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A generalized Perrin polynomial sequence and its two-dimensional recurrences

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Abstract In this work, the generating matrices for the positive integers of the Perrin polynomial sequence will be investigated, as well as the generalization of their matrix form, extending to the field of non-positive integers. The similarity of Perrin's sequence with Padovan's is highlighted, differentiating it only in relation to its initial terms. And yet, there is a discussion about the two-dimensional, or Gaussian, recurring relations of Perrin's numbers, related to Padovan's numbers, based on his one-dimensional model. Finally, we introduce new relationships derived from two-dimensional recurrence and some identities resulting from it.

Keywords Perrin polynomial sequence · generating matrix · two-dimensional relationships · recurrence relationships

Mathematics Subject Classification (2010) 11B37 · 11B39 · 11B83

1 Introduction

The Perrin sequence is a sequence similar to the Padovan sequence, linear and third order, changing only its initial values, thus maintaining its recurrence, both sequences have identities surrounding them as seen in [1]. Thus, we will denote in this work the

Perrin's numbers as being $Pe(n)$, and Padovan's as being $Pd(n)$.

Discovered by the French mathematician Olivier Raoul Perrin (1841-1910), an engineer who in his spare time liked to produce scientific works, Perrin's numbers are of great importance, particularly in graph theory, and are therefore used to discover the coordinates of taxis in urban networks confidentially [17].

The elements of the Padovan sequence have the same recurrence, but with initial values equal to $Pd(0) = Pd(1) = Pd(2) = 1$, presenting the first we have as: 1,1,1,2,2,3,4,5,7,9,12. Thus, we have the recurrence given by $Pe(n) = Pe(n-2) + Pe(n-3), n \geq 3, n \in \mathbb{N}$, with initial values equal to $Pe(0) = 3, Pe(1) = 0, Pe(2) = 2$, presenting its first terms as: 3,0,2,3,2,5,5,7,10,12,17.

Investigating Perrin's first prime numbers, we have 2, 3, 2, 5, 5, 7, 17,... With that, Perrin [12] investigated the sequence and made a discovery, in which case n is a prime number, then n can divide $Pe(n)$, and are then known as Perrin's pseudo-cousins [8].

Given these two sequences and their respective terms, we can still establish an algebraic relationship between them. With that, we have:

$$Pe(n) = 2Pd(n-4) + 3Pd(n-5), n \geq 5.$$

Proof. Demonstrating by the principle of finite induction, we have that:
For $n = 5$:

$$\begin{aligned} &Pe(5) = 2Pd(1) + 3Pd(0) \\ &Pe(5) = 5 \end{aligned}$$

Thus, equality is valid.
Assuming it is valid for any $n = k, k \in \mathbb{N}$:

$$Pe(k) = 2Pd(k-4) + 3Pd(k-5)$$

Verifying that it is valid for $n = k + 1$:

$$Pe(k+1) = 2Pd(k+1-4) + 3Pd(k+1-5).$$

□

This equation is derived from their respective matrix formulas for the numbers of Padovan and Perrin [3,4,6,13–16,19]. It should be noted that this relationship will be used later in this work, since these sequences are similar. Thus, we will introduce in this work, the Perrin's polynomial sequence, its matrix form, as well as the two-dimensional relations.

2 Perrin's polynomial sequence

The study of linear and recurrent sequences has been known for a long time, in which, there are many relationships and generalizations about these numbers. With regard to the generalization of sequences, their polynomial and matrix representations are evidenced, which was initially carried out by mathematicians Ernest Erich Jacobsthal (1881-1965) and Eugene Charles Catalan (1814-1894) in 1983 around the Fibonacci sequence. On the other hand, many researchers have been working around other sequences, in [9], there is the definition of Perrin's polynomial sequence, denoted by $Pe_n(x)$, with $x \in \mathbb{R}$, such that:

$$\begin{cases} Pe_n(x) = x^2 Pe_{n-2}(x) + Pe_{n-3}(x), n \geq 3. \\ Pe_0(x) = 3x, Pe_1(x) = 0, Pe_2(x) = 2. \end{cases}$$

Thus, this recurrence generates the following terms:

$3x, 0, 2, 3x, 2x^2, (3x^2+2), (2x^4+3x), (3x^4+4x^2), (2x^6+3x^3+3x^2+2), \ldots$

