An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXVIII, 2022, f. 2
https://doi.org/10.47743/anstim.2022.00011

Multiplicative Lie skew derivations on triangular algebras

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Abstract Let $\mathcal{U}$ be a triangular algebra. Under certain assumptions, we show that every multiplicative Lie σ-derivation $L : \mathcal{U} \to \mathcal{U}$ is of the form $\delta + \tau$, where δ is an additive σ-derivation on $\mathcal{U}$ and τ is a map from $\mathcal{U}$ to its center $Z_\sigma(\mathcal{U})$ that vanishes on Lie product. As an application, we study the multiplicative Lie σ-derivation on nest algebras and block upper triangular matrix algebras.

Keywords derivation · triangular algebra

Mathematics Subject Classification (2010) 16W25 · 15A78

1 Introduction

Let $\mathcal{A}$ be an algebra over a commutative ring R with unity. For any $x, y \in \mathcal{A}$, $[x, y] = xy - yx$ denotes the Lie product and suppose that $\sigma : \mathcal{A} \to \mathcal{A}$ is an automorphism on $\mathcal{A}$. A map (not necessarily linear) $L : \mathcal{A} \to \mathcal{A}$ is called a multiplicative σ-derivation if $L(xy) = L(x)y + \sigma(x)L(y)$ and L is called a multiplicative Lie σ-derivation if $L([x, y]) = L(x)y + \sigma(x)L(y) - L(y)x - \sigma(y)L(x)$ for all $x, y \in \mathcal{A}$. In addition, if L is linear on $\mathcal{A}$, then L is said to be a σ-derivation and Lie σ-derivation on $\mathcal{A}$ respectively. Obviously, if $\sigma = I_{\mathcal{A}}$ the identity mapping on $\mathcal{A}$, then σ-derivation and Lie σ-derivation is called derivation and Lie derivation respectively. Also, for given $x_0 \in \mathcal{A}$, a map $L : \mathcal{A} \to \mathcal{A}$ is called an inner σ-derivation if $L(x) = \sigma(x)x_0 - x_0x$ for some fixed $x_0 \in \mathcal{A}$ and for all $x \in \mathcal{A}$.

The study derivations and related maps with different algebraic structures are vast and diverse area of research (see [1–6, 9–11] and references therein). The problem of characterizing the relationship between derivations and the decomposition of Lie derivations have received a fair amount of attention of many mathematicians on various kind of algebras (see [9, 14] and references therein). Firstly, it was Martindale [9] who studied the characterization of Lie derivation on primitive rings with nontrivial idempotents. Recently, Yu and Zhang [14] obtained a characterization of multiplicative Lie derivation on triangular algebras and proved that every multiplicative Lie derivation on triangular algebra can be decomposed as a sum of a derivation and a central map that vanishes at all the commutators under some assumptions.

Furthermore, many algebraists studied the characterizations of different σ-derivations on triangular algebras following suitable techniques (see [8,13] and references therein). In the year 2011, Han and Wei [8] studied Jordan σ-derivation on triangular algebras $\mathcal{U}$ with $e\mathcal{U}e$ and $f\mathcal{U}f$ having only trivial idempotents and proved that any Jordan σ-derivation of $\mathcal{U}$ is a σ-derivation. Also, Yang and Zhu [13] gave characterizations of additive Lie σ-derivations on triangular algebras with only trivial idempotents. Note that the authors actually studied more general maps (Lie (α,β)-derivations). Furthermore, Wang [12] proposed the definition of the weak loyal condition and proved that every Jordan σ-derivation of a triangular algebra that exists either a left weak loyal bimodule or a right weak loyal bimodule, is a σ-derivation.

In the light of above mention references, we carried out the study of multiplicative Lie σ-derivation on triangular algebras having nontrivial idempotents and prove that it has standard form. As a consequence of our main result, we mention its application to nest algebras and block upper triangular matrix algebras.

2 Preliminaries

Throughout, let $\mathcal{R}$ be a commutative ring with identity and $\mathcal{U}$ be a triangular algebra over $\mathcal{R}$. Denote the triangular algebra consisting of A, B and M by $\mathcal{U} = \begin{bmatrix} \mathrm{A} & \mathrm{M} \\ 0 & \mathrm{B} \end{bmatrix}$. Then $\mathcal{U}$ is an associative and noncommutative $\mathcal{R}$-algebra. The center of $\mathcal{U}$ is

$$Z(\mathcal{U}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a \in Z(\mathrm{A}), b \in Z(\mathrm{B});\ am = mb\ \forall\ m \in \mathrm{M} \right\}.$$

