An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXVIII, 2022, f. 2
https://doi.org/10.47743/anstim.2022.00012

The Hadamard-type k-step Fibonacci sequences

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Abstract In this work, we define the Hadamard-type product of two polynomials by the aid of the Hadamard product of two polynomials. Then we obtain the Hadamard-type k-step Fibonacci sequence by using Hadamard-type product of characteristic polynomials of the Fibonacci sequence and the k-step Fibonacci sequence. Also, we derive relationships between the Hadamard-type k-step Fibonacci sequences and the generating matrices for these sequences. Finally, we give some properties of these sequences such as the Binet formula, the combinatorial representations, the generating function, the exponential representation.

Keywords the Hadamard product · the k-step Fibonacci sequence · matrix · representation

Mathematics Subject Classification (2010) 11K31 · 11C20 · 15A15

1 Introduction

The well-known Fibonacci sequence $\{f_n\}$ is defined recursively by the equation

$$f_{n+1} = f_n + f_{n-1}$$

for $n \geq 1$, where $f_0 = 0$, $f_1 = 1$.

The k-step Fibonacci sequence $\left\{F_n^k\right\}$ $(k \geq 2)$ is defined by initial values

$F_0^k = \cdots = F_{k-2}^k = 0$, $F_{k-1}^k = 1$ and recurrence relation

$$F_{n+k}^k = F_{n+k-1}^k + F_{n+k-2}^k + \cdots + F_n^k \qquad \text{for } n \geq 0.$$

Note that the characteristic polynomials of Fibonacci sequence and the k-step Fibonacci sequence are $f(x) = x^2 - x - 1$ and $F^k(x) = x^k - x^{k-1} - \cdots - x - 1$, respectively.

The Hadamard product of two polynomials

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$$

in $\mathbb{R}[x]$ is defined to be

$$(p * q)(x) = a_k b_k x^k + a_{k-1}b_{k-1}x^{k-1} + \cdots + a_1 b_1 x + a_0 b_0,$$

where $k = \min(n, m)$ [8].

Consider the k-step homogeneous linear recurrence sequence $\{a_n\}$,

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [10], Kalman derived a number of closed-form formulas for the sequence $\{a_n\}$ by matrix method as follows:

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

where

$$A = [a_{i,j}]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix}.$$

In the literature, many of the numbers obtained by using homogeneous linear recurrence relations and their miscellaneous properties have been studied by some authors; see, for example [1,4–7,9,13,14]. In this work, we define the Hadamard-type product of two polynomials and the Hadamard-type k-step Fibonacci sequence. Firstly, we obtain relationships between the Hadamard-type k-step Fibonacci sequences and the generating matrices for these sequences. Then using the roots of characteristic polynomial of the Hadamard-type k-step Fibonacci sequence, we produce the Binet formula for the Hadamard-type k-step Fibonacci numbers. Also, we give the combinatorial representations, the exponential representation and the sums of these sequences.

2 The Hadamard-type *k*-step Fibonacci sequences

Let mth and nth degrees two polynomials be given by

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0,$$
$$g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0.$$

We define the Hadamard-type product of polynomials f and g as follows:

$$f(x) * g(x) = \sum_{i=0}^{\infty} (a_i * b_i)\, x^i, \text{ where } a_i * b_i = \begin{cases} a_i b_i & \text{if} \quad a_i b_i \neq 0 \\ a_i + b_i & \text{if} \quad a_i b_i = 0 \end{cases}.$$

Now we consider a new k-step sequence which is defined by using Hadamard-type product of characteristic polynomials of Fibonacci sequence and the k-step Fibonacci sequence and is called the Hadamard-type k-step Fibonacci sequence. This sequence is defined by integer constants $HF_k(0) = \cdots = HF_k(k-2) = 0$, $HF_k(k-1) = 1$ and the recurrence relation

$$HF_k(n+k) = HF_k(n+k-1) + \cdots + HF_k(n+2) - HF_k(n+1) - HF_k(n) \tag{2.1}$$

for $n \geq 0$, where $k \geq 3$.
By relation (2.1), we obtain the generating matrix of the sequence $\{HF_k(n)\}$ as follows:

$$FM_k^h = f_{ij} = \begin{bmatrix} 1 & \cdots & 1 & 1 & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}_{k \times k}.$$

