)+2^{r-1}P(r,n+3)-3)]\\ &-P(r,0)j_1-(P(r,0)+P(r,1))j_2\\ &-(P(r,0)+P(r,1)+P(r,2))j_3.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{l=0}^{n} BhP_l^r &= \tfrac{1}{3\cdot 2^{r-1}-1}[P(r,n+1)+j_1P(r,n+2)\\ &\quad +j_2P(r,n+3)+j_3P(r,n+4)\\ &\quad +2^{r-1}(P(r,n)+j_1P(r,n+1)\\ &\quad +j_2P(r,n+2)+j_3P(r,n+3))\\ &\quad -3(1+j_1+j_2+j_3)]-2j_1-(3+2^{r+1})j_2\\ &\quad -(3+2\cdot 4^r+2^{r+2})j_3\\ &= \tfrac{BhP_{n+1}^r+2^{r-1}BhP_n^r-3(1+j_1+j_2+j_3)}{3\cdot 2^{r-1}-1}\\ &\quad -\left(2j_1+(3+2^{r+1})j_2+(3+2\cdot 4^r+2^{r+2})j_3\right).\end{aligned}$$

□

Theorem 3.7 (*convolution identity*) *Let $m \geq 2, n \geq 1$, $r \geq 1$. Then*

$$\begin{aligned}2BhP_{m+n}^r = 2^{r-1}&BhP_{m-1}^r BhP_n^r + 2^{2r-2}BhP_{m-2}^r BhP_{n-1}^r\\ &+P(r,m+n)+P(r,m+n+6)\\ &-P(r,m+n+2)-P(r,m+n+4)-2j_3BhP_{m+n+3}^r.\end{aligned}$$

Proof. Using (1.1) and (1.2), we have

$$\begin{aligned}
&2^{r-1}BhP^r_{m-1}BhP^r_n + 2^{2r-2}BhP^r_{m-2}BhP^r_{n-1}\\
&= 2^{r-1}(P(r,m-1) + j_1P(r,m) + j_2P(r,m+1) + j_3P(r,m+2))\\
&\quad\cdot(P(r,n) + j_1P(r,n+1) + j_2P(r,n+2) + j_3P(r,n+3))\\
&\quad+2^{2r-2}(P(r,m-2) + j_1P(r,m-1) + j_2P(r,m) + j_3P(r,m+1))\\
&\quad\cdot(P(r,n-1) + j_1P(r,n) + j_2P(r,n+1) + j_3P(r,n+2))\\
&= 2^{r-1}(P(r,m-1)P(r,n) + j_1P(r,m-1)P(r,n+1)\\
&\quad+j_2P(r,m-1)P(r,n+2) + j_3P(r,m-1)P(r,n+3))\\
&\quad+j_1P(r,m)P(r,n) + P(r,m)P(r,n+1) + j_3P(r,m)P(r,n+2)\\
&\quad+j_2P(r,m)P(r,n+3) + j_2P(r,m+1)P(r,n)\\
&\quad+j_3P(r,m+1)P(r,n+1)\\
&\quad+P(r,m+1)P(r,n+2) + j_1P(r,m+1)P(r,n+3)\\
&\quad+j_3P(r,m+2)P(r,n) + j_2P(r,m+2)P(r,n+1)\\
&\quad+j_1P(r,m+2)P(r,n+2) + P(r,m+2)P(r,n+3))\\
&\quad+2^{2r-2}(P(r,m-2)P(r,n-1) + j_1P(r,m-2)P(r,n)\\
&\quad+j_2P(r,m-2)P(r,n+1) + j_3P(r,m-2)P(r,n+2)\\
&\quad+j_1P(r,m-1)P(r,n-1) + P(r,m-1)P(r,n)\\
&\quad+j_3P(r,m-1)P(r,n+1) + j_2P(r,m-1)P(r,n+2)\\
&\quad+j_2P(r,m)P(r,n-1) + j_3P(r,m)P(r,n) + P(r,m)P(r,n+1)\\
&\quad+j_1P(r,m)P(r,n+2) + j_3P(r,m+1)P(r,n-1)\\
&\quad+j_2P(r,m+1)P(r,n) + j_1P(r,m+1)P(r,n+1)\\
&\quad+P(r,m+1)P(r,n+2)).
\end{aligned}$$