However, if M is faithful as a left A-module and also as a right B-module, then the conditions $a \in Z(\mathrm{A})$ and $b \in Z(\mathrm{B})$ become redundant and can be deleted. Define two natural projections $\pi_{\mathrm{A}} : \mathcal{U} \to \mathrm{A}$ and $\pi_{\mathrm{B}} : \mathcal{U} \to \mathrm{B}$ by $\pi_{\mathrm{A}}\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = a$ and $\pi_{\mathrm{B}}\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = b$. Moreover, $\pi_{\mathrm{A}}(Z(\mathcal{U})) \subseteq Z(\mathrm{A})$ and $\pi_{\mathrm{B}}(Z(\mathcal{U})) \subseteq Z(\mathrm{B})$ and there exists a unique algebraic isomorphism $\xi : \pi_{\mathrm{A}}(Z(\mathcal{U})) \to \pi_{\mathrm{B}}(Z(\mathcal{U}))$ such that $am = m\xi(a)$ for all $m \in \mathrm{M}$.

Let us assume that $\mathcal{U}$ has an idempotent $e \neq 0, 1$ and let f denote the idempotent $1 - e$. In this case, $\mathcal{U}$ can be represented in the so called Peirce decomposition form

$$\mathcal{U} = e\mathcal{U}e + e\mathcal{U}f + f\mathcal{U}f,$$

where $e\mathcal{U}e$ and $f\mathcal{U}f$ are subalgebras of $\mathcal{U}$ which are isomorphic to A and B, with unitary elements e and f respectively, $e\mathcal{U}f$ is an $(e\mathcal{U}e, f\mathcal{U}f)$-bimodule isomorphic to M which is faithful as left $e\mathcal{U}e$-module as well as right $f\mathcal{U}f$-module. That means $exe \cdot e\mathcal{U}f = 0$ implies $exe = 0$ and $e\mathcal{U}f \cdot fxf = 0$ implies $fxf = 0$ for all $x \in \mathcal{U}$. It should be remarked that $\pi_{\mathrm{A}}(Z(\mathcal{U}))$ and $\pi_{\mathrm{B}}(Z(\mathcal{U}))$ are isomorphic to $eZ(\mathcal{U})e$ and $fZ(\mathcal{U})f$ respectively.

Suppose that σ is an automorphism of $\mathcal{U}$ and denote $E = \sigma(e),\ F = \sigma(f) = 1 - E$, then E and F are also idempotents of $\mathcal{U}$. Since $F\mathcal{U}E = \sigma(f)\sigma(\mathcal{U})\sigma(e) = \sigma(f\mathcal{U}e) = \{0\}$, we can represent $\mathcal{U}$ as a triangular algebra of the form

$$\mathcal{U} = E\mathcal{U}E + E\mathcal{U}F + F\mathcal{U}F.$$

Subalgebras $e\mathcal{U}e$ and $E\mathcal{U}E$ are isomorphic, and subalgebras $f\mathcal{U}f$ and $F\mathcal{U}F$ are isomorphic. Likewise the bimodules $e\mathcal{U}f$ and $E\mathcal{U}F$ are isomorphic. Since $e\mathcal{U}f$ is faithful as a left $e\mathcal{U}e$-module and as a right $f\mathcal{U}f$-module, it follows that $E\mathcal{U}F$ is also faithful as a left $E\mathcal{U}E$-module and as a right $F\mathcal{U}F$-module.

An important decomposition of a triangular algebra $\mathcal{U}$ according to the automorphism σ can be given as

$$\mathcal{U} = E\mathcal{U}e + E\mathcal{U}f + F\mathcal{U}e + F\mathcal{U}f.$$

It can be easily seen that $E\mathcal{U}e \subseteq e\mathcal{U}e$ and $F\mathcal{U}f \subseteq F\mathcal{U}F$. As $e\mathcal{U}f$ is a faithful left $E\mathcal{U}e$ and $E\mathcal{U}F$ is a faithful right $F\mathcal{U}f$-module this implies that $Exe \cdot e\mathcal{U}f = \{0\}$. This shows that $Exe = 0$ and $E\mathcal{U}F \cdot Fxf = \{0\}$ implies $Fxf = 0$.

Automorphisms of triangular algebras have been extensively considered in Wang [12]. Now we mention some important results which are essential for our further discussions.

Lemma 2.1 *[7, Lemma 5.2.1] Every automorphism σ of $\mathcal{U}$ can be decomposed as $\sigma = \sigma_a \circ \bar{\sigma}$ where $a \in \mathcal{U}$ and $\bar{\sigma}$ is an automorphism of $\mathcal{U}$ with $\bar{\sigma}(e) \in e\mathcal{U}e + f\mathcal{U}f$ where $\bar{\sigma}(e)e \in e\mathcal{U}e, \bar{\sigma}(e)f \in f\mathcal{U}f$ are idempotents.*

Now it is easy to prove the following result. So, we omit the proof.