In [9], some determinant and permanent representations of the associated polynomials of Perrin numbers are presented using various Hessenberg matrices, such as the C_n matrix which, when finding its determinant, has Perrin polynomial recurrence. Thus, we have:

$$C_n = \begin{bmatrix} 0 & ix^2 & 0 & 0 & 0 & 0 \\ \frac{2i}{x^2} & 0 & ix^2 & 0 & 0 & 0 \\ \frac{-3x}{x^4} & i & 0 & ix^2 & 0 & 0 \\ 0 & \frac{-1}{x^4} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & i & 0 & ix^2 \\ 0 & 0 & 0 & \frac{-1}{x^4} & i & 0 \end{bmatrix},$$

where $i = \sqrt{-1}$ and $det(C_n) = Pe_n(x)$. However, in this article, we will cover other matrices of the Perrin polynomial sequence. And yet, in order to explore and study the generalization process for the field of integers, some algebraic manipulations are performed on the original recurrence, in order to obtain the recurrence formula for the non-positive integer terms of Perrin's polynomial sequence. Thus, we have:

Definition 2.1 *The recurrence formula of the Perrin polynomial sequence, for non-positive integers, with $n \geq 0$, is given by:*

$$Pe_{-n}(x) = Pe_{-n+3}(x) - x^2 Pe_{-n+1}(x).$$

With that, the negative terms of the Perrin polynomial sequence are presented in Table 1.

Pe_{-1}	$2-3x^3$
Pe_{-2}	$-2x^2+3x^5$
Pe_{-3}	$3x+2x^4-3x^7$
Pe_{-4}	$2-3x^2-3x^3-2x^6+3x^9$
Pe_{-5}	$-4x^2+3x^4+6x^5+2x^8-3x^{11}$
$\vdots$	$\vdots$

Table 1 Non-positive integer indices of the Perrin polynomial sequence.

That said, we will study the matrices of the Perrin polynomial sequence, and its generalized matrix form for the field of non-positive integers.

3 Perrin's polynomial matrices

In the literature, you can find the matrix form of the Perrin sequence in [10], in which, this matrix form when raised to n-th power carries the numbers of the Perrin sequence, as performed in [5]. On the other hand, in [9], matrices of Perrin's polynomial sequence are presented, since Perrin's polynomials were associated with Hessenberg's matrices.

A polynomial sequence was studied primarily by the mathematician Eugene Charles Catalan (1814–1894) and Ernest Erich Jacobsthal (1881–1965), for the Fibonacci numbers [2]. With that, we have the generalization of the initial terms of a sequence, forming a polynomial function. As soon as we can still establish its matrix form of this polynomial sequence, obtaining the generalized terms of this sequence, without needing to know the recurring terms.

However, in this section, other matrix forms of the Perrin polynomial sequence will be presented. Based on [19], an initialization vector will be added, that is, a line vector that contains the initial values of this sequence. Furthermore, the base matrix (Q) studied in the sequence of Padovan [13, 15] will be used to multiply it by the initialization vector. Thus, we have the initialization vector of the Perrin polynomial sequence containing three initialization entries, since three initial values are used in this sequence. With that, the product of this vector will be realized with the Padovan matrix, since this sequence has the same properties, changing only its initial values, according to permutations that occurred.

Theorem 3.1 *The matrix form of the Perrin polynomial sequence is given by:*

For $[2\ 0\ 3x]\begin{bmatrix}0&1&0\\1&0&1\\1&0&0\end{bmatrix} = [Pe_3(x)\ Pe_2(x)\ Pe_1(x)]$*, there is that:*

$$[2\ 0\ 3x]\begin{bmatrix}0&1&0\\1&0&1\\1&0&0\end{bmatrix}^n = [Pe_{n+2}(x)\ Pe_{n+1}(x)\ Pe_n(x)],\ n \geq 1.$$

Proof. Using the principle of finite induction, we have:

For $n = 1$:

$$[2\ 0\ 3x]\begin{bmatrix}0&1&0\\1&0&1\\1&0&0\end{bmatrix} = [3x\ 2\ 0] = [Pe_3(x)\ Pe_2(x)\ Pe_1(x)]\,,$$

so the equality is valid.

Assuming it is valid for $n = k, k \in \mathbb{N}$, we have to:

$$[2\ 0\ 3x]\begin{bmatrix}0&1&0\\1&0&1\\1&0&0\end{bmatrix}^k = [Pe_{k+2}(x)\ Pe_{k+1}(x)\ Pe_k(x)]\,.$$

Thus, it will also be valid for $n = k + 1$:

$$\begin{aligned}
[2\ 0\ 3x] \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{k+1} &= [2\ 0\ 3x] \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{k} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
&= [Pe_{k+2}(x)\ Pe_{k+1}(x)\ Pe_k(x)] \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
&= [Pe_{k+1}(x) + Pe_k(x)\ Pe_{k+2}(x)\ Pe_{k+1}(x)] \\
&= [Pe_{k+3}(x)\ Pe_{k+2}(x)\ Pe_{k+1}(x)] .
\end{aligned}$$

□

Swapping the lines of the initialization vector and the rows and columns of the base matrix, we obtain five more Perrin matrices. In which, its demonstrations are similar to the Theorem 3.1.

