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Extended zero divisor graph on commutative semigroup

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Abstract In this paper, we study the extended zero divisor graph $\mathcal{EG}(\mathcal{S})$ associated to a commutative semigroup $\mathcal{S}$ with zero. Also, we characterize all extended zero divisor graphs with three vertices.

Keywords connected · diameter · girth · extended zero divisor graph · reduced semigroup

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1 Introduction

Throughout this paper, $\mathcal{S}$ denotes a commutative semigroup with zero element whose binary operation is multiplication. We denote by $\mathcal{Z}(\mathcal{S}), Nil(\mathcal{S})$ and $\mathcal{I}(\mathcal{S})$ the set of zero-divisors of $\mathcal{S}$, the set of nilpotent elements of $\mathcal{S}$ and the set of idempotent elements of $\mathcal{S}$, respectively. The semigroup $\mathcal{S}$ is said to be *reduced* if it has no non-zero nilpotent element. Moreover, for the subset $\mathcal{A}$ of $\mathcal{S}$, we let $\mathcal{A}^* = \mathcal{A} \setminus \{0\}$.
Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be a graph, where $\mathcal{V}(\mathcal{G})$ denote the set of vertices and $\mathcal{E}(\mathcal{G})$ be the set of edges of $\mathcal{G}$. We say that $\mathcal{G}$ is *connected* if there exists a path between any two distinct vertices of $\mathcal{G}$. For vertices a and b of $\mathcal{G}$, $d(a, b)$ denotes the length of a shortest path from a to b. In particular, $d(a, a) = 0$ and $d(a, b) = \infty$ if there is no such path. The *diameter* of $\mathcal{G}$, denoted by $diam(\mathcal{G}) = sup\{d(a, b) \mid a, b \in \mathcal{V}(\mathcal{G})\}$. A *cycle* in a graph $\mathcal{G}$ is a path that begins and ends at the same vertex. The *girth* of $\mathcal{G}$, denoted by $gr(\mathcal{G})$, is the length of a shortest cycle in $\mathcal{G}$, $(gr(\mathcal{G}) = \infty$ if $\mathcal{G}$ contains no cycle). A *complete graph* $\mathcal{G}$ is a graph where all distinct vertices are adjacent. The complete graph with $|V(\mathcal{G})| = n$ is denoted by $\mathcal{K}_n$. A graph $\mathcal{G}$ is said to be *complete k-partite* if there is a partition $\bigcup_{i=1}^{k} \mathcal{V}_i = \mathcal{V}(\mathcal{G})$, such that $u - v \in \mathcal{E}(\mathcal{G})$ if and only if u and v are in different part of partition. If

$|\mathcal{V}_i| = n_i$, then $\mathcal{G}$ is denoted by $\mathcal{K}_{n_1,n_2,\cdots,n_k}$ and in particular $\mathcal{G}$ is called *complete bipartite* if $k = 2$. $\mathcal{K}_{1,n}$ is said to be a *star graph* and $\overline{\mathcal{G}}$ denotes the *complement graph* of $\mathcal{G}$. A graph $\mathcal{H} = (\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ is said to be a *subgraph* of $\mathcal{G}$, if $\mathcal{V}(\mathcal{H}) \subseteq \mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{H}) \subseteq \mathcal{E}(\mathcal{G})$. Moreover, $\mathcal{H}$ is said to be *induced subgraph* of $\mathcal{G}$ if $\mathcal{V}(\mathcal{H}) \subseteq \mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{H}) = \{u - v \in \mathcal{E}(\mathcal{G}) \mid u, v \in \mathcal{V}(\mathcal{H})\}$ and is denoted by $\mathcal{G}[\mathcal{V}(\mathcal{H})]$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two disjoint graphs. The *join* of $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted by $\mathcal{H}_1 \vee \mathcal{H}_2$, is a graph with vertex set $\mathcal{V}(\mathcal{H}_1 \vee \mathcal{H}_2) = \mathcal{V}(\mathcal{H}_1) \cup \mathcal{V}(\mathcal{H}_2)$ and edge set $\mathcal{E}(\mathcal{H}_1 \vee \mathcal{H}_2) = \mathcal{E}(\mathcal{H}_1) \cup \mathcal{E}(\mathcal{H}_2) \cup \{u - v \mid u \in \mathcal{V}(\mathcal{H}_1), v \in \mathcal{V}(\mathcal{H}_2)\}$. Also $\mathcal{G}$ is called a *null graph* if $\mathcal{E}(\mathcal{G}) = \emptyset$. For a graph $\mathcal{G}$, a complete subgraph of $\mathcal{G}$ is called a *clique*. The *clique number* $\omega(\mathcal{G})$ is the greatest integer $n \geq 1$ such that $\mathcal{K}_n \subseteq \mathcal{G}$, and $\omega(\mathcal{G}) = \infty$ if $\mathcal{K}_n \subseteq \mathcal{G}$ for all $n \geq 1$. The *chromatic number* $\chi(\mathcal{G})$ of a graph $\mathcal{G}$ is the minimum number of colours needed to colour all the vertices of $\mathcal{G}$ such that every two adjacent vertices get different colours. A graph $\mathcal{G}$ is *perfect* if $\chi(\mathcal{H}) = \omega(\mathcal{H})$ for every induced subgraph $\mathcal{H}$ of $\mathcal{G}$. Graph-theoretic terms are presented as they appear in R. Diestel [10].

The investigation of graphs associated to semigroups had been an exciting research topic in the past 20 years. There are several other graphs associated to ring and semigroup, for instance, see [1–3,5,6,11–14].

The zero-divisor graph of $\mathcal{S}$ is a simple undirected graph $\Gamma(\mathcal{S})$ with the set of vertices $\mathcal{Z}(\mathcal{S})^*$ and distinct vertices $x, y \in \mathcal{Z}(\mathcal{S})^*$ are adjacent if and only if $xy = 0$. It was proved that $\Gamma(\mathcal{S})$ is connected and $diam(\Gamma(\mathcal{S}))$ is at most three. If $\Gamma(\mathcal{S})$ contains a cycle, then $gr(\Gamma(\mathcal{S}))$ is at most four, see [4,7–9].

In this paper, we introduce and study the notion of extended zero divisor graph for a commutative semigroup $\mathcal{S}$ and denote it by $\mathcal{EG}(\mathcal{S})$. The graph $\mathcal{EG}(\mathcal{S})$ is a simple undirected graph with the set of vertices $\mathcal{Z}(\mathcal{S})^*$ and two distinct vertices u and v of $\mathcal{EG}(\mathcal{S})$ are adjacent if and only if either $\mathcal{S}u \cap ann_{\mathcal{S}}(v) \neq (0)$ or $\mathcal{S}v \cap ann_{\mathcal{S}}(u) \neq (0)$. In Section 2, we investigate some basic properties of $\mathcal{EG}(\mathcal{S})$ by means of $\Gamma(\mathcal{S})$. We also show that if $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$, then $\Gamma(\mathcal{S})$ is a subgraph of $\mathcal{EG}(\mathcal{S})$ and $\mathcal{EG}(\mathcal{S})$ is connected, $diam(\mathcal{EG}(\mathcal{S}))$ is at most two.

2 Fundamental properties of extended zero divisor graph $\mathcal{EG}(\mathcal{S})$

In this section, we study fundamental properties of $\mathcal{EG}(\mathcal{S})$.

Definition 2.1 *Let $\mathcal{S}$ be a commutative semigroup. An extended zero divisor graph $\mathcal{EG}(\mathcal{S})$ is an (undirected) graph with set of vertices $\mathcal{Z}(\mathcal{S})^*$ and any two distinct vertices u and v of $\mathcal{EG}(\mathcal{S})$ are adjacent if and only if either $\mathcal{S}u \cap ann_{\mathcal{S}}(v) \neq (0)$ or $\mathcal{S}v \cap ann_{\mathcal{S}}(u) \neq (0)$.*

Theorem 2.2 *If $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$, then $\Gamma(\mathcal{S}) \leq \mathcal{EG}(\mathcal{S})$.*

Proof. Clearly, $\mathcal{V}(\Gamma(\mathcal{S})) = \mathcal{V}(\mathcal{EG}(\mathcal{S}))$ and let $u - v$ be an edge in $\Gamma(\mathcal{S})$. Then $uv = 0$ and $0 \neq su \in \mathcal{S}u \cap ann_{\mathcal{S}}(v)$ for some $s \in \mathcal{S} \setminus \mathcal{Z}(\mathcal{S})$, i.e., $\mathcal{S}u \cap ann_{\mathcal{S}}(v) \neq (0)$. Thus u and v are also adjacent in $\mathcal{EG}(\mathcal{S})$ and hence $\Gamma(\mathcal{S}) \leq \mathcal{EG}(\mathcal{S})$.