Lemma 2.2 *Let σ be an automorphism of $\mathcal{U}$ having the form $\sigma = \sigma_a \circ \bar{\sigma}$ such as in Lemma 2.1. Then $L : \mathcal{U} \to \mathcal{U}$ is a (Lie) σ-derivation if and only if the map $\bar{L} : \mathcal{U} \to \mathcal{U}$ defined by $\bar{L}(x) = aL(x)$ is a (Lie) $\bar{\sigma}$-derivation.*

Proof. Let L be σ-derivation. Then

$$\begin{aligned} L(xy) &= L(x)y + \sigma(x)L(y) \\ &= L(x)y + a^{-1}\bar{\sigma}(x)aL(y) \end{aligned}$$

for all $x, y \in \mathcal{U}$. Multiplying by a from the left side, we get

$$\begin{aligned} aL(xy) &= aL(x)y + aa^{-1}\bar{\sigma}(x)aL(y) \\ \bar{L}(xy) &= \bar{L}(x)y + \bar{\sigma}(x)\bar{L}(y) \end{aligned}$$

for all $x, y \in \mathcal{U}$. Hence $\bar{L}$ is a $\bar{\sigma}$-derivation. The converse can be proved in similar manner. Moreover, in the case of a Lie $\bar{\sigma}$-derivation an analogous proof can be applied.

□

Remark 2.1 [6, Proposition 2.5] Let $\mathcal{A}$ be an algebra and σ an automorphism of $\mathcal{A}$. The σ-center of $\mathcal{A}$ is $Z_\sigma(\mathcal{A}) = \{\lambda \in \mathcal{A} : \sigma(x)\lambda = \lambda x, \forall x \in \mathcal{A}\}$. It is a subspace of $\mathcal{A}$ which is invariant under σ. A linear map from $\mathcal{A}$ to $Z_\sigma(\mathcal{A})$ will be called σ-central.

The σ-center of a triangular algebra $\mathcal{U} = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B})$ has form

$$Z_\sigma(\mathcal{U}) = \{a + b \in \mathcal{U} : am = \sigma(m)b, \forall m \in \mathrm{M}\}.$$

Moreover, $\pi_{\mathrm{A}}(Z_\sigma(\mathcal{U})) \subseteq Z_\sigma(\mathrm{A})$ and $\pi_{\mathrm{B}}(Z_\sigma(\mathcal{U})) \subseteq Z_\sigma(\mathrm{B})$ and there exists a unique algebra isomorphism $\eta : \pi_{\mathrm{A}}(Z_\sigma(\mathcal{U})) \to \pi_{\mathrm{B}}(Z_\sigma(\mathcal{U}))$ such that $\sigma(m)\eta(a) = am$, for all $a \in \pi_{\mathrm{A}}(Z_\sigma(\mathcal{U})), m \in \mathrm{M}$.

3 Main results

Many authors studied σ-derivations on different rings and algebras. Here we study multiplicative Lie σ-derivation on triangular algebra $\mathcal{U}$. Actually, we prove the following result:

Theorem 3.1 *Let $\mathcal{U} = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B})$ be a 2-torsion free triangular algebra with nontrivial idempotents consisting of algebras A, B & faithful (A, B)-bimodule M and σ be an automorphism of $\mathcal{U}$. Suppose that $\mathcal{U}$ satisfying $\pi_{\mathrm{A}}(Z_\sigma(\mathcal{U})) = Z_\sigma(\mathrm{A})$ and $\pi_{\mathrm{B}}(Z_\sigma(\mathcal{U})) = Z_\sigma(\mathrm{B})$. Then every multiplicative Lie σ-derivation $L : \mathcal{U} \to \mathcal{U}$ has the form $L = \delta + \tau$, where $\delta : \mathcal{U} \to \mathcal{U}$ is an additive σ-derivation and τ is a mapping from $\mathcal{U}$ to its center which annihilates all Lie products.*

To develop the proof of our main result, we begin with the following lemmas:

Lemma 3.2 *Let $L : \mathcal{U} \to \mathcal{U}$ be a multiplicative Lie σ-derivation. Then there exist an inner σ-derivation $d : \mathcal{U} \to \mathcal{U}$ and a multiplicative Lie σ-derivation $L' : \mathcal{U} \to \mathcal{U}$ such that $L = d + L'$ and $EL'(e)f = FL'(e)e = 0$.*

Proof. Let $x_0 = EL'(e)f - FL'(e)e \in \mathcal{U}$ and let us define $d : \mathcal{U} \to \mathcal{U}$ as an inner σ-derivation $d(x) = \sigma(x)x_0 - x_0x$ for all $x \in \mathcal{U}$. It is easy to observe that $L' = L - d$ is multiplicative Lie σ-derivation. Accordingly,

$$\begin{aligned}
L'(e) &= L(e) - d(e) \\
&= L(e) - \{\sigma(e)(EL(e)f - FL(e)e) - (EL(e)f - FL(e)e)e\} \\
&= L(e) - EL(e)f - FL(e)e \\
&= EL(e)e + FL(e)f,
\end{aligned}$$

that is $L'(e) \in E\mathcal{U}e + F\mathcal{U}f$. This implies that $EL'(e)f = FL'(e)e = 0$.