The companion matrix FM_k^h is called Hadamard-type k-Fibonacci matrix. Let $HF_k(n)$ be denoted by HF_k^n. By induction on n, we get
(i). For $k = 3$,

$$\left(FM_3^h\right)^n = \begin{bmatrix} HF_3^{n+2} & -HF_3^{n+1} - HF_3^n & -HF_3^{n+1} \\ HF_3^{n+1} & -HF_3^n - HF_3^{n-1} & -HF_3^n \\ HF_3^n & -HF_3^{n-1} - HF_3^{n-2} & -HF_3^{n-1} \end{bmatrix},$$

(ii). For $k = 4$,

$$\left(FM_4^h\right)^n = \begin{bmatrix} HF_4^{n+3} & HF_4^{n+4} - HF_4^{n+3} & -HF_4^{n+2} - HF_4^{n+1} & -HF_4^{n+2} \\ HF_4^{n+2} & HF_4^{n+3} - HF_4^{n+2} & -HF_4^{n+1} - HF_4^n & -HF_4^{n+1} \\ HF_4^{n+1} & HF_4^{n+2} - HF_4^{n+1} & -HF_4^n - HF_4^{n-1} & -HF_4^n \\ HF_4^n & HF_4^{n+1} - HF_4^n & -HF_4^{n-1} - HF_4^{n-2} & -HF_4^{n-1} \end{bmatrix},$$

(iii). For $k \geq 5$,

$$\left(FM_k^h\right)^n = \begin{bmatrix} HF_k^{n+k-1} & HF_k^{n+k} - HF_k^{n+k-1} & & HF_k^{n+k-2} - HF_k^{n+k-3} - HF_k^{n+k-4} \\ HF_k^{n+k-2} & HF_k^{n+k-1} - HF_k^{n+k-2} & & HF_k^{n+k-3} - HF_k^{n+k-4} - HF_k^{n+k-5} \\ \vdots & \vdots & & \vdots \\ HF_k^{n+1} & HF_k^{n+2} - HF_k^{n+1} & FM_k^{h,*} & HF_k^n - HF_k^{n-1} - HF_k^{n-2} \\ HF_k^n & HF_k^{n+1} - HF_k^n & & HF_k^{n-1} - HF_k^{n-2} - HF_k^{n-3} \end{bmatrix} \tag{2.2}$$

$$\begin{bmatrix} -HF_k^{n+k-2} - HF_k^{n+k-3} & -HF_k^{n+k-2} \\ -HF_k^{n+k-3} - HF_k^{n+k-4} & -HF_k^{n+k-3} \\ \vdots & \vdots \\ -HF_k^n - HF_k^{n-1} & -HF_k^n \\ -HF_k^{n-1} - HF_k^{n-2} & -HF_k^{n-1} \end{bmatrix}$$

where $FM_k^{h,*}$ is a $(k) \times (k-5)$ matrix as follows:

$$\begin{bmatrix} HF_k^{n+k+1}-HF_k^{n+k}-HF_k^{n+k-1} & HF_k^{n+k+2}-HF_k^{n+k+1}-HF_k^{n+k}-HF_k^{n+k-1} & \cdots \\ HF_k^{n+k}-HF_k^{n+k-1}-HF_k^{n+k-2} & HF_k^{n+k+1}-HF_k^{n+k}-HF_k^{n+k-1}-HF_k^{n+k-2} & \cdots \\ \vdots & \vdots & \vdots \\ HF_k^{n+3} - HF_k^{n+2} - HF_k^{n+1} & HF_k^{n+4} - HF_k^{n+3} - HF_k^{n+2} - HF_k^{n+1} & \cdots \\ HF_k^{n+2} - HF_k^{n+1} - HF_k^n & HF_k^{n+3} - HF_k^{n+2} - HF_k^{n+1} - HF_k^n & \cdots \end{bmatrix}$$

$$\begin{bmatrix} HF_k^{n+2k-5}-HF_k^{n+2k-6}-\cdots-HF_k^{n+k}-HF_k^{n+k-1} \\ HF_k^{n+2k-6}-HF_k^{n+2k-7}-\cdots-HF_k^{n+k-1}-HF_k^{n+k-2} \\ \vdots \\ HF_k^{n+k-3}-HF_k^{n+k-4}-\cdots-HF_k^{n+2}-HF_k^{n+1} \\ HF_k^{n+k-4}-HF_k^{n+k-5}-\cdots-HF_k^{n+1}-HF_k^n \end{bmatrix}$$

for $n \geq 3$. Also, It is easy to see that $\det FM_k^h = (-1)^k$.