□

Corollary 2.3 *If $\mathcal{S}$ be a monoid with $1 \neq 0$, then $\Gamma(\mathcal{S}) \leq \mathcal{EG}(\mathcal{S})$.*

The proof of the following lemma is trivial.

Lemma 2.4 *Let $\mathcal{S}$ be a reduced semigroup with $\mathcal{Z}(\mathcal{S})^* \neq \emptyset$, and let $u \in \mathcal{V}(\mathcal{EG}(\mathcal{S}))$. Then*
(i) $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(u^n)$ for each positive integer $n \geq 2$,
(ii) $\mathcal{S}u \cap ann_{\mathcal{S}}(u) = (0)$.

The following lemma indicates serval useful properties of $\mathcal{EG}(\mathcal{S})$ and plays an important role in this section.

Lemma 2.5 *Let $\mathcal{S}$ be a commutative semigroup with $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$.*

(i) If $u - v$ is an edge of $\Gamma(\mathcal{S})$ for some $u, v \in \mathcal{Z}(\mathcal{S})^*$, then $u - v$ is an edge of $\mathcal{EG}(\mathcal{S})$.
(ii) If $u - v$ is not an edge of $\mathcal{EG}(\mathcal{S})$ for some $u, v \in \mathcal{Z}(\mathcal{S})^*$, then $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(v)$. Moreover, if $\mathcal{S}$ is reduced semigroup, then the converse is also true.
(iii) If $ann_{\mathcal{S}}(u) \nsubseteq ann_{\mathcal{S}}(v)$ or $ann_{\mathcal{S}}(v) \nsubseteq ann_{\mathcal{S}}(u)$ for some $u, v \in \mathcal{Z}(\mathcal{S})^*$, then $u - v$ is an edge of $\mathcal{EG}(\mathcal{S})$.
(iv) If $\mathcal{S}u \cap ann_{\mathcal{S}}(u) \neq (0)$ for some $u \in \mathcal{Z}(\mathcal{S})^*$, then u is adjacent to all other vertex in $\mathcal{EG}(\mathcal{S})$. In particular, if $u \in \mathcal{N}(\mathcal{S})^*$, then u is adjacent to every other vertex of $\mathcal{EG}(\mathcal{S})$.
(v) $\mathcal{EG}(\mathcal{S})[Nil(\mathcal{S})^*]$ is a complete subgraph of $\mathcal{EG}(\mathcal{S})$.

Proof. (i) Directly follows from Theorem 2.2.
(ii) If $u-v$ not an edge of $\mathcal{EG}(\mathcal{S})$ for some $u, v \in \mathcal{Z}(\mathcal{S})^*$, then $\mathcal{S}u \cap ann_{\mathcal{S}}(v) = (0)$ and $\mathcal{S}v \cap ann_{\mathcal{S}}(u) = (0)$. Thus $(\mathcal{S}u)ann_{\mathcal{S}}(v) \subseteq \mathcal{S}u \cap ann_{\mathcal{S}}(v) = (0)$ and $(\mathcal{S}v)ann_{\mathcal{S}}(u) \subseteq \mathcal{S}v \cap ann_{\mathcal{S}}(u) = (0)$ and hence $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(v)$. If $\mathcal{S}$ is a reduced semigroup, then by Lemma 2.4(ii), $\mathcal{S}u \cap ann_{\mathcal{S}}(u) = (0)$ for every $u \in \mathcal{Z}(\mathcal{S})^*$. This together with the fact $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(v)$ yield that $u - v$ is not an edge of $\mathcal{EG}(\mathcal{S})$.
(iii) It is clear by part (ii).
(iv) Assume that $\mathcal{S}u \cap ann_{\mathcal{S}}(u) \neq (0)$ for some $u \in \mathcal{Z}(\mathcal{S})^*$, and let v be another vertex of $\mathcal{EG}(\mathcal{S})$. If v is not adjacent to u, then by part (ii), $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(v)$ and hence $\mathcal{S}u \cap ann_{\mathcal{S}}(v) \neq (0)$, a contradiction.
(v) Clearly true by (iv).

□

Theorem 2.6 *Let $\mathcal{S}$ be a commutative semigroup with $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$. Then $\mathcal{EG}(\mathcal{S})$ is connected and $diam(\mathcal{EG}(\mathcal{S})) \leq 2$. Moreover if $\mathcal{EG}(\mathcal{S})$ contains a cycle, then $gr(\mathcal{EG}(\mathcal{S})) \leq 5$.*

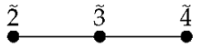
Proof. Let u and v any two distinct vertices of $\mathcal{EG}(\mathcal{S})$. We have the following cases.
Case(i) If $uv = 0$, then $\mathcal{S}u \cap ann_{\mathcal{S}}(v) \neq (0)$ and $u - v$ is an edge of $\mathcal{EG}(\mathcal{S})$.
Case(ii) If $uv \neq 0, u^2 = 0 = v^2$ or $uv \neq 0, u^2 = 0 \neq v^2$ or $uv \neq 0, u^2 \neq 0 =$

v^2, then by Lemma 2.5(iv), $u - v$ is an edge of $\mathcal{EG}(\mathcal{S})$.
Case(iii) If $uv \neq 0, v^2 \neq 0$ and $u^2 \neq 0$, then there exist $v_1, u_1 \in \mathcal{Z}(\mathcal{S})^*$ such that $vv_1 = 0 = uu_1$. If $u_1 = v_1$, then $\mathcal{S}u \cap ann_{\mathcal{S}}(u_1) \neq (0)$ and $\mathcal{S}v \cap ann_{\mathcal{S}}(u_1) \neq (0)$. Thus $u - u_1 - v$ is a path in $\mathcal{EG}(\mathcal{S})$.
Subcase(iii)(a) If $u_1 \neq v_1$ and $u_1v_1 \neq 0$, then $u - u_1v_1 - v$ is a path in $\mathcal{EG}(\mathcal{S})$.
Subcase(iii)(b) If $u_1 \neq v_1, u_1v_1 = 0$ and $uv_1 \neq 0$ then $\mathcal{S}u \cap ann_{\mathcal{S}}(v) \neq (0)$ and $u - v$ is a path in $\mathcal{EG}(\mathcal{S})$. If $u_1 \neq v_1, uv_1 = 0$, then $u - v_1 - v$ is a path in $\mathcal{EG}(\mathcal{S})$.
Thus in all the cases $diam(\mathcal{EG}(\mathcal{S})) \leq 2$. We know that $gr(\mathcal{G}) \leq 2(diam(\mathcal{G}))+1$, and hence $gr(\mathcal{EG}(\mathcal{S})) \leq 5$.

□

Corollary 2.7 *Let $\mathcal{S}$ be a reduced semigroup. Then $\Gamma(\mathcal{S}) \leq \mathcal{EG}(\mathcal{S})$ and $diam(\mathcal{EG}(\mathcal{S})) \leq 2$.*

Remark 2.1 Let $\mathcal{S}$ be a non-reduced semigroup such that $\mathcal{Z}(\mathcal{S}) = \mathcal{S}$. Then the conclusions of the above theorems do not have to be true.



$\mathcal{EG}(\mathcal{S}_1) = \Gamma(\mathcal{S}_1)$
Figure 2.1

$\mathcal{EG}(\mathcal{S}_2)$
Figure 2.2

$\Gamma(\mathcal{S}_2)$
Figure 2.3

Example 2.1 Let $\mathcal{S}_1 = \{\tilde{0}, \tilde{2}, \tilde{3}, \tilde{4}\}$ and $\mathcal{S}_2 = \{\tilde{0}, \tilde{3}, \tilde{6}\}$ be semigroups with multiplicative operation modulo 6 and modulo 9 respectively. Clearly, $\mathcal{Z}(\mathcal{S}_1) = \mathcal{S}_1$ and $\mathcal{Z}(\mathcal{S}_2) = \mathcal{S}_2$. It can be easily seen from the Figures 2.1, 2.2 & 2.3, that $\mathcal{EG}(\mathcal{S}_1)$ is connected while $\mathcal{EG}(\mathcal{S}_2)$ is a disconnected graph and $\Gamma(\mathcal{S}_2)$ is not a subgraph of $\mathcal{EG}(\mathcal{S}_2)$.