□

Accordingly, it suffices to consider only those multiplicative Lie σ-derivation $L : \mathcal{U} \to \mathcal{U}$ satisfying $EL(e)f = FL(e)e = 0$.

Lemma 3.3 *If* $L : \mathcal{U} \to \mathcal{U}$ *is a multiplicative Lie* σ*-derivation satisfying* $EL(e)f = FL(e)e = 0$*, then* $L(e), L(f) \in Z_\sigma(\mathcal{U})$.

Proof. According to our assumption we have $L(e) = EL(e)e + FL(e)f$. As we know that $m = [e, m]$,

$$\begin{aligned} L(m) &= L([e, m]) \\ &= L(e)m + \sigma(e)L(m) - L(m)e - \sigma(m)L(e). \end{aligned} \tag{3.1}$$

Left multiplication by E and right multiplication by f in the above expression, we find that $EL(e)em = \sigma(m)FL(e)f$. Hence $L(e) = EL(e)e + FL(e)f \in Z_\sigma(\mathcal{U})$. In the similar way, we can show that $L(f) \in Z_\sigma(\mathcal{U})$.

□

Lemma 3.4 *Let* $\mathcal{U}$ *be* 2*-torsion free and* $L : \mathcal{U} \to \mathcal{U}$ *a multiplicative Lie* σ*-derivation satisfying* $EL(e)f = FL(e)e = 0$*. Then*

$$L(a) = EL(a)e + FL(a)f, \ L(m) = EL(m)f, \ L(b) = EL(b)e + FL(b)f$$

and

1. $L(am) = L(a)m + \sigma(a)L(m) - \sigma(m)L(a)$,
2. $L(mb) = L(m)b + \sigma(m)L(b) - L(b)m$

for all $a \in \mathrm{A}, \ m \in \mathrm{M}, \ b \in \mathrm{B}$.

Proof. Using $0 = [a, e]$, we get

$$\begin{aligned} 0 &= L([a, e]) \\ &= L(a)e + \sigma(a)L(e) - L(e)a - \sigma(e)L(a). \end{aligned} \tag{3.2}$$

Left multiplication by E and right multiplication by f in (3.2) yields $EL(a)f = \sigma(a)EL(e)f = 0$. Also, $FL(a)e = 0$ and this leads to $L(a) = EL(a)e + FL(a)f$. Similarly, we have $L(b) = EL(b)e + FL(b)f$.

Now multiplying by E and right multiplication by e in (3.1) yields $EL(m)e = -\sigma(m)FL(e)e = 0$. Again, multiplying by F and right multiplication by f in (3.1) yields $FL(m)f = FL(e)e\sigma(m) = 0$. Multiplying by F and right multiplication by e in (3.1) yields $2FL(m)e = 0$. Hence it follows that $FL(m)e = 0$. This proves that $L(m) = EL(m)f$.

Next, since $am = [a, m]$, we can write

$$\begin{aligned} L(am) &= L([a, m]) \\ &= L(a)m + \sigma(a)L(m) - L(m)a - \sigma(m)L(a) \\ &= L(a)m + \sigma(a)L(m) - \sigma(m)L(a) \end{aligned}$$

for all $a \in \mathrm{A}, \ m \in \mathrm{M}$. This proves (1). With similar steps, we can prove (2).

□

Lemma 3.5 $EL(\mathrm{B})e \subseteq Z_\sigma(\mathrm{A})$ *and* $FL(\mathrm{A})f \subseteq Z_\sigma(\mathrm{B})$.

Proof. Let $a \in \mathrm{A},\ b \in \mathrm{B}$. Using the fact $[a,b]=0$, we get

$$\begin{aligned} 0 &= L([a,b]) \\ &= L(a)b + \sigma(a)L(b) - L(b)a - \sigma(b)L(a). \end{aligned} \tag{3.3}$$

On multiplying left by E and right multiplication by e in (3.3) yields $EL(b)ea = \sigma(a)EL(b)e$. By definition of $Z_\sigma(\mathrm{A})$, we have $EL(b)e \in Z_\sigma(\mathrm{A})$. Again, multiplying left by F and right multiplication by f in (3.3), we have $FL(a)f \in Z_\sigma(\mathrm{B})$.