We now give the Binet formula for the Hadamard-type k-Fibonacci sequences.

Lemma 2.1 *The characteristic equation of the Hadamard-type k-Fibonacci sequences $x^k - x^{k-1} - \cdots - x^2 + x + 1 = 0$ does not have multiple roots.*

Proof. Let $f(x) = x^k - x^{k-1} - \cdots - x^2 + x + 1$. For $k = 3$, we reach the equation $x^3 - x^2 + x + 1$. Then, the roots of this equation are obtained as $x_1 = 0.771\,84 - 1.115\,1i$, $x_2 = 0.771\,84 + 1.115\,1i$ and $x_3 = -0.543\,69$. Similarly, we reach the equation $x^4 - x^3 - x^2 + x + 1$ for $k = 4$. Then, we obtain the roots of this equation as follows:

$x_1 = 1.192\,4$–$0.547\,88i$, x_2=$1.192\,4$+$0.547\,88i$, x_3=$-0.692\,44-0.318\,15i$ and $x_4 = -0.692\,44 + 0.318\,15i$.

Thus, it is easily seen that the equation $f(x) = 0$ does not have multiple roots for $k = 3$ and $k = 4$. Now, we consider the proof for the case $k \geq 5$. Suppose that $g(x) = (x-1) f(x) = x^{k+1} - 2x^k + 2x^2 - 1$. In this case, we obtain

$$x^k = \frac{1 - 2x^2}{x - 2}. \tag{2.3}$$

On the other hand, we may write $g^{'}(x) = (k+1)x^k - 2kx^{k-1} + 4x$ and thus, we get

$$x^k = \frac{-4x^2}{(k+1)x - 2k}. \tag{2.4}$$

From (2.3) and (2.4), we reach the equation

$$(2k-2)x^3 + (8-4k)x^2 - (k+1)x + 2k. \tag{2.5}$$

By equation (2.5), we get $k = 1 + \frac{4x^2-2x+2}{-2x^3+4x^2+x-2}$. Using appropriate softwares such as mathematica wolfram 10.0 [15], we obtain that there is no solution for $k \geq 5$. Since k are integers such that $k \geq 5$, it is a contradiction. Therefore, the equation $f(x) = 0$ does not have multiple roots.

□

If x_1, x_2, ..., x_k are eigenvalues of the matrix FM_k^h, then by Lemma 2.1, it is known that x_1, x_2, ..., x_k are distinct. Let V_k be $k \times k$ Vandermonde matrices as follows:

$$V_k = \begin{bmatrix} (x_1)^{k-1} & (x_2)^{k-1} & \cdots & (x_k)^{k-1} \\ (x_1)^{k-2} & (x_2)^{k-2} & \cdots & (x_k)^{k-2} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & & x_k \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Let

$$U_k(i,j) = \begin{bmatrix} x_1^{n+k-i} \\ x_2^{n+k-i} \\ \vdots \\ x_k^{n+k-i} \end{bmatrix}$$

and supoose that $V_k(i,j)$ is a $k \times k$ matrix obtained from V_k by replacing the jth column of V_k by $U_k(i,j)$.

Theorem 2.2 *Let* $\left(FM_k^h\right)^n = \left[f_{i,j}^{h,k,n}\right]$, *then*

$$f_{i,j}^{h,k,n} = \frac{\det V_k(i,j)}{\det V_k},$$

for $n \geq 3$ *and* $k \geq 3$.

Proof. Since the eigenvalues of the matrix FM_k^h are distinct, the matrixFM_k^h is diagonalizable. Thus, we easily see that $FM_k^h V_k = V_k P_k$, where $P_k = (x_1, x_2, \ldots, x_k)$. Since V_k is invertible, we may write $(V_k)^{-1} FM_k^h V_k = P_k$.