Lemma 2.8 *Let u and v be distinct adjacent vertices in $\mathcal{EG}(\mathcal{S})$ such that $uv \neq 0$. Also assume that $w \in ann_{\mathcal{S}}(uv) \setminus \{u, v\}$ such that $wu \neq 0 \neq wv$. Then $u - w - v$ is a path in $\mathcal{EG}(\mathcal{S})$, which is not a path in $\Gamma(\mathcal{S})$ and thus $C: u - w - v - u$ is a cycle in $\mathcal{EG}(\mathcal{S})$ of length three and each edge of C is not an edge of $\Gamma(\mathcal{S})$.*

Proof. Assume that there are distinct elements u, v and $w \in \mathcal{S}^*$ such that $w \in ann_{\mathcal{S}}(uv) \setminus \{u, v\}$ and $wu \neq 0 \neq wv$ and $uv \neq 0$. Then $\mathcal{S}v \cap ann_{\mathcal{S}}(w) \neq 0$. Thus w is adjacent to v in $\mathcal{EG}(\mathcal{S})$. Since $vw \neq 0$, we have w is not adjacent to v in $\Gamma(\mathcal{S})$. Similarly, w is adjacent to u in $\mathcal{EG}(\mathcal{S})$ and w is not adjacent to u in $\Gamma(\mathcal{S})$. Therefore, $C: u - w - v - u$ is a cycle in $\mathcal{EG}(\mathcal{S})$ of length three and each edge of C is not an edge of $\Gamma(\mathcal{S})$.

□

Theorem 2.9 *Let u and v be distinct adjacent vertices of $\mathcal{EG}(\mathcal{S})$ such that $uv \neq 0, u^2v \neq 0$ and $uv^2 \neq 0$. Then there exists $w \in \mathcal{Z}(\mathcal{S})^*$ such that either $w-u-v$ or $w-v-u$ is a path in $\mathcal{EG}(\mathcal{S})$ which is not a path in $\Gamma(\mathcal{S})$. Also in this case $gr(\mathcal{EG}(\mathcal{S})) = 3$.*

Proof. Assume that u is adjacent to v in $\mathcal{EG}(\mathcal{S})$ and $uv \neq 0$. Then either $\mathcal{S}u \cap ann_{\mathcal{S}}(v) \neq (0)$ or $\mathcal{S}v \cap ann_{\mathcal{S}}(u) \neq (0)$. If $\mathcal{S}u \cap ann_{\mathcal{S}}(v) \neq (0)$, then there exists $s \in \mathcal{S}^*$ such that $su \neq 0$ and $suv = 0$. Clearly, $v \neq s \neq u$ and $\mathcal{S}s \cap ann_{\mathcal{S}}(v) \neq 0,\ \mathcal{S}u \cap ann_{\mathcal{S}}(v) \neq 0,\ \mathcal{S}u \cap ann_{\mathcal{S}}(s) \neq 0$. Thus $v-u-s$ is a path in $\mathcal{EG}(\mathcal{S})$ which is not a path in $\Gamma(\mathcal{S})$ and $u-v-s-u$ is a cycle in $\mathcal{EG}(\mathcal{S})$. If $\mathcal{S}v \cap ann_{\mathcal{S}}(u) \neq (0)$, then there exists $s' \in \mathcal{S}^*$ such that $s'v \neq 0$ and $s'vu = 0$. Clearly, $u \neq s' \neq v$ and $\mathcal{S}s' \cap ann_{\mathcal{S}}(u) \neq 0,\ \mathcal{S}v \cap ann_{\mathcal{S}}(u) \neq 0,\ \mathcal{S}v \cap ann_{\mathcal{S}}(s')$. Thus $u-v-s'$ is a path in $\mathcal{EG}(\mathcal{S})$ which is not a path in $\Gamma(\mathcal{S})$ and $s'-v-u-s'$ is a cycle in $\mathcal{EG}(\mathcal{S})$. Hence $gr(\mathcal{EG}(\mathcal{S})) = 3$.

□

Lemma 2.10 *Let $\mathcal{S}$ be a reduced semigroup and u and v be distinct adjacent vertices of $\mathcal{EG}(\mathcal{S})$ such that $uv \neq 0$. Then there exists $w \in ann_{\mathcal{S}}(uv) \setminus \{u, v\}$ such that either $w-u-v$ or $w-v-u$ is a path in $\mathcal{EG}(\mathcal{S})$ which is not a path in $\Gamma(\mathcal{S})$. Also $C : w-u-v-w$ is a cycle in $\mathcal{EG}(\mathcal{S})$*

Proof. Since $\mathcal{S}$ is reduced, $uv \neq 0$ implies that $u^2v \neq 0 \neq v^2u$. Now by Theorem 2.9 the claim follows immediately.

□

The next theorem directly follows from Lemma 2.10.

Theorem 2.11 *Let $\mathcal{S}$ be a reduced semigroup and $\Gamma(\mathcal{S}) \neq \mathcal{EG}(\mathcal{S})$. Then $gr(\mathcal{EG}(\mathcal{S})) = 3$.*

Theorem 2.12 *Let $\mathcal{S}$ be a semigroup such that $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$. If there exists a vertex in $\mathcal{EG}(\mathcal{S})$ which is adjacent to every other vertex, then one of the following statements hold:*
(i) $Nil(\mathcal{S})^* \neq \emptyset$,
(ii) $\mathcal{I}(\mathcal{S})^* \neq \emptyset$.

Proof. If $Nil(\mathcal{S}) \neq (0)$, then by Lemma 2.5$(iv)$, we are done. Suppose that $u \in \mathcal{Z}(\mathcal{S})^*$ which is adjacent to every other vertex and $Nil(\mathcal{S}) = (0)$. If $u^2 \neq u$, then u is adjacent to u^2 and either $\mathcal{S}u \cap ann_S(u^2) \neq (0)$ or $\mathcal{S}u^2 \cap ann_S(u) \neq (0)$, which contradicts Lemma 2.4(ii). Thus $u^2 = u$. Hence $\mathcal{I}(\mathcal{S})^* \neq \emptyset$.

□

Corollary 2.13 *Let $\mathcal{S}$ be a reduced semigroup. If there exists a vertex x of $\mathcal{EG}(\mathcal{S})$ which is adjacent to every other vertex, then $x^2 = x$.*

Proof. Directly follows from the Theorem 2.12.

□

Theorem 2.14 *Let $\mathcal{S}$ be a reduced semigroup. Then $\mathcal{EG}(\mathcal{S})$ is a star graph if and only if there exists a non-trivial idempotent e such that $ann_{\mathcal{S}}(e) = \mathcal{Z}(\mathcal{S}) \setminus \{e\}$ and $uv \neq 0$ for all $u \neq v$ in $\mathcal{Z}(\mathcal{S})^* \setminus \{e\}$.*

Proof. Assume that $e \in \mathcal{Z}(\mathcal{S})^*$ such that e is adjacent to every other vertex. Then $e^2 = e$ because if $e^2 \neq e$, then $e^2 \in \mathcal{Z}(\mathcal{S})$ and e is adjacent to e^2, thus $\mathcal{S}e \cap ann_{\mathcal{S}}(e^2) \neq (0)$ or $\mathcal{S}e^2 \cap ann_{\mathcal{S}}(e) \neq (0)$ and we get a contradiction. Therefore $e^2 = e$. Now we will show that $ann_{\mathcal{S}}(e) = \mathcal{Z}(\mathcal{S}) \setminus \{e\}$. Trivially $ann_{\mathcal{S}}(e) \subseteq \mathcal{Z}(\mathcal{S}) \setminus \{e\}$. Now assume on contrary that $ann_{\mathcal{S}}(e) \subsetneq \mathcal{Z}(\mathcal{S}) \setminus \{e\}$. Then there exists $u \in \mathcal{Z}(\mathcal{S}) \setminus \{e\}$ such that $ue \neq 0$. Now e is adjacent to u, therefore either $\mathcal{S}e \cap ann_{\mathcal{S}}(u) \neq (0)$ or $\mathcal{S}u \cap ann_{\mathcal{S}}(e) \neq (0)$.
$Case(i)$ If $\mathcal{S}e \cap ann_{\mathcal{S}}(u) \neq (0)$, then there exists $s \in \mathcal{S}^*$ such that $se \neq 0,\ seu = 0$. Clearly, $u \neq s \neq e$ and $\mathcal{S}u \cap ann_{\mathcal{S}}(s) \neq (0)$ and $s - u$ is an edge in $\mathcal{EG}(\mathcal{S})$, a contradiction.
$Case(ii)$ If $\mathcal{S}u \cap ann_{\mathcal{S}}(e) \neq (0)$, then there exists $s \in \mathcal{S}^*$ such that $su \neq 0,\ sue = 0$. Clearly, $u \neq s \neq e$, $\mathcal{S}u \cap ann_{\mathcal{S}}(s) \neq (0)$ and $s - u$ is an edge in $\mathcal{EG}(\mathcal{S})$, a contradiction.
Hence $ue = 0$ and $ann_{\mathcal{S}}(e) = \mathcal{Z}(\mathcal{S}) \setminus \{e\}$. Clearly, $uv \neq 0$, for all $u \neq v$ in $\mathcal{Z}(\mathcal{S}) \setminus \{e\}$.
Converse part holds trivially.