□

Now we define a map $\chi : \mathcal{U} \to \mathcal{U}$ in the following way:

$$\chi(a) = EL(a)e - \eta^{-1}(FL(a)f),\ \chi(m) = EL(m)f,\ \chi(b) = FL(b)f - \eta(EL(b)e) \tag{3.4}$$

for all $a \in \mathrm{A},\ m \in \mathrm{M},\ b \in \mathrm{B}$. Then let us define $\gamma = L - \chi$. Thus $\gamma : \mathcal{U} \to Z_\sigma(\mathcal{U})$. By Lemma 3.4, we have

$$\gamma(a) = \eta^{-1}(FL(a)f) + FL(a)f,\ \gamma(m) = 0,\ \gamma(b) = EL(b)e + \eta(EL(b)e). \tag{3.5}$$

Lemma 3.6 *Let $\mathcal{U}$ be a triangular algebra and σ an automorphism of $\mathcal{U}$. Assume that $\chi : \mathcal{U} \to \mathcal{U}$ is a map such that*

$$\chi(a) = EL(a)e - \eta^{-1}(FL(a)f),\ \chi(m) = EL(m)f,\ \chi(b) = FL(b)f - \eta(EL(b)e)$$

for all $a \in \mathrm{A},\ m \in \mathrm{M},\ b \in \mathrm{B}$. Then

1. $\chi(am) = \chi(a)m + \sigma(a)\chi(m)$,
2. $\chi(mb) = \chi(m)b + \sigma(m)\chi(b)$,
3. $\chi(a_1a_2) = \chi(a_1)a_2 + \sigma(a_1)\chi(a_2)$,
4. $\chi(b_1b_2) = \chi(b_1)b_2 + \sigma(b_1)\chi(b_2)$.

Proof. (1) The left multiplication by E and the right multiplication by f in (1) of Lemma 3.4 yields $\chi(am) = \chi(a)m + \sigma(a)\chi(m)$ for all $a \in \mathrm{A},\ m \in \mathrm{M}$.
(2) Similar to (1).
(3) For $a_1, a_2 \in \mathrm{A},\ m \in \mathrm{M}$ using (1) we find

$$\chi(a_1a_2m) = \chi(a_1a_2)m + \sigma(a_1a_2)\chi(m).$$

On the other way,

$$\begin{aligned} \chi(a_1a_2m) &= \chi(a_1)a_2m + \sigma(a_1)\chi(a_2m) \\ &= \chi(a_1)a_2m + \sigma(a_1)\chi(a_2)m + \sigma(a_1)\sigma(a_2)\chi(m). \end{aligned}$$

From the above two expressions we have

$$\{\chi(a_1a_2) - \chi(a_1)a_2 - \sigma(a_1)\chi(a_2)\} \cdot \mathrm{M} = \{0\},$$
$$E\{\chi(a_1a_2) - \chi(a_1)a_2 - \sigma(a_1)\chi(a_2)\}e \cdot \mathrm{M} = \{0\}.$$

Since $E\mathcal{U}e \subseteq \mathrm{A}$ and M is faithful as a left A-module, we find that

$$\chi(a_1a_2) = \chi(a_1)a_2 + \sigma(a_1)\chi(a_2)$$

for all $a_1, a_2 \in \mathrm{A},\ m \in \mathrm{M}$.
(4) Similar to (3).

$\square$

Lemma 3.7 *For any $a \in \mathrm{A}, m \in \mathrm{M}$ and $b \in \mathrm{B}$, we have*

1. $\chi(exf) = E\chi(x)f$,
2. $\chi(a+m) - \chi(a) - \chi(m) \in Z(\mathcal{U})$,
3. $\chi(m+b) - \chi(m) - \chi(b) \in Z(\mathcal{U})$.

Proof. (1) For any $x \in \mathcal{U}$, we have

$$\begin{aligned}\chi(exf) &= L([x,f]) \\ &= L(x)f + \sigma(x)L(f) - L(f)x - \sigma(f)L(x).\end{aligned}$$

Since $\chi(exf) \in \mathrm{M}$, it follows that $\chi(exf) = EL(x)f = E\chi(x)f$ for all $x \in \mathcal{U}$.
(2) For any $a \in \mathrm{A}$ and $m, m_1 \in \mathrm{M}$, using $[a+m, m_1] = [a, m_1]$.

$$\begin{aligned}\chi(am_1) &= \chi([a+m, m_1]) \\ &= L([a+m, m_1]) \\ &= L(a+m)m_1 + \sigma(a+m)L(m_1) - L(m_1)(a+m) - \sigma(m_1)L(a+m) \\ &= \chi(a+m)m_1 + \sigma(a+m)\chi(m_1) - \chi(m_1)(a+m) - \sigma(m_1)\chi(a+m).\end{aligned} \tag{3.6}$$