Theorem 3.2 *For* $[2\ 3x\ 0] \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = [Pe_3(x)\ Pe_1(x)\ Pe_2(x)]$, *there is that:*

$$[2\ 3x\ 0] \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}^{n} = [Pe_{n+2}(x)\ Pe_n(x)\ Pe_{n+1}(x)],\ n \geq 1.$$

Theorem 3.3 *For* $[0\ 2\ 3x] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [Pe_2(x)\ Pe_3(x)\ Pe_1(x)]$, *there is that:*

$$[0\ 2\ 3x] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n} = [Pe_{n+1}(x)\ Pe_{n+2}(x)\ Pe_n(x)],\ n \geq 1.$$

Theorem 3.4 *For* $[0\ 3x\ 2] \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = [Pe_2(x)\ Pe_1(x)\ Pe_3(x)]$, *there is that:*

$$[0\ 3x\ 2] \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{n} = [Pe_{n+1}(x)\ Pe_n(x)\ Pe_{n+2}(x)],\ n \geq 1.$$

Theorem 3.5 *For* $[3x\ 0\ 2] \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = [Pe_1(x)\ Pe_2(x)\ Pe_3(x)]$, *there is that:*

$$[3x\ 0\ 2] \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{n} = [Pe_n(x)\ Pe_{n+1}(x)\ Pe_{n+2}(x)],\ n \geq 1.$$

Theorem 3.6 *For* $[3x\ 2\ 0]\begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix} = [Pe_1(x)\ Pe_3(x)\ Pe_2(x)]$*, there is that:*
$[3x\ 2\ 0]\begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix}^n = [Pe_n(x)\ Pe_{n+2}(x)\ Pe_{n+1}(x)],\ n \geq 1.$

4 The generalization of Perrin's polynomial matrix form

In order to obtain the generalized matrix form of the Perrin polynomial sequence, an extension was made to the negative indices of this sequence. Similarly to the previous section, the same initialization vector will be used and the inverse of the base matrix will be calculated, causing the theorems below.

Theorem 4.1 *The generalized matrix form of the Perrin polynomial sequence for the field of non-positive integers is given by:*
For $[2\ 0\ 3x]\begin{bmatrix}0&0&1\\1&0&0\\0&1&-1\end{bmatrix} = [Pe_1(x)\ Pe_0(x)\ Pe_{-1}(x)]$*, there is that:*
$[2\ 0\ 3x]\begin{bmatrix}0&0&1\\1&0&0\\0&1&-1\end{bmatrix}^n = [Pe_{-n+2}(x)\ Pe_{-n+1}(x)\ Pe_{-n}(x)]$*, for* $n > 0$.

Proof. Using the same principle performed previously, finite induction, we have:

For $n = 1$, there is that:

$$[2\ 0\ 3x]\begin{bmatrix}0&0&1\\1&0&0\\0&1&-1\end{bmatrix} = [0\ 3x\ 2-3x] = [Pe_1(x)\ Pe_0(x)\ Pe_{-1}(x)]\,,$$

validating equality.

Assuming it is valid for $n = k, k \in \mathbb{N}$, we have to:

$$[2\ 0\ 3x]\begin{bmatrix}0&0&1\\1&0&0\\0&1&-1\end{bmatrix}^k = [Pe_{-k+2}(x)\ Pe_{-k+1}(x)\ Pe_{-k}(x)]\,.$$

Now, we will verify that it is valid for $n = k+1$, we have to:

$$\begin{aligned}
[2\ 0\ 3x]\begin{bmatrix}0&0&1\\1&0&0\\0&1&-1\end{bmatrix}^{k+1} &= [2\ 0\ 3x]\begin{bmatrix}0&0&1\\1&0&0\\0&1&-1\end{bmatrix}^{k}\begin{bmatrix}0&0&1\\1&0&0\\0&1&-1\end{bmatrix}\\
&= [Pe_{-k+2}(x)\ Pe_{-k+1}(x)\ Pe_{-k}(x)]\begin{bmatrix}0&0&1\\1&0&0\\0&1&-1\end{bmatrix}\\
&= [Pe_{-k+1}(x)\ Pe_{-k}(x)\ Pe_{-k+2}(x) - Pe_{-k}(x)]\\
&= [Pe_{-k+1}(x)\ Pe_{-k}(x)\ Pe_{-k-1}(x)]\,.
\end{aligned}$$

□

For the following Theorems, their demonstrations are similar to the Theorem 4.1 demonstration, thus validating the theorems.