□

Theorem 2.15 *Let $\mathcal{S}$ be a non reduced semigroup such that $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$. Then $\mathcal{EG}(\mathcal{S})$ is a star graph if and only if one of the following statements hold:*
(i) $|\mathcal{Z}(\mathcal{S})| = 3$
(ii) $Nil(\mathcal{S})$ *is a prime ideal of* $\mathcal{S}$ *and* $|Nil(\mathcal{S})| = 2$.

Proof. First assume that $\mathcal{EG}(\mathcal{S})$ is a star graph. By Lemma 2.5(v), $|Nil(\mathcal{S})| \leq 3$. We have the following cases.
Case(i) If $|Nil(\mathcal{S})| = 2$, then $\mathcal{Z}(\mathcal{S}) \neq Nil(\mathcal{S})$ (Since $\mathcal{EG}(\mathcal{S})$ is a star graph). Let $u \in \mathcal{N}(\mathcal{S})^*$. Then by Lemma 2.5$(iv)$, u is adjacent to all vertices contained in $\mathcal{Z}(\mathcal{S}) \setminus \{0, u\}$. If $vw = 0$ for some $v, w \in \mathcal{Z}(\mathcal{S}) \setminus \{0, u\}$, then v is adjacent to w, a contradiction. Thus $ann_{\mathcal{S}}(v) = \{0, u\}$ for all $v \in \mathcal{Z}(\mathcal{S})^*$. Now we show that $Nil(\mathcal{S})$ is a prime ideal of $\mathcal{S}$. Suppose on the contrary that there exist two ideals $\mathcal{U}, \mathcal{V}$ of $\mathcal{S}$ such that $\mathcal{UV} \subseteq Nil(\mathcal{S})$ and $\mathcal{U} \nsubseteq Nil(\mathcal{S}), \mathcal{V} \nsubseteq Nil(\mathcal{S})$. Let $x \in \mathcal{U} \setminus Nil(\mathcal{S})$ and $y \in \mathcal{V} \setminus Nil(\mathcal{S})$ such that $xy \in Nil(\mathcal{S})$. It is easy to see that $x \neq y, xy \neq 0$ and $x, y \in \mathcal{Z}(\mathcal{S})$, $xy \in \mathcal{S}x \cap ann_{\mathcal{S}}(y)$. Therefore x is adjacent to y in $\mathcal{EG}(\mathcal{S})$, a contradiction.
Case(ii) If $|Nil(\mathcal{S})| = 3$, then by Lemma 2.5(iv), $\mathcal{EG}(\mathcal{S}) = \mathcal{K}_2$ and $\mathcal{Z}(\mathcal{S}) = Nil(\mathcal{S}) = 3$.
Conversely, Suppose that one of the conditions (i) and (ii) holds. If (i) holds, then $\mathcal{EG}(\mathcal{S}) = \mathcal{K}_2$ and we are done. If (ii) holds, then there exists exactly one element of $u \in Nil(\mathcal{S})^*$ and by Lemma 2.5(iv), u is adjacent to all other vertices of $\mathcal{EG}(\mathcal{S})$. Since $Nil(\mathcal{S})$ is a prime ideal, $ann_{\mathcal{S}}(v) = \{0, u\}$, for every $v \in \mathcal{Z}(\mathcal{S})^*$. Now we will show that there is no adjacency between any two distinct vertices of $\mathcal{Z}(\mathcal{S}) \setminus Nil(\mathcal{S})$. Assume on contrary that $v, w \in \mathcal{Z}(\mathcal{S}) \setminus Nil(\mathcal{S})$ are

adjacent in $\mathcal{EG}(\mathcal{S})$. Then either $\mathcal{S}v \cap ann_{\mathcal{S}}(w) \neq (0)$ or $\mathcal{S}w \cap ann_{\mathcal{S}}(v) \neq (0)$. Without loss of generality, assume that $\mathcal{S}v \cap ann_{\mathcal{S}}(w) \neq (0)$. Then there exists $s \in \mathcal{S}$ such that $sv \neq 0$ and $svw = 0$. Thus $svw \in Nil(\mathcal{S})$, since $Nil(\mathcal{S})$ is a prime ideal and $w \in \mathcal{Z}(\mathcal{S}) \setminus Nil(\mathcal{S})$ implies $sv \in Nil(\mathcal{S})$. Again since $Nil(\mathcal{S})$ is a prime ideal and $v \in \mathcal{Z}(\mathcal{S}) \setminus Nil(\mathcal{S})$ we find that $s \in Nil(\mathcal{S})$ and $sv = 0$, a contradiction. Hence $\mathcal{EG}(\mathcal{S})$ is a star graph.

□

Theorem 2.16 *Let $\mathcal{S}$ be a non-reduced semigroup such that $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$. If $\mathcal{EG}(\mathcal{S})$ is a star graph, then $\mathcal{S}$ cannot be written as the direct product of semigroups.*

Proof. Let $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2$ where $\mathcal{S}_j$ is a semigroup for $1 \leq j \leq 2$. Clearly $\mathcal{Z}(\mathcal{S}_j) \neq \mathcal{S}_j$ $1 \leq j \leq 2$ and assume that $\alpha_j \in \mathcal{Z}(\mathcal{S}_j) \setminus \mathcal{S}_j$ for $1 \leq j \leq 2$. Then for every $\alpha \in Nil(\mathcal{S})^*$ $\mathcal{EG}(\mathcal{S})[\{\alpha, (\alpha_1, 0), (0, \alpha_2)\}]$ forms a triangle, a contradiction.

□

Theorem 2.17 *Let $\mathcal{S}$ be a reduced semigroup such that $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$ and let $\mathcal{EG}(\mathcal{S})$ be a complete graph. Then, $z^2 = z$ for each $z \in \mathcal{Z}(\mathcal{S})$.*

Proof. Assume on contrary that $z \in \mathcal{Z}(\mathcal{S})^*$ and $z^2 \neq z$. By Lemma 2.2(i), $ann_{\mathcal{S}}(z) = ann_{\mathcal{S}}(z^2)$. By Lemma 2.5($ii$), z is not adjacent to z^2 in $\mathcal{EG}(\mathcal{S})$, which is a contradiction. Hence $z = z^2$ for every $z \in \mathcal{Z}(\mathcal{S})$.