Then, the matrix FM_k^h is similar to P_k and $\left(FM_k^h\right)^n V_k = V_k \left(P_k\right)^n$ for $n \geq 3$ and $k \geq 3$. Then we can write the following linear system of equations:

$$\begin{cases} f_{i,1}^{h,k,n} x_1^{k-1} + f_{i,2}^{h,k,n} x_1^{k-2} + \cdots + f_{i,k}^{h,k,n} = x_1^{n+k-i} \\ f_{i,1}^{h,k,n} x_2^{k-1} + f_{i,2}^{h,k,n} x_2^{k-2} + \cdots + f_{i,k}^{h,k,n} = x_2^{n+k-i} \\ \vdots \\ f_{i,1}^{h,k,n} x_k^{k-1} + f_{i,2}^{h,k,n} x_k^{k-2} + \cdots + f_{i,k}^{h,k,n} = x_k^{n+k-i} \end{cases}$$

Therefore, for each $i, j = 1, 2, \ldots, k$, we obtain

$$f_{i,j}^{h,k,n} = \frac{\det V_k \left(i, j\right)}{\det V_k}.$$

□

Theorem 2.2 gives immediately:

Corollary 2.3 *Let $HF_k \left(n\right)$ be the nth the Hadamard-type k-step Fibonacci number, then*

$$HF_k \left(n\right) = \frac{\det V_k \left(k, 1\right)}{\det V_k} = -\frac{\det V_k \left(k-1, k\right)}{\det V_k}$$

for $k \geq 3$.

Now we concentrate on finding the permanental representations of the Hadamard-type k-step Fibonacci numbers.

Definition 2.4 *A $u \times v$ real matrix $M = [m_{i,j}]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.*

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1) \times (v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k} x_j + m_{j,k} x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [2], Brualdi and Gibson obtained that $per\left(M\right) = per\left(N\right)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Let $\alpha \geq k$ ($k \geq 3$) be a integer and suppose that $L_{\alpha,k} = \left[l_{i,j}^{\alpha,k}\right]$ is the $\alpha \times \alpha$ super-diagonal matrix, defined by

$$l_{i,j}^{\alpha,k} = \begin{cases} 1 & \begin{array}{c} \text{if } i = u \text{ and } \ j = u \text{ for } 1 \leq u \leq \alpha, \\ i = u \text{ and } j = u+1 \text{ for } 1 \leq u \leq \alpha - 1, \\ \vdots \\ i = u \text{ and } j = u+k-3 \text{ for } 1 \leq u \leq \alpha - k + 3 \end{array} \\ -1 & \begin{array}{c} \text{and} \\ i = u \text{ and } j = u-1 \text{ for } 2 \leq u \leq \alpha, \\ \text{if } i = u \text{ and } j = u+k-2 \text{ for } 1 \leq u \leq \alpha - k + 2 \end{array} \\ 0 & \begin{array}{c} \text{and} \\ i = u \text{ and } j = u+k-1 \text{ for } 1 \leq u \leq \alpha - k + 1, \\ \text{otherwise} \end{array} \end{cases}$$

that is,

$$L_{\alpha,k} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccccccc} & & & & & & (k)\,\text{th} & & & & & \\ & & & & & & \downarrow & & & & & \end{array} \\ \begin{bmatrix} 1 & 1 & \cdots & 1 & -1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 1 & -1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & -1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & -1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \ddots & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

Then we have the following Theorem.

Theorem 2.5 *For $\alpha \geq k$ and $k \geq 3$,*

$$perL_{\alpha,k} = HF_k(\alpha + k - 1).$$

Proof. The assertion may be proved by induction on α. Let the equation be hold for $\alpha \geqslant k$, then we show that the equation holds for $\alpha+1$. If we expand the $perL_{\alpha,k}$ by the Laplace expansion of permanent according to the first row, then we obtain $perL_{\alpha,k}$

$$perL_{\alpha+1,k} = perL_{\alpha,k} + perL_{\alpha-1,k} + \cdots + perL_{\alpha-k+3,k} - perL_{\alpha-k+2,k} - perL_{\alpha-k+1,k}.$$

Since $perL_{\alpha,k}$=$HF_k(\alpha+k-1)$, $perL_{\alpha-1,k}$=$HF_k(\alpha+k-2)$,..., $perL_{\alpha-k+3,k}$=$HF_k(\alpha+2)$, $perL_{\alpha-k+2,k} = HF_k(\alpha+1)$ and $perL_{\alpha-k+1,k} = HF_k(\alpha)$, it is easy to see that $perL_{\alpha+1,k} = HF_k(\alpha+k)$. Thus the result of the theorem holds.