Theorem 4.2 *For* $[2\ 3x\ 0]\begin{bmatrix}0&1&0\\0&-1&1\\1&0&0\end{bmatrix} = [Pe_1(x)\ Pe_{-1}(x)\ Pe_0(x)]$, *there is that:* $[2\ 3x\ 0]\begin{bmatrix}0&1&0\\0&-1&1\\1&0&0\end{bmatrix}^n = [Pe_{-n+2}(x)\ Pe_{-n}(x)\ Pe_{-n+1}(x)]$, *for* $n > 0$.

Theorem 4.3 *For* $[0\ 2\ 3x]\begin{bmatrix}0&1&0\\0&0&1\\1&0&-1\end{bmatrix} = [Pe_0(x)\ Pe_1(x)\ Pe_{-1}(x)]$, *there is that:* $[0\ 2\ 3x]\begin{bmatrix}0&1&0\\0&0&1\\1&0&-1\end{bmatrix}^n = [Pe_{-n+1}(x)\ Pe_{-n+2}(x)\ Pe_{-n}(x)]$, *for* $n > 0$.

Theorem 4.4 *For* $[0\ 3x\ 2]\begin{bmatrix}0&0&1\\1&-1&0\\0&1&0\end{bmatrix} = [Pe_0(x)\ Pe_{-1}(x)\ Pe_{-1}(x)]$, *there is that:* $[0\ 3x\ 2]\begin{bmatrix}0&0&1\\1&-1&0\\0&1&0\end{bmatrix}^n = [Pe_{-n+1}(x)\ Pe_{-n}(x)\ Pe_{-n+2}(x)]$, *for* $n > 0$.

Theorem 4.5 *For* $[3x\ 0\ 2]\begin{bmatrix}-1&1&0\\0&0&1\\1&0&0\end{bmatrix} = [Pe_{-1}(x)\ Pe_0(x)\ Pe_1(x)]$, *there is that:* $[3x\ 0\ 2]\begin{bmatrix}-1&1&0\\0&0&1\\1&0&0\end{bmatrix}^n = [Pe_{-n}(x)\ Pe_{-n+1}(x)\ Pe_{-n+2}(x)]$, *for* $n > 0$.

Theorem 4.6 *For* $[3x\ 2\ 0]\begin{bmatrix}-1 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} = [Pe_{-1}(x)\ Pe_1(x)\ Pe_0(x)]$, *there is that:* $[3x\ 2\ 0]\begin{bmatrix}-1 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}^n = [Pe_{-n}(x)\ Pe_{-n+2}(x)\ Pe_{-n+1}(x)]$, *for* $n > 0$.

5 Perrin's two-dimensional recurring relations

When we insert an imaginary unit, denoted by i, where $i^2 = -1$, in the recurrence formula of a linear sequence, called Gaussian numbers (n, m), where $n, m \in \mathbb{N}$ [7]. Based on this, in this section, recurring relations in two dimensions of Perrin's numbers, called recurrent two-dimensional or Gaussian relations, will be introduced, based on the work of Oliveira, Alves and Paiva [11]. Before starting these studies, it can be seen that there will be relationships between these numbers and Padovan's numbers, given their similarity in their original recurrence.

Definition 5.1 *The numbers described in the form of* $Pe(n, m)$ *will represent the two-dimensional Perrin sequence terms, satisfying the recurring conditions, being* $n, m \in \mathbb{N}$*:*

$$\begin{cases} Pe(n, m) = Pe(n-2, m) + Pe(n-3, m) \\ Pe(n, m) = Pe(n, m-2) + Pe(n, m-3) \end{cases}$$

Given these relationships, we can then establish some initial values, and thus study some properties that will be discussed later. So, we have to: $Pe(0, 0) = 3, Pe(0, 1) = 3 + 2i, Pe(1, 0) = 0, Pe(1, 1) = 2i, Pe(0, 2) = 3 + 3i, Pe(2, 0) = 2, Pe(2, 1) = 2 + 2i, Pe(1, 2) = 3i$ and
$Pe(2, 2) = 2 + 3i$.