□

Theorem 2.18 *Let $\mathcal{S}$ be a non-reduced semigroup such that $\mathcal{Z}(\mathcal{S}) \neq (\mathcal{S})$. Then the following statements are equivalent:*

(i) $\mathcal{EG}(\mathcal{S}) = \Gamma(\mathcal{S})$.
(ii) *If $uv \neq 0$ for some $u, v \in \mathcal{Z}(\mathcal{S})$, then $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(v)$ and $ann_{\mathcal{S}}(u)$ is a prime ideal of $\mathcal{S}$.*

Proof. (i) $\Rightarrow$ (ii) Assume that $uv \neq 0$, for some $u, v \in \mathcal{Z}(\mathcal{S})$. Since $\Gamma(\mathcal{S}) = \mathcal{EG}(\mathcal{S})$, by Lemma 2.5($ii$), we obtain $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(v)$. We now show that $ann_{\mathcal{S}}(u)$ is a prime ideal of $\mathcal{S}$. Suppose that $ab \in ann_{\mathcal{S}}(u)$, and $a, b \notin ann_{\mathcal{S}}(u)$. Thus $au \neq 0 \neq bu, a, b \in \mathcal{Z}(\mathcal{S})$. By Lemma 2.5($iv$), $u, v \notin Nil(\mathcal{S})$ and hence $u \neq a$ or $u \neq b$. Without loss of generality we may assume that $u \neq b$. Thus $au \in \mathcal{S}u \cap ann_{\mathcal{S}}(b)$ and so $ub = 0$, a contradiction. Hence $ann_{\mathcal{S}}(u)$ is a prime ideal of $\mathcal{S}$.
(ii) $\Rightarrow$ (i) If $uv = 0$ for all $u, v \in \mathcal{Z}(\mathcal{S})$, then $\Gamma(\mathcal{S})$ is complete and by Theorem 2.2, $\mathcal{EG}(\mathcal{S})$ is complete. To complete the proof, we prove that if $uv \neq 0$, then $\mathcal{S}u \cap ann_{\mathcal{S}}(v) = 0 = \mathcal{S}v \cap ann_{\mathcal{S}}(u)$. Since $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(v)$, it is sufficient if we show $\mathcal{S}u \cap ann_{\mathcal{S}}(u) = 0 = \mathcal{S}v \cap ann_{\mathcal{S}}(v)$. If $u \in ann_{\mathcal{S}}(u)$, then $u \in ann_{\mathcal{S}}(v)$ and we get $uv = 0$, a contradiction. Thus $u \notin ann_{\mathcal{S}}(u)$, also if $\mathcal{S}u \cap ann_{\mathcal{S}}(u) \neq (0)$, then there exists $su \neq 0$ such that $su(u) = su^2 = 0$. Since $u^2 \notin ann_{\mathcal{S}}(u)$ ($ann_{\mathcal{S}}(u)$ is a prime ideal), $s \in ann_{\mathcal{S}}(u)$, a contradiction. Thus $\mathcal{S}u \cap ann_{\mathcal{S}}(u) = (0)$. Similarly, $\mathcal{S}v \cap ann_{\mathcal{S}}(v) = (0)$.

□

Theorem 2.19 *Let $\mathcal{S}$ be a non reduced semigroup such that $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$. If $\mathcal{EG}(\mathcal{S}) = \Gamma(\mathcal{S}) = \mathcal{K}_m \vee \overline{\mathcal{K}_n}$, then $\mathcal{V}(\mathcal{K}_m) \cup \{0\}$ is a prime ideal of $\mathcal{S}$.*

Proof. First assume that $\mathcal{EG}(\mathcal{S}) = \Gamma(\mathcal{S}) = \mathcal{K}_m \vee \overline{\mathcal{K}_n}$. Since $\Gamma(\mathcal{S}) = \mathcal{K}_m \vee \overline{\mathcal{K}_n}$, every vertices of $\mathcal{K}_m$ is adjacent to all other vertices and there is no adjacency between any two distinct vertices of $\overline{\mathcal{K}_n}$. Therefore $uv \neq 0$ for every $u, v \in \mathcal{V}(\overline{\mathcal{K}_n})$ and by Lemma 2.6(ii), $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(v)$ and also $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(v) = \mathcal{V}(\mathcal{K}_m) \cup \{0\}$. By Theorem 2.18, $\mathcal{V}(\mathcal{K}_m) \cup \{0\}$ is a prime ideal.

□

Theorem 2.20 *Let $\mathcal{S}$ be a non reduced semigroup such that $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$. If $ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))$ is a prime ideal of $\mathcal{S}$, then $\mathcal{EG}(\mathcal{S}) = \Gamma(\mathcal{S}) = \mathcal{K}_m \vee \overline{\mathcal{K}_n}$.*

Proof. Suppose that $ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))$ is a prime ideal of $\mathcal{S}$. Thus $uv \neq 0$ for all $u, v \in \mathcal{Z}(\mathcal{S}) \setminus ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))$ and $uv = 0$ for all $u, v \in ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))$. Now, it is not difficult to see that $\Gamma(\mathcal{S})[ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))]$ and $\Gamma(\mathcal{S})[\mathcal{Z}(\mathcal{S}) \setminus ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))]$ are subgraphs of $\mathcal{EG}(\mathcal{S})$ such that $\Gamma(\mathcal{S})[ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))^*] = \mathcal{K}_{|ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))^*|}$ and $\Gamma(\mathcal{S})[\mathcal{Z}(\mathcal{S}) \setminus ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))] = \overline{\mathcal{K}_{|\mathcal{Z}(\mathcal{S}) \setminus ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))|}}$ and $\Gamma(\mathcal{S}) = \Gamma(\mathcal{S})[ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))] \vee \Gamma(\mathcal{S})[\mathcal{Z}(\mathcal{S}) \setminus ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))]$. Finally we prove that $\Gamma(\mathcal{S}) = \mathcal{EG}(\mathcal{S})$. Trivially, $uv \neq 0$ if and only if $u, v \in \mathcal{Z}(\mathcal{S}) \setminus ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))$. Since $ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))$ is prime and $uv \neq 0$ we find $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(v) = ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))$. Thus $ann_{\mathcal{S}}(u)$ is a prime ideal and by Theorem 2.18, $\Gamma(\mathcal{S}) = \mathcal{EG}(\mathcal{S})$.

□

Theorem 2.21 *Let $\mathcal{S}$ be a non-reduced semigroup such that $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$. Then the following statements are equivalent:*

(i) *$\mathcal{EG}(\mathcal{S})$ is a star graph.*
(ii) *$gr(\mathcal{EG}(\mathcal{S})) = \infty$.*
(iii) *$\Gamma(\mathcal{S}) = \mathcal{EG}(\mathcal{S})$ and $gr(\Gamma(\mathcal{S})) = \infty$.*
(iv) *either $|\mathcal{Z}(\mathcal{S})| = |Nil(\mathcal{S})| = 3$ or $ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))$ is a prime ideal of $\mathcal{S}$, $|ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))| = 2$ and $|\mathcal{Z}(\mathcal{S})| = n$.*
(v) *$\mathcal{EG}(\mathcal{S}) = \mathcal{K}_{1,1}$ or $\mathcal{EG}(\mathcal{S}) = \mathcal{K}_{1,n-2}$.*

Proof. $(i) \Rightarrow (ii)$ This implication is obvious.

$(ii) \Rightarrow (iii)$ If $u \in Nil(\mathcal{S})^*$, then u is adjacent to every other vertex of $\mathcal{EG}(\mathcal{S})$. Since $gr(\mathcal{EG}(\mathcal{S})) = \infty$ implies that $\mathcal{EG}(\mathcal{S})$ is a star graph. Since $\Gamma(\mathcal{S})$ is a connected subgraph of $\mathcal{EG}(\mathcal{S})$, we conclude that $\mathcal{EG}(\mathcal{S}) = \Gamma(\mathcal{S})$ and $gr(\Gamma(\mathcal{S})) = \infty$.

$(iii) \Rightarrow (iv)$ Since $\mathcal{S}$ is a non-reduced semigroup, one may easily check that $\Gamma(\mathcal{S})$ is a star graph. Clearly either $|Nil(\mathcal{S})| = 3$ or $|Nil(\mathcal{S})| = 2$. If $|Nil(\mathcal{S})| = 3$, then $\mathcal{Z}(\mathcal{S}) = Nil(\mathcal{S})$. If $|Nil(\mathcal{S})| = 2$, then $u \in Nil(\mathcal{S})^*$ is adjacent to every other vertex. Since $\mathcal{EG}(\mathcal{S}) = \Gamma(\mathcal{S})$ are star graphs and $Nil(\mathcal{S} = 2$, we have $u^2 = 0$. Thus $\{0, u\} = ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))$ and for any $v \neq u \neq w$ $vw \neq 0$, $ann_{\mathcal{S}}(v) = ann_{\mathcal{S}}(w) = ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))$ by Theorem 2.20,

the result holds.

$(iv) \Rightarrow (v)$ is clear by Theorem 2.20.

$(v) \Rightarrow (i)$ Clearly true.