□

Let $\alpha > k$ and let $Y_{\alpha,k} = \left[y_{i,j}^{\alpha,k}\right]$ be the $\alpha \times \alpha$ matrix, defined by

$$y_{i,j}^{\alpha,k} = \begin{cases} \text{if } i = u \text{ and } \ j = u \text{ for } 1 \leq u \leq \alpha, \\ i = u \text{ and } j = u+1 \text{ for } 1 \leq u \leq \alpha - k + 1, \\ i = u \text{ and } j = u+2 \text{ for } 1 \leq u \leq \alpha - k + 1, \\ \vdots \\ 1 \qquad i = u \text{ and } j = u + k - 3 \text{ for } 1 \leq u \leq \alpha - k + 1, \\ i = u \text{ and } j = u - 1 \text{ for } 2 \leq u \leq \alpha - k + 1, \\ \text{and} \\ i = \alpha - k + 1 \text{ and } j = \alpha, \\ \text{if } i = u \text{ and } j = u + k - 2 \text{ for } 1 \leq u \leq \alpha - k \\ -1 \qquad \text{and} \\ i = u \text{ and } j = u + k - 1 \text{ for } 1 \leq u \leq \alpha - k, \\ 0 \qquad \text{otherwise.} \end{cases}.$$

Now we define the $\alpha \times \alpha$ matrix $D_{\alpha,k} = \left[d_{i,j}^{\alpha,k}\right]$ as follows:

$$D_{\alpha,k} = \begin{bmatrix} & & \overset{(\alpha-k+1)\text{th}}{\downarrow} & & \\ 1 & \cdots & 1 & 0 & \cdots \ 0 \\ 1 & & & & \\ 0 & & Y_{\alpha-1,k} & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}.$$

Then we can give the following Theorem by using the permanental representations.

Theorem 2.6 (i). *For $\alpha > k$,*

$$perY_{\alpha,k} = HF_k(\alpha).$$

(ii). *For $\alpha > k+1$,*

$$perD_{\alpha,k} = \sum_{i=1}^{\alpha-1} HF_k(i).$$

Proof. (i). To prove this Theorem, suppose that the equation is hold for $\alpha > k$, then we show that the equation holds for $k+1$. If we expand the $perY_{\alpha,k}$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\begin{aligned} perY_{\alpha+1,k} &= perY_{\alpha,k} + perY_{\alpha-1,k} + \cdots + perY_{\alpha-k+3,k} - perY_{\alpha-k+2,k} - perY_{\alpha-k+1,k} \\ &= HF_k(\alpha) + HF_k(\alpha-1) + \cdots + HF_k(\alpha-k+3) - HF_k(\alpha-k+2) - HF_k(\alpha-k+1). \end{aligned}$$

So, we have the conclusion.

(ii) . If we extend the $D_{\alpha,k}$ with respect to the first row, we write

$$perD_{\alpha,k} = perD_{\alpha-1,k} + perY_{\alpha-1,k}.$$

By the results of Theorem 2.5 and Theorem 2.6. (i) and induction on α, the conclusion is easily seen.

□

Let the notation $M \circ K$ denotes the Hadamard product of M and K. A matrix M is called convertible if there is an $u \times u$ (1, -1)-matrix K such that per $M = \det(M \circ K)$.

Let $\alpha > k+1$ and let R be the $\alpha \times \alpha$ matrix, defined by

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Then the following useful results are obtained.

Corollary 2.7 *For $\alpha > k+1$,*

$$\det\left(L_{\alpha,k} \circ R\right) = HF_k\left(\alpha + k - 1\right),$$
$$\det\left(Y_{\alpha,k} \circ R\right) = HF_k\left(\alpha\right)$$

and

$$\det\left(D_{\alpha,k} \circ R\right) = \sum_{i=1}^{\alpha-1} HF_k\left(i\right).$$

Let $K\left(k_1, k_2, \ldots, k_v\right)$ be a $v \times v$ companion matrix as follows:

$$K\left(k_1, k_2, \ldots, k_v\right) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For more information about the companion matrix, see [11,12]

Theorem 2.8 *(Chen and Louck [3]).The (i,j) entry $k_{i,j}^{(u)}\left(k_1, k_2, \ldots, k_v\right)$ in the matrix $K^u\left(k_1, k_2, \ldots, k_v\right)$ is given by the following formula:*

$$k_{i,j}^{(u)}\left(k_1, k_2, \ldots, k_v\right) = \sum_{(t_1,t_2,\ldots,t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v} \quad (2.6)$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = u - i + j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v} = \frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ is a multinomial coefficient, and the coefficients in (2.6) are defined to be 1 if $u = i - j$.