Lemma 5.2 *The following properties are valid for Perrin's two-dimensional sequence:*
(a) $Pe(n, 0) = Pe(n)$,
(b) $Pe(0, m) = 3Pe(m) + Pe(m + 1)i$,
(c) $Pe(n, 1) = Pe(n) + 2Pd(n)i$,
(d) $Pe(1, m) = Pe(m + 1)i$.

Proof. Demonstrating the first property $Pe(n, 0) = Pe(n)$, for $m = 0$ and varying $n = (0, 1, 2, 3, \ldots)$, we have:

$$\begin{aligned} &Pe(n, 0) = Pe(n-2, 0) + Pe(n-3, 0) : \\ &Pe(3, 0) = Pe(1, 0) + Pe(0, 0) = 3 = Pe(3); \\ &Pe(4, 0) = Pe(2, 0) + Pe(1, 0) = 2 = Pe(4); \end{aligned}$$

$$\begin{aligned}
Pe(5,0) &= Pe(3,0) + Pe(2,0) = 5 = Pe(5);\\
Pe(6,0) &= Pe(4,0) + Pe(3,0) = 5 = Pe(6);\\
&\vdots\\
Pe(n-3,0) &= Pe(n-5,0) + Pe(n-6,0) = Pe(n-2);\\
Pe(n-2,0) &= Pe(n-4,0) + Pe(n-5,0) = Pe(n-2);\\
Pe(n-1,0) &= Pe(n-3,0) + Pe(n-4,0) = Pe(n-1);\\
Pe(n,0) &= Pe(n-2,0) + Pe(n-3,0)\\
&= Pe(n-2) + Pe(n-3) = Pe(n).
\end{aligned}$$

For the second property, $Pe(0,m) = 3Pe(m) + Pe(m+1)i$, for $n = 0$ and $m = (0,1,2,3,\ldots)$, is such that:

$$\begin{aligned}
Pe(0,m) &= Pe(0,m-2) + Pe(0,m-3):\\
Pe(0,3) &= Pe(0,1) + Pe(0,0) = 6 + 2i = 3Pe(3) + Pe(4)i;\\
Pe(0,4) &= Pe(0,2) + Pe(0,1) = 6 + 5i = 3Pe(4) + Pe(5)i;\\
Pe(0,5) &= Pe(0,3) + Pe(0,2) = 9 + 5i = 3Pe(5) + Pe(6)i;\\
Pe(0,6) &= Pe(0,4) + Pe(0,3) = 12 + 7i = 3Pe(6) + Pe(7)i;\\
&\vdots\\
Pe(0,n-3) &= Pe(0,n-5) + Pe(0,n-6) = 3Pe(n-3) + Pe(n-2)i;\\
Pe(0,n-2) &= Pe(0,n-4) + Pe(0,n-5) = 3Pe(n-2) + Pe(n-1)i;\\
Pe(0,n-1) &= Pe(0,n-3) + Pe(0,n-4) = 3Pe(n-1) + Pe(n)i;\\
Pe(0,n) &= Pe(0,n-2) + Pe(0,n-3)\\
&= Pe(n-2) + Pe(n-1)i + Pe(n-3) + Pe(n-2)i\\
&= Pe(n) + Pe(n+1)i.
\end{aligned}$$

The third property $Pe(n,1) = Pe(n) + 2Pd(n)i$, is demonstrated evaluating recurrence for $m = 0$ and $n = (0,1,2,3,\ldots)$. Soon:

$$\begin{aligned}
Pe(n,1) &= Pe(n-2,1) + Pe(n-3,1):\\
Pe(3,1) &= Pe(1,1) + Pe(0,1) = 3 + 4i = Pe(3) + 2Pd(3)i;\\
Pe(4,1) &= Pe(2,1) + Pe(1,1) = 2 = 4i = Pe(4) + 2Pd(4)i;\\
Pe(5,1) &= Pe(3,1) + Pe(2,1) = 3 + 6i = Pe(5) + 2Pd(5)i;\\
Pe(6,1) &= Pe(4,1) + Pe(3,1) = 5 + 8i = Pe(6) + 2Pd(6)i;\\
&\vdots\\
Pe(n-3,1) &= Pe(n-5,1) + Pe(n-6,1) = Pe(n-3) + 2Pd(n-3)i;\\
Pe(n-2,1) &= Pe(n-4,1) + Pe(n-5,1) = Pe(n-2) + 2Pd(n-2)i;\\
Pe(n-1,1) &= Pe(n-3,1) + Pe(n-4,1) = Pe(n-1) + 2Pd(n-1)i;
\end{aligned}$$

$$\begin{aligned}Pe(n,1) &= Pe(n-2,1) + Pe(n-3,1)\\ &= Pe(n-2) + 2Pd(n-2)i + Pe(n-3) + 2Pd(n-3)i\\ &= Pe(n) + 2Pd(n)i.\end{aligned}$$