□

Theorem 2.22 *Let $\mathcal{S}$ be a commutative semigroup such that $\mathcal{Z}(\mathcal{S}) = \mathcal{S}$. Then $\mathcal{EG}(\mathcal{S})$ contains an isolated vertex if and only if $\mathcal{S}^2 = (0)$.*

Proof. Suppose that $x \in \mathcal{Z}(\mathcal{S})^*$ is an isolated vertex in $\mathcal{EG}(\mathcal{S})$. Then $x^2 = 0$, because if $x^2 \neq 0$, then there exists $y \in \mathcal{S} \setminus \{0, x\}$ such that $xy = 0$. Thus $\mathcal{S}x \cap ann_{\mathcal{S}}(y) \neq (0)$ and $x - y$ is an edge of $\mathcal{EG}(\mathcal{S})$, which is a contradiction. Therefore $x^2 = 0$ and $\mathcal{S}x \cap ann_{\mathcal{S}}(s) = \mathcal{S}s \cap ann_{\mathcal{S}}(x) = (0)$ for all $s \in \mathcal{S}$. This implies that $ann_{\mathcal{S}}(x) = ann_{\mathcal{S}}(s)$ for all $s \in \mathcal{S}^*$. If $z \in \mathcal{S}^*$, then there exists $z' \in \mathcal{S}^*$ such that $zz' = 0$. As $ann_{\mathcal{S}}(x) = ann_{\mathcal{S}}(z')$ gives $z \in ann_{\mathcal{S}}(x)$. Thus $ann_{\mathcal{S}}(x) = \mathcal{S}$. Since $ann_{\mathcal{S}}(x) = ann_{\mathcal{S}}(s)$ for all $s \in \mathcal{S}$, we have $\mathcal{S}^2 = (0)$. Hence proved.
Conversely, suppose that $\mathcal{S}^2 = (0)$. If $u \in \mathcal{S}^*$, then $\mathcal{S}u \cap ann_{\mathcal{S}}(s) = \mathcal{S}s \cap ann_{\mathcal{S}}(u) = (0)$ for all $s \in \mathcal{S}^*$, which yield that u is an isolated vertex of $\mathcal{EG}(\mathcal{S})$.

□

Corollary 2.23 *Let $\mathcal{S}$ be a commutative semigroup such that $\mathcal{Z}(\mathcal{S}) = \mathcal{S}$. Then $\mathcal{EG}(\mathcal{S}) \cong \overline{\mathcal{K}_n}$ if and only if $\mathcal{S}^2 = (0)$ and $|\mathcal{S}^*| = n$.*

Theorem 2.24 *Let $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \mathcal{S}_k$, where $k \geq 2$ is a positive integer and $\mathcal{Z}(\mathcal{S}_j)^* = \emptyset$ for every $1 \leq j \leq k$. Then $\mathcal{EG}(\mathcal{S})$ is a complete $(2^k - 2)$-partite graph.*

Proof. Assume that $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \mathcal{S}_k$, where $k \geq 2$ is a positive integer and $\mathcal{Z}(\mathcal{S}_j)^* = \emptyset$ for every $1 \leq j \leq k$. Let $\alpha, \beta \in \mathcal{V}(\mathcal{EG}(\mathcal{S}))$. Then $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_k)$, where $\alpha_j, \beta_j \in \mathcal{S}_j$, for every $1 \leq j \leq k$. We define a relation $\precsim$ on $\mathcal{V}(\mathcal{EG}(\mathcal{S}))$ as follows: $\alpha \precsim \beta$, whenever $\alpha_j = 0$ if and only if $\beta_j = 0$, for every $1 \leq j \leq k$. Clearly $\precsim$ is an equivalence relation on $\mathcal{V}(\mathcal{EG}(\mathcal{S}))$. For any $\alpha \in \mathcal{V}(\mathcal{EG}(\mathcal{S}))$, $\tilde{\alpha}$ denote the equivalence class of α. Let $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_k) \in \mathcal{V}(\mathcal{EG}(\mathcal{S}))$. Then for every components γ_j, $1 \leq j \leq k$, we have either $\gamma_j = 0$ or $\gamma_j \neq 0$. Thus, the number of all selection of γ is $(2^k - 2)$ (for other selection $\gamma \notin \mathcal{V}(\mathcal{EG}(\mathcal{S}))$). Assume that $\tilde{\alpha}$ and $\tilde{\beta}$ are two distinct equivalence classes. It is sufficient to show, that there is no adjacency between any two distinct elements of $\tilde{\alpha}$ and each vertices of $\tilde{\alpha}$ is adjacent to all vertices of $\tilde{\beta}$. To see this let $\alpha', \alpha'' \in \tilde{\alpha}$ and $\beta \in \tilde{\beta}$. Then there exist $\alpha'_j, \alpha''_j, \beta_j \in \mathcal{S}_j$ such that $\alpha' = (\alpha'_1, \alpha'_2, \cdots, \alpha'_k)$, $\alpha'' = (\alpha''_1, \alpha''_2, \cdots, \alpha''_k)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_k)$, for all $1 \leq j \leq k$. Thus $\alpha'_j = 0$ if and only if $\alpha''_j = 0$, for all $1 \leq j \leq k$. This implies that $ann_{\mathcal{S}}(\alpha') = ann_{\mathcal{S}}(\alpha'')$ and by Lemma 2.5(ii), α' and α'' are not adjacent in $\mathcal{EG}(\mathcal{S})$). Also since α' not related to β' by $\precsim$. Then $\alpha'_j = 0 \neq \beta'_j$, for some $1 \leq j \leq k$. Hence $ann_{\mathcal{S}}(\alpha') \neq ann_{\mathcal{S}}(\beta)$. By Lemma 2.5$(ii)$, α' is adjacent to β in $\mathcal{EG}(\mathcal{S})$.

□

Corollary 2.25 *Let $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \mathcal{S}_k$ where $k \geq 2$ be a positive integer and $\mathcal{Z}(\mathcal{S}_j)^* = \emptyset$ for every $1 \leq j \leq k$. Then $\omega(\mathcal{EG}(\mathcal{S})) = \chi(\mathcal{EG}(\mathcal{S})) = 2^k - 2$.*

Proof. Directly follows from the Theorem 2.24.

□

In order to achieve the goal, we need a celebrated Kuratowski's theorem from graph theory [15, Theorem 6.2.2].

Theorem 2.26 *(Kuratowski's Theorem) A Graph $\mathcal{G}$ is planar if and only if it contains no subdivision of either $\mathcal{K}_{3,3}$ or $\mathcal{K}_5$.*

Proposition 2.27 *Let $\mathcal{S}$ be a commutative semigroup such that $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$ and $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3$. Then $\mathcal{EG}(\mathcal{S})$ is not planar.*

Proof. Let $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3 \setminus \mathcal{Z}(\mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3)$. Then the vertices $(\alpha_1, 0, 0), (0, \alpha_2, 0), (0, 0, \alpha_3), (\alpha_1, \alpha_2, 0)$ and $(\alpha_1, 0, \alpha_3)$ form $\mathcal{K}_5$ as a subgraph of $\mathcal{EG}(\mathcal{S})$.

□

Lemma 2.28 *Let $\mathcal{S}$ be a non reduced semigroup such that $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$ and $\mathcal{EG}(\mathcal{S})$ is planar. Then the following statements hold:*

(*i*) *If $\mathcal{EG}(\mathcal{S})$ is a infinite graph, then $\mathcal{S}$ can not be written as the finite direct products of semigroup.*
(*ii*) *If $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2$, then $|Nil(\mathcal{S})| = 2$.*

Proof. (*i*) Suppose on contrary that $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2$, for some semigroup $\mathcal{S}_1, \mathcal{S}_2$. Without loss of generality we may assume that $a \in Nil(\mathcal{S}_1)^*$ and $(\alpha, \beta) \in \mathcal{S} \setminus \mathcal{Z}(\mathcal{S})$. If $\mathcal{Z}(\mathcal{S}_2) = \infty$, then the set of vertices $\{(\alpha, 0), (a, 0), (a, \beta), (0, \beta), (0, b), (0, c)\}$ forms $\mathcal{K}_{3,3}$, where $b, c \in \mathcal{Z}(\mathcal{S}_2)^*$, and we get a contradiction. If $\mathcal{Z}(\mathcal{S}_1) = \infty$, then the vertices of the set $\{(0, \beta), (a, 0), (a, \beta), (\alpha, 0), (b, 0), (c, 0)\}$ forms $\mathcal{K}_{3,3}$, where $b, c \in \mathcal{Z}(\mathcal{S}_1)^*$, a contradiction.
(*ii*) Assume on contrary that $|Nil(\mathcal{S})| \geq 3$. Without loss of generality we may assume that $a \in Nil(\mathcal{S})^*$, $b \in Nil(\mathcal{S}_1)^*$ and $a \neq b$, $(\alpha, \beta) \in \mathcal{S} \setminus \mathcal{Z}(\mathcal{S})$. It can be easily seen that $\mathcal{EG}(\mathcal{S})[\{(\alpha, 0), (0, \beta), (b, \beta), (b, 0), a\}]$ forms $\mathcal{K}_5$, a contradiction.