Now we concentrate on finding combinatorial representation for the Hadamard-type k-step Fibonacci numbers.

Corollary 2.9 *For $k \geq 3$, let $HF_k(\alpha)$ be the αth the Hadamard-type k-step Fibonacci number. Then*

i.

$$HF_k(\alpha) = \sum_{(t_1,t_2\ldots,t_k)} \binom{t_1+\cdots+t_k}{t_1,\ldots,t_k} (-1)^{t_{k-1}+t_k}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (k)\, t_k = \alpha - k + 1$.

ii.

$$HF_k(\alpha) = -\sum_{(t_1,t_2\ldots,t_k)} \frac{t_k}{t_1+t_2+\cdots+t_k} \times \binom{t_1+\cdots+t_k}{t_1,\ldots,t_k} (-1)^{t_{k-1}+t_k}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (k)\, t_k = \alpha + 1$.

Proof. If we take $v = k$, $i = k$ and $j = 1$, for case *i.* and $i = k-1$, $j = k$, for case *ii.* in Theorem 2.8, then the proof is immediately seen from (2.2). □

3 The generating function of the Hadamard-type *k*-step Fibonacci sequences

The generating function of the Hadamard-type k-step Fibonacci sequence is given by:

$$g_k(x) = \frac{x^{k-1}}{1 - x - x^2 - \cdots - x^{k-2} + x^{k-1} + x^k}.$$

Now an exponential representation for the Hadamard-type k-step Fibonacci numbers can be given with the following Theorem.

Theorem 3.1 *The Hadamard-type k-step Fibonacci numbers have the following exponential representation:*

$$g_k(x) = x^{k-1} \exp\left(\sum_{i=1}^{\infty} \frac{(x)^i}{i} \left(1 + x + \cdots + x^{k-3} - x^{k-2} - x^{k-1}\right)^i\right)$$

where $k \geq 3$.

Proof. Since

$$\begin{aligned}\ln g_k(x) &= \ln \frac{x^{k-1}}{1-x-x^2-\cdots-x^{k-2}+x^{k-1}+x^k} \\ &= \ln x^{k-1} - \ln\left(1-x-x^2-\cdots-x^{k-2}+x^{k-1}+x^k\right)\end{aligned}$$

and

$$\begin{aligned}-\ln\left(1-x-x^2-\cdots-x^{k-2}+x^{k-1}+x^k\right) = -[&-x\left(1+x+\cdots+x^{k-3}-x^{k-2}-x^{k-1}\right)- \\ &\frac{1}{2}x^2\left(1+x+\cdots+x^{k-3}-x^{k-2}-x^{k-1}\right)^2-\cdots- \\ &\frac{1}{n}x^n\left(1+x+\cdots+x^{k-3}-x^{k-2}-x^{k-1}\right)^n-\cdots]\end{aligned}$$

It is clear that

$$\ln\frac{g_k(x)}{x^{k-1}} = \sum_{i=1}^{\infty}\frac{(x)^i}{i}\left(1+x+\cdots+x^{k-3}-x^{k-2}-x^{k-1}\right)^i.$$

Thus we have the conclusion.

□

4 The sums of the Hadamard-type *k*-step Fibonacci sequences

Now we consider the sums of the Hadamard-type k-step Fibonacci numbers. Let the sums of the Hadamard-type k-step Fibonacci numbers from 1 to α be denoted by S_α, that is,

$$S_\alpha = \sum_{i=1}^{\alpha} HF_k(\alpha)$$

and suppose that A_k^h is the $(k+1)\times(k+1)$ matrix, such that

$$A_k^h = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & FM_k^h & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

If we use induction on α, then we obtain

$$\left(A_k^h\right)^\alpha = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ S_{\alpha+k-2} & & & \\ S_{\alpha+k-3} & & \left(FM_k^h\right)^\alpha & \\ \vdots & & & \\ S_{\alpha-1} & & & \end{bmatrix}.$$

5 Conclusions

We defined the Hadamard-type product of two polynomials and the Hadamard-type k-step Fibonacci sequence. Firstly, we obtained the generating matrices for these sequences. Also, we gave the Binet formula, the combinatorial representations, the exponential representation and the sums of these sequences.

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Received: 17.VI.2021 / Revised: 01.VII.2021 / Accepted: 06.VII.2021

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