On the other way, we have

$$\begin{aligned}\chi(am_1) &= \chi([a, m_1]) \\ &= L([a, m_1]) \\ &= L(a)m_1 + \sigma(a)L(m_1) - L(m_1)a - \sigma(m_1)L(a) \\ &= \chi(a)m_1 + \sigma(a)\chi(m_1) - \chi(m_1)a - \sigma(m_1)\chi(a).\end{aligned} \tag{3.7}$$

Now from (3.6) and (3.7), we find that

$$\{\chi(a+m) - \chi(a)\}m_1 = \sigma(m_1)\{\chi(a+m) - \chi(a)\}.$$

This implies that $\chi(a+m) - \chi(a) - E(\chi(a+m) - \chi(a))f \in Z_\sigma(\mathcal{U})$. Applying Lemma 3.7(1), we arrive at $\chi(a+m) - \chi(a) - \chi(m) \in Z_\sigma(\mathcal{U})$ for all $a \in \mathrm{A},\ m \in \mathrm{M}$. Similarly, we can show that $\chi(b+m) - \chi(b) - \chi(m) \in Z_\sigma(\mathcal{U})$ for all $b \in \mathrm{B},\ m \in \mathrm{M}$.

$\square$

Lemma 3.8 *L is additive on A, M and B.*

Proof. For any $m_1, m_2 \in \mathrm{M}$,

$$\begin{aligned}\chi(m_1+m_2) &= \chi([e+m_1, f+m_2]) \\ &= L([e+m_1, f+m_2]) \\ &= L(e+m_1)(f+m_2)+\sigma(e+m_1)L(f+m_2) \\ &\quad -L(f+m_2)(e+m_1)-\sigma(f+m_2)L(e+m_1) \\ &= L([m_1,f])+L([e,m_2]) \\ &= \chi([m_1,f])+\chi([e,m_2]) \\ &= \chi(m_1)+\chi(m_2).\end{aligned}$$

Now we check the additivity on A as follows:

$$\begin{aligned}\chi(a_1m+a_2m) &= \chi(a_1m)+\chi(a_2m) \\ &= \chi(a_1)m+\sigma(a_1)\chi(m)+\chi(a_2)m+\sigma(a_2)\chi(m). \quad (3.8)\end{aligned}$$

On the other hand,

$$\begin{aligned}\chi(a_1m+a_2m) &= \chi((a_1+a_2)m) \\ &= \chi(a_1+a_2)m+\sigma(a_1+a_2)\chi(m). \quad (3.9)\end{aligned}$$

Now from (3.8) and (3.9), we find that

$$\{\chi(a_1+a_2)-\chi(a_1)-\chi(a_2)\}\cdot \mathrm{M} = \{0\}.$$

Since M is faithful as left A-module. Therefore, $\chi(a_1+a_2) = \chi(a_1)+\chi(a_2)$ for all $a_1, a_2 \in \mathrm{A}$. Similarly, we can check the additivity on B i.e., $\chi(b_1+b_2) = \chi(b_1)+\chi(b_2)$ for all $b_1, b_2 \in \mathrm{B}$.

□

Lemma 3.9 *For any* $a \in \mathrm{A}, m \in \mathrm{M}$ *and* $b \in \mathrm{B}$, *we have* $\chi(a+m+b) - \chi(a)-\chi(m)-\chi(b) \in Z_\sigma(\mathcal{U})$.

Proof. For any $a \in \mathrm{A}, m \in \mathrm{M}$ and $b \in \mathrm{B}$, we have

$$\begin{aligned}\chi([a+m+b, m_1]) &= L([a+m+b, m_1]) \\ &= L(a+m+b)m_1+\sigma(a+m+b)L(m_1) \\ &\quad -L(m_1)(a+m+b)-\sigma(m_1)L(a+m+b) \\ &= \chi(a+m+b)m_1+\sigma(a+m+b)\chi(m_1) \\ &\quad -\chi(m_1)(a+m+b)-\sigma(m_1)\chi(a+m+b). \quad (3.10)\end{aligned}$$

On the other hand,

$$\begin{aligned}\chi([a+m+b, m_1]) &= \chi([a,m_1])+\chi([b,m_1]) \\ &= L([a,m_1])+L([b,m_1]) \\ &= L(a)m_1+\sigma(a)L(m_1)-L(m_1)a-\sigma(m_1)L(a) \\ &\quad +L(b)m_1+\sigma(b)L(m_1)-L(m_1)b-\sigma(m_1)L(b) \\ &= \chi(a)m_1+\sigma(a)\chi(m_1)-\chi(m_1)a-\sigma(m_1)\chi(a) \\ &\quad +\chi(b)m_1+\sigma(b)\chi(m_1)-\chi(m_1)b-\sigma(m_1)\chi(b). \\ & \qquad (3.11)\end{aligned}$$