□

Lemma 2.29 *Let $\mathcal{S}$ be a semigroup such that $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$ and $u - v$ is an edge of $\mathcal{EG}(\mathcal{S})$. Then either $u - w$ is an edge of $\mathcal{EG}(\mathcal{S})$ or $v - w$ is an edge of $\mathcal{EG}(\mathcal{S})$ for every $w \in \mathcal{Z}(\mathcal{S}) \setminus \{u, v, 0\}$.*

Proof. Assume that $u - v$ is an edge of $\mathcal{EG}(\mathcal{S})$. If $u - w$ is not an edge of $\mathcal{EG}(\mathcal{S})$, then by Lemma 2.5(*ii*), $ann_{\mathcal{S}}(u) = ann_{\mathcal{S}}(w)$. If $ann_{\mathcal{S}}(v) \neq ann_{\mathcal{S}}(w)$, then by Lemma 2.5(*ii*), $v - w$ is an edge of $\mathcal{EG}(\mathcal{S})$ and if $ann_{\mathcal{S}}(v) = ann_{\mathcal{S}}(w)$, then $v - w$ is an edge of $\mathcal{EG}(\mathcal{S})$.

□

Theorem 2.30 *Let $\mathcal{S}$ be a non-reduced semigroup such that $|Nil(\mathcal{S})| = 2$ and $|\mathcal{I}(\mathcal{S})| = 1$. Then the following statements are equivalent:*
(i) $ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S}))$ is a prime ideal of $\mathcal{S}$.
(ii) $\mathcal{EG}(\mathcal{S})$ is a planar graph.

Proof. $(i) \Rightarrow (ii)$ Directly follows from the Theorem 2.20.

$(ii) \Rightarrow (i)$ Let $u \in Nil(\mathcal{S})^*$. We prove that $vw \neq 0$ for every $w, v \in \mathcal{Z}(\mathcal{S})\backslash\{0, u\}$. Assume on contrary, that $wv = 0$ for some $v, w \in \mathcal{Z}(\mathcal{S})\backslash\{0, u\}$. Since $|\mathcal{I}(\mathcal{S})| = 1$, $w^j \neq w^{j+1}$ and $v^j \neq v^{j+1}$, for some $1 \leq j \leq 2$. Thus $\mathcal{EG}(\mathcal{S})[\{v, v^2, v^3, w, w^2, w^3\}]$ forms $\mathcal{K}_{3,3}$, a contradiction. Therefore $ann_{\mathcal{S}}(\mathcal{Z}(\mathcal{S})) = Nil(\mathcal{S})$ and $ann_{\mathcal{S}}(v) = \{0, u\}$, for every $v \in \mathcal{Z}(\mathcal{S}) \setminus \{0, u\}$. Hence $\Gamma(\mathcal{S}) = \mathcal{EG}(\mathcal{S}) = \mathcal{K}_1 \vee \overline{\mathcal{K}_{\mathcal{Z}(\mathcal{S})\setminus\{0,u\}}}$ and by Theorem 2.20, the result holds.

□

3 Extended zero divisor graph with three vertices

In this section, we characterize all extended zero divisor graphs $\mathcal{EG}(\mathcal{S})$ with three vertices. Throughout this section, we assume that $\mathcal{Z}(\mathcal{S}) = \{0, x, y, z\}$. Since $\Gamma(\mathcal{S})$ is connected, there exists a vertex z such that z is adjacent to both x and y in $\Gamma(\mathcal{S})$. Therefore without loss of generality, we may assume that $xz = 0 = yz$. If $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$, then $\Gamma(\mathcal{S}) \leq \mathcal{EG}(\mathcal{S})$, and so $\mathcal{EG}(\mathcal{S})$ is connected, which implies that $\mathcal{EG}(\mathcal{S})$ is isomorphic to $\mathcal{K}_{1,2}$ or $\mathcal{K}_3$.

Theorem 3.1 *Let $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$. Then $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_{1,2}$(with center z) if and only if $\{0, z\}$ is a prime ideal of $\mathcal{S}$.*

Proof. First suppose that $\{0, z\}$ is a prime ideal of $\mathcal{S}$. Then there is no edge between x and y in $\Gamma(\mathcal{S})$. On the other hand, since $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$ and z is adjacent to x and y in $\Gamma(\mathcal{S})$, we find that z is adjacent to x and y in $\mathcal{EG}(\mathcal{S})$. It is easy to see that x is not adjacent to y in $\mathcal{EG}(\mathcal{S})$ and z is adjacent to x and y in $\mathcal{EG}(\mathcal{S})$. Therefore $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_{1,2}$.
Conversely, suppose that $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_{1,2}$(with center z). Clearly z is a cut vertex of $\Gamma(\mathcal{S})$. Then by Lemma 3.35 of [4], $\{0, z\}$ is an ideal of $\mathcal{S}$. Now if $\{0, z\}$ is not prime, then $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_3$.

□

Theorem 3.2 *Let $\mathcal{Z}(\mathcal{S}) \neq \mathcal{S}$. Then $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_3$ if and only if there exist $s_1, s_2 \in \mathcal{S} \setminus \{0, z\}$ such that $s_1 s_2 \in \{0, z\}$.*

Proof. Suppose that $s_1, s_2 \in \mathcal{S}\setminus\{0, z\}$ such that $s_1 s_2 \in \{0, z\}$. Now we have the following cases.
Case(i) If $s_1 s_2 = 0$ and $s_1 \neq s_2$, then $s_1, s_2 \in \mathcal{Z}(\mathcal{S}) \setminus \{0, z\}$. Without loss of generality we may assume that $s_1 = x, s_2 = y$. Thus $x - y$ is an edge in $\mathcal{EG}(\mathcal{S})$. If $s_1 = s_2$, then $s_1^2 = 0$. Since $s_1 \in \mathcal{Z}(\mathcal{S})\backslash\{0, z\}$, we have either $x^2 = 0$ or $y^2 = 0$. If $x^2 = 0$, then either $xy = 0$ or $xy \neq 0$. Thus $\mathcal{S}y \cap ann_{\mathcal{S}}(x) \neq (0)$

and $x-y$ is an edge of $\mathcal{EG}(\mathcal{S})$. Similarly if $y^2=0$, then $\mathcal{S}x\cap ann_{\mathcal{S}}(y)\neq(0)$ and $x-y$ is an edge of $\mathcal{EG}(\mathcal{S})$.
Case(ii) Let $s_1s_2=z$. Since $zx=0$ implies $s_1s_2x=0$, i.e., $s_1(s_2x)=0$. If $s_2x=0$, then either $xy=0$ or $x^2=0$. Thus $\mathcal{S}y\cap ann_{\mathcal{S}}(x)\neq 0$ and $x-y$ is an edge of $\mathcal{EG}(\mathcal{S})$. If $s_2x\neq 0$, then either $s_1=x$ or $s_1=y$. If $s_1=x$, then $\mathcal{S}x\cap ann_{\mathcal{S}}(y)\neq 0$ and if $s_1=y$, then $\mathcal{S}y\cap ann_{\mathcal{S}}(x)\neq 0$. Thus all of the above cases, $x-y$ is an edge of $\mathcal{EG}(\mathcal{S})$. Since $\Gamma(\mathcal{S})\leq\mathcal{EG}(\mathcal{S})$ and z is adjacent to both vertices in $\Gamma(\mathcal{S})$, we have $\mathcal{EG}(\mathcal{S})\cong\mathcal{K}_3$.
Conversely, assume that $\mathcal{EG}(\mathcal{S})\cong\mathcal{K}_3$ and $s_1s_2\notin\{0,z\}$ for all $s_1,s_2\in\mathcal{S}\setminus\{0,z\}$. Then $xy\neq 0$ and $x-z-y$ is only a path in $\Gamma(\mathcal{S})$. Thus z is a cut vertex of $\Gamma(\mathcal{S})$ and by Lemma 3.25 of [4], $\{0,z\}$ is an ideal of $\mathcal{S}$. Finally, $\{0,z\}$ is a prime ideal of $\mathcal{S}$. Then by Theorem 3.1, $\mathcal{EG}(\mathcal{S})\cong\mathcal{K}_{1,2}$, a contradiction.