Finishing the fourth and last property, we then have to $Pe(1,m) = Pe(m+1)i$, for $n = 1$ and $m = (0,1,2,3,\ldots)$. With that, it is clear that:

$$\begin{aligned}Pe(1,m) &= Pe(1,m-2) + Pe(1,m-3):\\ Pe(1,3) &= Pe(1,1) + Pe(1,0) = 2i = Pe(4)i;\\ Pe(1,4) &= Pe(1,2) + Pe(1,1) = 5i = Pe(5)i;\\ Pe(1,5) &= Pe(1,3) + Pe(1,2) = 5i = Pe(6)i;\\ Pe(1,6) &= Pe(1,4) + Pe(1,3) = 7i = Pe(7)i;\\ &\vdots\\ Pe(1,m-3) &= Pe(1,m-5) + Pe(1,m-6) = Pe(m-2)i;\\ Pe(1,m-2) &= Pe(1,m-4) + Pe(1,m-5) = Pe(m-1)i;\\ Pe(1,m-1) &= Pe(1,m-3) + Pe(1,m-4) = Pe(m)i;\\ Pe(1,m) &= Pe(1,m-2) + Pe(1,m-3)\\ &= Pe(m-1)i + Pe(m-2)i\\ &= Pe(m+1)i.\end{aligned}$$

This validates the properties described in the Lemma above.

□

However, we can establish a two-dimensional relationship, as performed primarily in the work of Harman [7]. Thus, we have the relationship of the Perrin sequence, with the insertion of an imaginary unit, in the function of the Padovan sequence.

Theorem 5.3 *For integers, $n, m \in \mathbb{N}$, the two-dimensional Perrin sequence terms in the form $Pe(n,m)$, are described by:*

$$Pe(n,m) = Pd(m)Pe(n) + Pe(m)Pd(n)i.$$

Proof. Thus, for $m = 2$, we have that:

$$\begin{aligned}Pe(n,2) &= Pe(n-2,2) + Pe(n-3,2);\\ Pe(3,2) &= Pe(1,2) + Pe(0,2) = 3 + 6i = Pe(3) + 3Pd(3)i;\\ Pe(4,2) &= Pe(2,2) + Pe(1,2) = 2 + 6i = Pe(4) + 3Pd(4)i;\\ Pe(5,2) &= Pe(3,2) + Pe(2,2) = 5 + 9i = Pe(5) + 3Pd(4)i;\\ Pe(6,2) &= Pe(4,2) + Pe(3,2) = 5 + 12i = Pe(6) + 3Pd(6)i;\\ &\vdots\end{aligned}$$

$$\begin{aligned}
Pe(n-3,2) &= Pe(n-5,2) + Pe(n-6,2) = Pe(n-3) + 3Pd(n-3)i;\\
Pe(n-2,2) &= Pe(n-4,2) + Pe(n-5,2) = Pe(n-2) + 3Pd(n-2)i;\\
Pe(n-1,2) &= Pe(n-3,2) + Pe(n-4,2) = Pe(n-1) + 3Pd(n-1)i;\\
Pe(n,2) &= Pe(n-2,2) + Pe(n-3,2)\\
&= Pe(n-2) + 3Pd(n-2)i + Pe(n-3) + 3Pd(n-3)i\\
&= Pe(n) + 3Pd(n)i\\
&= Pe(n) + Pe(3)Pd(n)i.
\end{aligned}$$

In view of this, one can prove the veracity of the Theorem 5.3, by applying it to $m = 0, 1, 2, 3, \ldots, k$, in the situation presented below:

$$\begin{aligned}
Pe(n,0) &= Pd(0)Pe(n) + Pe(1)Pd(n)i\\
Pe(n,1) &= Pd(1)Pe(n) + Pe(2)Pd(n)i\\
Pe(n,2) &= Pd(2)Pe(n) + Pe(3)Pd(n)i\\
Pe(n,3) &= Pd(3)Pe(n) + Pe(4)Pd(n)i\\
&\vdots\\
Pe(n,k-3) &= Pd(k-3)Pe(n) + Pe(k-2)Pd(n)i\\
Pe(n,k-2) &= Pd(k-2)Pe(n) + Pe(k-1)Pd(n)i\\
Pe(n,k-1) &= Pd(k-1)Pe(n) + Pe(k)Pd(n)i\\
Pe(n,k) &= Pe(k-2) + Pe(k-3)\\
&= Pd(k-2)Pe(n) + Pe(k-1)Pd(n)i + Pd(k-3)Pe(n)+\\
&\quad Pe(k-2)Pd(n)i\\
&= Pd(k)Pe(n) + Pe(k+1)Pd(n)i.
\end{aligned}$$