Now from (3.10) and (3.11), we find that

$$\{\chi(a+m+b)-\chi(a)-\chi(m)-\chi(b)\}m_1=\sigma(m_1)\{\chi(a+m+b)-\chi(a)-\chi(m)-\chi(b)\}.$$

Hence, $\chi(a+m+b)-\chi(a)-\chi(b)-E(\chi(a+m)-\chi(a)-\chi(b))f \in Z_\sigma(\mathcal{U})$. Applying (1) of Lemma 3.7, we have $\chi(a+m+b)-\chi(a)-\chi(m)-\chi(b) \in Z_\sigma(\mathcal{U})$ for all $a \in \mathrm{A},\ b \in \mathrm{B},\ m \in \mathrm{M}$.

□

In view of Lemma 3.9, we define a map $\phi : \mathcal{U} \to Z(\mathcal{U})$ by $\phi(x) = \chi(x) - \chi(exe) - \chi(exf) - \chi(fxf)$ for all $x \in \mathcal{U}$. Clearly, $\phi(x) \in Z_\sigma(\mathcal{U})$ and $\phi(\mathrm{A}) = \phi(\mathrm{M}) = \phi(\mathrm{B}) = 0$. Now, define another map $\delta : \mathcal{U} \to \mathcal{U}$ by $\delta(x) = \chi(x)-\phi(x)$ for all $x \in \mathcal{U}$. It is easy to observe that

$$\delta(a+m+b) = \delta(a) + \delta(m) + \delta(b)$$

for all $a \in \mathrm{A}, m \in \mathrm{M}$ and $b \in \mathrm{B}$. Then

$$L(x) = \chi(x) + \gamma(x) = \delta(x) + \phi(x) + \gamma(x) = \chi(x) + \tau(x)$$

for all $x \in \mathcal{U}$, where $\gamma + \phi = \tau$.

Proof of Theorem 3.1. For any $x, y \in \mathcal{U}$, suppose that $x = a_1 + m_1 + b_1$ and $x = a_2 + m_2 + b_2$ where $a_1, a_2 \in \mathrm{A},\ b_1, b_2 \in \mathrm{B}$ and $m_1, m_2 \in \mathrm{M}$. We have

$$\begin{aligned}
\delta(x+y) &= \delta(a_1 + m_1 + b_1 + a_2 + m_2 + b_2)\\
&= \delta(a_1 + a_2) + \delta(m_1 + m_2) + \delta(b_1 + b_2)\\
&= \delta(a_1) + \delta(m_1) + \delta(b_1) + \delta(a_2) + \delta(m_2) + \delta(b_2)\\
&= \delta(x) + \delta(y).
\end{aligned}$$

Also, using Lemma 3.6 and 3.9, we find

$$\begin{aligned}
\delta(xy) &= \delta((a_1 + m_1 + b_1)(a_2 + m_2 + b_2))\\
&= \delta(a_1a_2 + a_1m_2 + m_1b_2 + b_1b_2)\\
&= \delta(a_1a_2) + \delta(a_1m_2) + \delta(m_1b_2) + \delta(b_1b_2)\\
&= \delta(a_1)a_2 + \sigma(a_1)\delta(a_2) + \delta(a_1)m_2 + \sigma(a_1)\delta(m_2)\\
&\quad +\delta(m_1)b_2 + \sigma(m_1)\delta(b_2) + \delta(b_1)b_2 + \sigma(b_1)\delta(b_2)\\
&= \delta(a_1 + m_1 + b_1)(a_2 + m_2 + b_2) + \sigma(a_1 + m_1 + b_1)\delta(a_2 + m_2 + b_2)\\
&= \delta(x)y + \sigma(x)\delta(y).
\end{aligned}$$

Hence, $\delta : \mathcal{U} \to \mathcal{U}$ is an additive σ-derivation. Now from (3.5), it is easy to observe that $\tau(x) \in Z_\sigma(\mathcal{U})$ and $\tau([x, y]) = 0$ for all $x, y \in \mathcal{U}$. According to the above conclusions, L has desired form.

□

The following example shows that the restrictions in Theorem 3.1 are necessary and can not be removed.