□

Theorem 3.3 *Let $\mathcal{Z}(\mathcal{S})=\mathcal{S}$. Then $\mathcal{EG}(\mathcal{S})\cong\overline{\mathcal{K}_3}$ if and only if $\mathcal{S}^2=(0)$.*

Proof. Suppose that $\mathcal{EG}(\mathcal{S})\cong\overline{\mathcal{K}_3}$. Since $xz=yz=0$, x is not adjacent to z in $\mathcal{EG}(\mathcal{S})$. Then $\mathcal{S}x\cap ann_{\mathcal{S}}(z)=\mathcal{S}x\cap ann_{\mathcal{S}}(z)=(0)$. This implies that $ann_{\mathcal{S}}(xz)=ann_{\mathcal{S}}(x)=ann_{\mathcal{S}}(z)$ and $xy=x^2=z^2=0$. Similarly, y is not adjacent to z in $\mathcal{EG}(\mathcal{S})$ gives $y^2=0$. Hence $\mathcal{S}^2=(0)$.
The converse implication is obvious.

□

Now suppose that $\mathcal{Z}(\mathcal{S})=\mathcal{S}=\{0,x,y,z\}$. In this case Theorem 2.2 need not be true. Since $\Gamma(\mathcal{S})$ is connected, we have $\Gamma(\mathcal{S})\cong\mathcal{K}_{1,2}$ or $\Gamma(\mathcal{S})\cong\mathcal{K}_3$. If $\Gamma(\mathcal{S})\cong\mathcal{K}_{1,2}$ then without loss of generality we may assume $xz=yz=0$ and $xy\neq 0$. $\Gamma(\mathcal{S})\cong\mathcal{K}_3$, then $xz=yz=0=xy$. In these cases we have the following theorems.

Theorem 3.4 *Let $\mathcal{S}$ be a commutative semigroup such that $\mathcal{Z}(\mathcal{S})=\mathcal{S}$ and $\Gamma(\mathcal{S})\cong\mathcal{K}_{1,2}$ (with center z). Then $\mathcal{EG}(\mathcal{S})\cong\mathcal{K}_3$ if and only if $(xy)^2=0$.*

Proof. Suppose that $\mathcal{EG}(\mathcal{S})\cong\mathcal{K}_3$ and assume on contrary that $(xy)^2\neq 0$. Thus $xy\in\{x,y\}$. If $xy=x$, then clearly, $\mathcal{S}x\cap ann_{\mathcal{S}}(y)=0=\mathcal{S}y\cap ann_{\mathcal{S}}(x)$ and there is no edge between x and y and we get a contradiction. Similarly, If $xy=y$, again we get a contradiction. Hence $(xy)^2=0$.
Conversely, suppose that $(xy)^2=0$. Thus $xy\in\{x,y,z\}$.
Case(i) If $xy=x$, then $\mathcal{S}y\cap ann_{\mathcal{S}}(x)\neq 0\neq\mathcal{S}x\cap ann_{\mathcal{S}}(z)$ and $\mathcal{S}y\cap ann_{\mathcal{S}}(z)\neq 0$.
Case(ii) If $xy=y$, then $\mathcal{S}x\cap ann_{\mathcal{S}}(y)\neq 0\neq\mathcal{S}x\cap ann_{\mathcal{S}}(z)$ and $\mathcal{S}y\cap ann_{\mathcal{S}}(z)\neq 0$.
Case(iii) If $xy=z$, then $\mathcal{S}x\cap ann_{\mathcal{S}}(y)\neq 0\neq\mathcal{S}x\cap ann_{\mathcal{S}}(z)$ and $\mathcal{S}y\cap ann_{\mathcal{S}}(z)\neq 0$.
Thus in all the cases, we get $\mathcal{EG}(\mathcal{S})\cong\mathcal{K}_3$.

□

Theorem 3.5 *Let $\mathcal{S}$ be a commutative semigroup such that $\mathcal{Z}(\mathcal{S}) = \mathcal{S}$ and $\Gamma(\mathcal{S}) \cong \mathcal{K}_{1,2}$ (with center z). Then $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_{1,2}$ if and only if $(xy)^2 \neq 0$.*

Proof. Suppose that $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_{1,2}$ and assume on contrary that $(xy)^2 = 0$. Then $xy \in \{x, y, z\}$.
Case(i) If $xy = x$, then $\mathcal{S}y \cap ann_{\mathcal{S}}(x) \neq 0 \neq \mathcal{S}x \cap ann_{\mathcal{S}}(z)$ and $\mathcal{S}y \cap ann_{\mathcal{S}}(z) \neq 0$.
Case(ii) If $xy = y$, then $\mathcal{S}x \cap ann_{\mathcal{S}}(y) \neq 0 \neq \mathcal{S}x \cap ann_{\mathcal{S}}(z)$ and $\mathcal{S}y \cap ann_{\mathcal{S}}(z) \neq 0$.
Case(iii) If $xy = z$, then $\mathcal{S}x \cap ann_{\mathcal{S}}(y) \neq 0 \neq \mathcal{S}x \cap ann_{\mathcal{S}}(z)$ and $\mathcal{S}y \cap ann_{\mathcal{S}}(z) \neq 0$.
Thus in all the cases we get $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_3$ which is a contradiction.
Conversely, suppose that $(xy)^2 \neq 0$. Thus $xy \in \{x, y\}$. If $xy = x$, then clearly, $\mathcal{S}x \cap ann_{\mathcal{S}}(y) = 0 = \mathcal{S}y \cap ann_{\mathcal{S}}(x)$ and $\mathcal{S}x \cap ann_{\mathcal{S}}(z) \neq 0 \neq \mathcal{S}y \cap ann_{\mathcal{S}}(z)$. Thus $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_{1,2}$. We get the similar result, if $xy = y$.

□

Theorem 3.6 *Let $\mathcal{S}$ be a commutative semigroup such that $\mathcal{Z}(\mathcal{S}) = \mathcal{S}$ and $\Gamma(\mathcal{S}) \cong \mathcal{K}_3$. Then the following statements hold:*

(i) *$\mathcal{EG}(\mathcal{S}) \cong \overline{\mathcal{K}_3}$ if and only if $\mathcal{S}^2 = \{0\}$.*
(ii) *$\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_3$ if and only if there exist at least two distinct elements $\alpha, \beta \in \mathcal{S}$ such that $\alpha^2 \neq 0 \neq \beta^2$.*
(iii) *$\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_{1,2}$ if and only if there exists exactly one element in $\alpha \in \mathcal{S}$ such that $\alpha^2 \neq 0$.*

Proof. (i) Directly follows by Corollary 2.23.
(ii) Without loss of generality we may assume that $x, y \in \mathcal{S}$ such that $x^2 \neq 0 \neq y^2$. Thus $\mathcal{S}x \cap ann_{\mathcal{S}}(y) \neq 0 \neq \mathcal{S}y \cap ann_{\mathcal{S}}(z)$ and $\mathcal{S}x \cap ann_{\mathcal{S}}(z) \neq 0$. Hence $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_3$.
Conversely, suppose that $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_3$. Assume on contrary that there do not exist at least two distinct elements $\alpha, \beta \in \mathcal{S}$ such that $\alpha^2 = 0 = \beta^2$. Since $\mathcal{S} = \{0, x, y, z\}$, we have the following cases.
Case(a). There exists only one element $\gamma \in \mathcal{S}$ such that $\gamma^2 \neq 0$. If $\gamma = x$, then $\mathcal{S}x \cap ann_{\mathcal{S}}(y) \neq 0 \neq \mathcal{S}x \cap ann_{\mathcal{S}}(z)$ and $\mathcal{S}y \cap ann_{\mathcal{S}}(z) = 0 = \mathcal{S}z \cap ann_{\mathcal{S}}(y)$. Thus $\mathcal{EG}(\mathcal{S}) \cong \mathcal{K}_{1,3}$, a contradiction. Similarly if $\gamma = y$ or $\gamma = z$, again we get a contradiction.
Case(b). There does not exist any element $\gamma \in \mathcal{S}$ such that $\gamma^2 \neq 0$. Then $\mathcal{S}^2 = 0$ and by Corollary 2.23, $\mathcal{EG}(\mathcal{S}) \cong \overline{\mathcal{K}_3}$ a contradiction. Hence there exist at least two elements $\alpha, \beta \in \mathcal{S}$ such that $\alpha^2 \neq 0 \neq \beta^2$.
(iii) Directly follows from (i) and (ii).

□

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