□

Corollary 5.4 *Perrin's two−dimensional relationship, as a function of Padovan's sequence, is given by:*

$$\begin{aligned}
Pe(n,m) = {} & Pd(m)[2Pd(n-4) + 3Pd(n-5)] + Pd(n)i[2Pd(m-4)\\
& + 3Pd(m-5)].
\end{aligned}$$

Proof. Assigning the relationship $Pe(n) = 2Pd(n-4) + 3Pd(n-5)$ in Perrin's two-dimensional formula, we can rewrite it in terms of Padovan:

$$\begin{aligned}
Pe(n,m) &= Pd(m)Pe(n) + Pe(m)Pd(n)i\\
&= Pd(m)[2Pd(n-4) + 3Pd(n-5)] + Pd(n)i[2Pd(m-4)+\\
&\quad 3Pd(m-5)].
\end{aligned}$$

□

6 Two-dimensional identities

In this section, some identities inherent to Perrin's two-dimensional sequence [18].

Identity 6.1 *The sum of $Pe(n,m)$ index numbers n odd is described for:*

$$\sum_{l=1}^{n} Pe(2l+1,m) = Pe(2n+4,m) - Pe(4,m).$$

Proof. Through the relationship $Pe(n+3,m) = Pe(n+1,m) + Pe(n,m)$, we can rewrite it as $Pe(n,m) = Pe(n+3,m) - Pe(n+1,m)$, one has to:

$$\begin{aligned}
Pe(3,m) &= Pe(6,m) - Pe(4,m)\\
Pe(5,m) &= Pe(8,m) - Pe(6,m)\\
Pe(7,m) &= Pe(10,m) - Pe(8,m)\\
Pe(9,m) &= Pe(12,m) - Pe(10,m)\\
&\vdots\\
Pe(2n-1,m) &= Pe(2n+2,m) - Pe(2n,m)\\
Pe(2n+1,m) &= Pe(2n+4,m) - Pe(2n+2,m).
\end{aligned}$$

Using successive cancellations, we have to:

$$\sum_{l=1}^{n} Pe(2l+1,m) = Pe(2n+4,m) - Pe(4,m).$$

□

Identity 6.2 *The sum of the $Pe(n,m)$ index numbers n non-null and even is described for:*

$$\sum_{l=1}^{n} Pe(2l,m) = Pe(2n+3,m) - Pe(5,m).$$

Proof. Similar to the previous demonstration, we have to:

$$\begin{aligned}
Pe(2,m) &= Pe(5,m) - Pe(3,m)\\
Pe(4,m) &= Pe(7,m) - Pe(5,m)\\
Pe(6,m) &= Pe(9,m) - Pe(7,m)\\
Pe(8,m) &= Pe(11,m) - Pe(9,m)\\
&\vdots\\
Pe(2n,m) &= Pe(2n+3,m) - Pe(2n+1,m).
\end{aligned}$$

Canceling some terms, we have to:

$$\sum_{l=1}^{n} Pe(2l,m) = Pe(2n+3,m) - Pe(5,m).$$

□

Writing this identity according to the Padovan sequence, based on $Pe(n) = 2Pd(n-4) + 3Pd(n-5)$, we have to:

$$\begin{aligned}\sum_{l=1}^{n} Pe(2l,m) &= 2Pd(2n-1,m) + 3Pd(2n-2,m) - 2Pd(1,m) - 3Pd(0,m)\\ &= 2(Pd(2n-1,m) - Pd(1,m)) + 3(Pd(2n-2,m) - Pd(0,m)).\end{aligned}$$

Identity 6.3 *The sum of the initial* n *terms of the numbers* $Pe(n,m)$*, for* $n > 0$ *is described by:*

$$\sum_{l=1}^{n} Pe(2,m) = Pe(2n+3,m) - Pe(5,m).$$

Proof. Analogously to the previous demonstration, we have to:

$$\begin{aligned}Pe(1,m) &= Pe(4,m) - Pe(2,m)\\ Pe(2,m) &= Pe(5,m) - Pe(3,m)\\ Pe(3,m) &= Pe(6,m) - Pe(4,m)\\ Pe(4,m) &= Pe(7,m) - Pe(5,m)\\ &\vdots\\ Pe(n,m) &= Pe(n+3,m) - Pe(n+1,m).\end{aligned}$$