Example 3.1 Let $\mathcal{U} = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B})$ be a triangular algebra where $\mathrm{A} = \mathrm{B} = \left\{ \begin{bmatrix} p & a \\ 0 & p \end{bmatrix} \middle| \, p, a \in \mathcal{R} \right\}$ and $\mathrm{M} = \mathcal{T}_2(\mathcal{R})$, the ring of 2×2 upper triangular matrices over $\mathcal{R}$. Define a map $\sigma : \mathcal{U} \to \mathcal{U}$ such that

$$\sigma\left(\begin{bmatrix} p & a & u & v \\ 0 & p & 0 & w \\ 0 & 0 & q & b \\ 0 & 0 & 0 & q \end{bmatrix}\right) = \begin{bmatrix} p & -a & -u & v \\ 0 & p & 0 & -w \\ 0 & 0 & q & -b \\ 0 & 0 & 0 & q \end{bmatrix}$$

is an automorphism. Define another map $L : \mathcal{U} \to \mathcal{U}$ such that

$$L\left(\begin{bmatrix} p & a & u & v \\ 0 & p & 0 & w \\ 0 & 0 & q & b \\ 0 & 0 & 0 & q \end{bmatrix}\right) = \begin{bmatrix} 0 & b & w & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then L is a Lie σ-derivation. Note that $\pi_{\mathrm{A}}(Z_\sigma(\mathcal{U})) = \pi_{\mathrm{B}}(Z_\sigma(\mathcal{U})) = \{0\}$. Since $\pi_{\mathrm{A}}(L(\mathrm{B})) \notin \pi_{\mathrm{A}}(Z_\sigma(\mathcal{U}))$ and hence by Theorem 3.1, L has no standard form.

4 Applications

In this section we give some applications of Theorem 3.1 to certain special classes of triangular algebras such as nest algebras and block upper triangular matrix algebras and obtain the following results:

Theorem 4.1 *Let X be an infinite dimensional Banach space over the real or complex field F and $\mathcal{N}$ be a nest on X which contains a nontrivial element complemented in X and let $\mathcal{T}(\mathcal{N})$ be a nest algebra. Let $L : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ be a multiplicative Lie σ-derivation. Then there exist an operator $T \in \mathcal{T}(\mathcal{N})$ and $\tau : \mathcal{T}(\mathcal{N}) \to FI$ is a functional that vanishes on all commutators of $\mathcal{T}(\mathcal{N})$ such that any multiplicative Lie σ-derivation has standard form.*

Proof. Let $N \in \mathcal{N}$ be the complemented element. Then $\mathcal{T}(\mathcal{N}) = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B})$ is a triangular algebra. It is easily seen that both conditions $\pi_{\mathrm{A}}(Z_\sigma(\mathcal{T}(\mathcal{N}))) = Z_\sigma(\mathrm{A})$ and $\pi_{\mathrm{B}}(Z_\sigma(\mathcal{T}(\mathcal{N}))) = Z_\sigma(\mathrm{B})$ are satisfied. Now the result follows from Theorem 3.1.

□

Theorem 4.2 *Let $\mathcal{N}$ be a nest of a Hilbert space $\mathcal{H}$ dimension greater than 2 and let $\mathcal{T}(\mathcal{N})$ be a nest algebra which contains nontrivial elements. Each multiplicative Lie σ-derivation $L : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ is the sum of an additive σ-derivation $\delta : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ and a functional $\tau : \mathcal{T}(\mathcal{N}) \to FI$ that vanishes on all commutators of $\mathcal{T}(\mathcal{N})$.*

Proof. Since $\mathcal{T}(\mathcal{N})$ contains only nontrivial elements so that associated nest algebra is a triangular algebra which satisfies the conditions of Theorem 3.1. This follows the result.

$\square$

If Hilbert space $\mathcal{H}$ is finite dimensional, then nest algebras are block upper triangular matrices algebras [7].

Theorem 4.3 *Let* $\mathrm{B}_n(\mathcal{R})$ *be a block upper triangular matrix algebra over a commutative ring* $\mathcal{R}$. *Then every multiplicative Lie* σ*-derivation* $L : \mathrm{B}_n(\mathcal{R}) \to \mathrm{B}_n(\mathcal{R})$ *is the sum of an additive* σ*-derivation* $\delta : \mathrm{B}_n(\mathcal{R}) \to \mathrm{B}_n(\mathcal{R})$ *and a functional* $\tau : \mathrm{B}_n(\mathcal{R}) \to \mathcal{R}I$ *that vanishes on all commutators of* $\mathrm{B}_n(\mathcal{R})$.

Acknowledgements The authors would like to thank the anonymous referees for careful reading and the helpful comments improving this paper. This research is supported by Dr. D. S. Kothari Postdoctoral Fellowship under University Grants Commission (Grant No. F.4-2/2006 (BSR)/MA/18-19/0014), awarded to the second author.

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Received: 05.IV.2021 / Revised: 28.III.2022 / Accepted: 29.III.2022

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