By simple calculations and formula (2.5) we get

$$\begin{aligned}
&2^{r-1}BhP^r_{m-1}BhP^r_n + 2^{2r-2}BhP^r_{m-2}BhP^r_{n-1}\\
&= 2^{r-1}P(r,m-1)P(r,n) + 2^{2r-2}(P(r,m-2)P(r,n-1)\\
&\quad+j_1(2^{r-1}P(r,m-1)P(r,n+1) + 2^{2r-2}P(r,m-2)P(r,n))\\
&\quad+j_2(2^{r-1}P(r,m-1)P(r,n+2) + 2^{2r-2}P(r,m-2)P(r,n+1))\\
&\quad+j_3(2^{r-1}P(r,m-1)P(r,n+3) + 2^{2r-2}P(r,m-2)P(r,n+2))\\
&\quad+j_1(2^{r-1}P(r,m)P(r,n) + 2^{2r-2}P(r,m-1)P(r,n-1))\\
&\quad+j_2(2^{r-1}P(r,m+1)P(r,n) + 2^{2r-2}P(r,m)P(r,n-1))\\
&\quad+j_3(2^{r-1}P(r,m)P(r,n+2) + 2^{2r-2}P(r,m-1)P(r,n+1))\\
&\quad+2^{r-1}P(r,m)P(r,n+1) + 2^{2r-2}P(r,m-1)P(r,n)\\
&\quad+2^{r-1}P(r,m+1)P(r,n+2) + 2^{2r-2}P(r,m)P(r,n+1)\\
&\quad+2^{r-1}P(r,m+2)P(r,n+3) + 2^{2r-2}P(r,m+1)P(r,n+2)\\
&\quad+j_1[2^{r-1}P(r,m+1)P(r,n+3) + 2^{2r-2}(P(r,m)P(r,n+2)\\
&\quad+2^{r-1}P(r,m+2)P(r,n+2) + 2^{2r-2}(P(r,m+1)P(r,n+1)]\\
&\quad+j_2[2^{r-1}P(r,m+2)P(r,n+1) + 2^{2r-2}P(r,m+1)P(r,n)\\
&\quad+2^{r-1}P(r,m)P(r,n+3) + 2^{2r-2}(P(r,m-1)P(r,n+2)]\\
&\quad+j_3[2^{r-1}P(r,m+1)P(r,n+1) + 2^{2r-2}P(r,m)P(r,n)\\
&\quad+2^{r-1}P(r,m+2)P(r,n) + 2^{2r-2}P(r,m+1)P(r,n-1)].
\end{aligned}$$

Using formula (2.5) again, we obtain

$$\begin{aligned}
&2^{r-1}BhP^r_{m-1}BhP^r_n + 2^{2r-2}BhP^r_{m-2}BhP^r_{n-1}\\
&= 2(P(r,m+n) + j_1P(r,m+n+1) + j_2P(r,m+n+2)\\
&\quad +j_3P(r,m+n+3)) - P(r,m+n) + P(r,m+n+2)\\
&\quad +P(r,m+n+4) + P(r,m+n+6)\\
&\quad +2\big(j_1P(r,m+n+5) + j_2P(r,m+n+4) + j_3P(r,m+n+3)\big)\\
&= 2BhP^r_{m+n} - P(r,m+n) + P(r,m+n+2)\\
&\quad +P(r,m+n+4) + P(r,m+n+6)\\
&\quad +2\big(j_1P(r,m+n+5) + j_2P(r,m+n+4) + j_3P(r,m+n+3)\big)\\
&= 2BhP^r_{m+n} - P(r,m+n) + P(r,m+n+2)\\
&\quad +P(r,m+n+4) - P(r,m+n+6)\\
&\quad +2j_3[P(r,m+n+3) + j_1P(r,m+n+4)\\
&\quad +j_2P(r,m+n+5) + j_3P(r,m+n+6)]\\
&= 2BhP^r_{m+n} - P(r,m+n) + P(r,m+n+2)\\
&\quad +P(r,m+n+4) - P(r,m+n+6) + 2j_3BhP^r_{m+n+3}.
\end{aligned}$$

Hence we get the result.

□

At the end we give the ordinary generating function of the bihyperbolic r-Pell numbers.

Theorem 3.8 *The generating function of the bihyperbolic r-Pell sequence $\{BhP^r_n\}$ is*

$$g(x) = \frac{BhP^r_0 + (BhP^r_1 - 2^rBhP^r_0)x}{1 - 2^rx - 2^{r-1}x^2}.$$

Proof. Let $g(x) = \sum_{n=0}^{\infty} BhP^r_n x^n$. By formula (3.3) we get

$$\begin{aligned}
&(1 - 2^rx - 2^{r-1}x^2)g(x)\\
&= (1 - 2^rx - 2^{r-1}x^2)\cdot(BhP^r_0 + BhP^r_1x + BhP^r_2x^2 + \cdots)\\
&= BhP^r_0 + BhP^r_1x + BhP^r_2x^2 + \cdots\\
&\quad -2^rBhP^r_0x - 2^rBhP^r_1x^2 - 2^rBhP^r_2x^3 - \cdots\\
&\quad -2^{r-1}BhP^r_0x^2 - 2^{r-1}BhP^r_1x^3 - 2^{r-1}BhP^r_2x^4 - \cdots.
\end{aligned}$$

Because the coefficients of x^n for $n \geq 2$ are equal to zero, we have

$$(1 - 2^rx - 2^{r-1}x^2)g(x) = BhP^r_0 + (BhP^r_1 - 2^rBhP^r_0)x.$$

Hence

$$g(x) = \frac{BhP^r_0 + (BhP^r_1 - 2^rBhP^r_0)x}{1 - 2^rx - 2^{r-1}x^2}.$$

By (3.2) we obtain

$$\begin{aligned}
BhP^r_0 &= 2 + j_1(1 + 2^{r+1}) + j_2(2^{r+1} + 2\cdot 4^r)\\
&\quad +j_3(2^{r-1} + 3\cdot 4^r + 2\cdot 8^r),\\
BhP^r_1 - 2^rBhP^r_0 &= 1 + j_12^r + j_2(4^r + 2^{r-1}) + j_3(8^r + 4^r).
\end{aligned}$$

□

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Authors

Dorota Bród (Corresponding author),
Rzeszow University of Technology,
Faculty of Mathematics and Applied Physics,
al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland,
E-mail: dorotab@prz.edu.pl,

Anetta Szynal-Liana,
Rzeszow University of Technology,
Faculty of Mathematics and Applied Physics,
al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland,
E-mail: aszynal@prz.edu.pl,

Iwona Włoch,
Rzeszow University of Technology,
Faculty of Mathematics and Applied Physics,
al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland,
E-mail: iwloch@prz.edu.pl