Performing some successive cancellations, we have to:

$$\sum_{l=1}^{n} Pe(l,m) = Pe(n+3,m) - Pe(2,m).$$

□

Identity 6.4 *The sum of the first seven consecutive terms of the numbers* $Pe(n)$*, is represented by:*

$$\sum_{l=0}^{6} Pe(n+l) = 2Pe(n+8).$$

Proof. So, we have to:

$$\begin{aligned}\sum_{l=0}^{6} Pe(n+l) &= Pe(n) + Pe(n+1) + Pe(n+2) + Pe(n+3) + Pe(n+4) + \\ &\quad Pe(n+5) + Pe(n+6) \\ &= Pe(n+3) + Pe(n+2) + Pe(n+3) + Pe(n+4) + \\ &\quad Pe(n+5) + Pe(n+6) \\ &= Pe(n+5) + Pe(n+6) + Pe(n+5) + Pe(n+6) \\ &= 2Pe(n+5) + Pe(n+6) \\ &= 2Pe(n+8).\end{aligned}$$

□

Identity 6.5 *The sum of the first seven consecutive terms of the numbers $Pd(n)$, is represented by:*

$$\sum_{l=0}^{6} Pd(n+l) = 2Pd(n+8).$$

Proof. So, we have to:

$$\begin{aligned}\sum_{l=0}^{6} Pd(n+l) &= Pd(n) + Pd(n+1) + Pd(n+2) + Pd(n+3) + \\ &\quad Pd(n+4) + Pd(n+5) + Pd(n+6) \\ &= Pd(n+3) + Pd(n+2) + Pd(n+3) + Pd(n+4) + \\ &\quad Pd(n+5) + Pd(n+6) \\ &= Pd(n+5) + Pd(n+6) + Pd(n+5) + Pd(n+6) \\ &= 2Pd(n+5) + Pd(n+6) \\ &= 2Pd(n+8).\end{aligned}$$

□

Identity 6.6 *The sum of the first seven consecutive terms of the numbers $Pe(n,m)$, is represented by:*

$$\sum_{l=0}^{6} Pe(n+l,m) = 2Pe(n+8,m).$$

Proof. We have to:

$$\begin{aligned}\sum_{l=0}^{6} Pe(n+l,m) &= Pe(n,m) + Pe(n+1,m) + \ldots + Pe(n+6,m) \\ &= Pd(m)Pe(n) + Pe(m+1)Pd(n)i\end{aligned}$$

$$
\begin{aligned}
&+ Pd(m)Pe(n+1) + Pe(m+1)Pd(n+1)i \\
&+ \ldots \\
&+ Pd(m)Pe(n+6) + Pe(m+1)Pd(n+6)i \\
=\ & Pd(m)[Pe(n) + Pe(n+1) + \ldots + Pe(n+6)] \\
&+ iPe(m+1)[Pd(n) + Pd(n+1) + \ldots + Pd(n+6)] \\
=\ & Pd(m)2Pe(n+8) + iPe(m+1)2Pd(n+8) \\
=\ & 2[Pd(m)Pe(n+8) + Pe(m+1)Pd(n+8)i] \\
=\ & 2Pe(n+8,m).
\end{aligned}
$$

□

7 Conclusion

In this work, it was possible to introduce Perrin's Gaussian numbers, also called two-dimensional numbers, by inserting an imaginary unit i. With that, we investigated some properties and their generalized relation of this recurrence, establishing a connection with the Padovan sequence. It is noteworthy that this interaction with Padovan's numbers was already foreseen, since Perrin's numbers are derived from Padovan, changing only their initial values.

A new relationship between these two sequences was established and introduced primarily in this research, based on the matrix forms of these numbers, already studied in other works previously.

It was also possible to expose the generalization of the Perrin sequence, thus presenting its polynomial sequence and realizing the extension of its recurrence to non-positive integer indices. Consequently, there were six generating matrices for Perrin's positive polynomial numbers and another six for non-positive integers.

For future work, the generalization of this sequence is sought, inserting more imaginary units, continuing the research process of these numbers. With that, we can develop the three-dimensional, four-dimensional relationships up to n-dimensional, highlighting other mathematical properties and theorems.

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