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# Recognition by the set of orders of vanishing elements and order of $PSL(3,p)$

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**Abstract** Let $G$ be a finite group. We say that an element $g$ of $G$ is a vanishing element if there exists an irreducible complex character $\chi$ of $G$ such that $\chi(g) = 0$. Denote by $Vo(G)$ the set of order of vanishing elements of $G$, and we prove that $G \cong PSL(3,p)$ if and only if $Vo(G) = Vo(PSL(3,p))$ and $|G| = |PSL(3,p)|$, where $p$ is a prime number.

**Keywords** vanishing element · finite simple groups · vanishing prime graph

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## 1 Introduction

Let $n$ be a positive integer. By $\pi(n)$ we mean the set of prime divisors of $n$. Let $G$ be a finite group and $\pi(G)$ be the set of prime divisors of $|G|$. Denote by $\omega(G)$, the set of element orders of $G$. Given a finite set $X$ of positive integers, the prime graph $\pi(X)$ is defined as a simple undirected graph whose vertices are the primes $p$ where exists an element of $X$ divisible by $p$, and two distinct vertices $p$ and $q$ are adjacent if and only if there exists an element divisible by $p$ and $q$ in $X$. We denote the graph $\prod(\omega(G))$ by $GK(G)$ and we call it the prime graph or Gruenberg-Kegel graph of $G$. The number of connected components of $GK(G)$ is denoted by $t(G)$, and the connected components of $GK(G)$ is denoted by $\pi_1, \pi_2, ..., \pi_{t(G)}$. If $2 \in \pi(G)$, then we always suppose that $2 \in \pi_1(G)$. $|G|$ can be expressed as a product of co-prime positive integers $OC_i$, $i = 1, 2, \ldots, t(G)$, where $\pi(OC_i) = \pi_i$. These $OC_i$'s are called the order components of $G$.
Let $n$ be a positive integer and $p$ be a prime number. Then $|n|_p$ denotes the $p$-part of $n$. If $G$ is a finite group and $q$ is a prime, then we denote by $S_q(G)$ a Sylow $q$-subgroup of $G$. Let $n$ and $m$ be positive integers and $p$ be a prime. We write $p^m||n$, if $p^m \mid n$ but $p^{m+1} \nmid n$.
It is well known that the set of values $cd(G) = \{\chi(1) : \chi \in Irr(G)\}$ has

a strong influence on the group structure of $G$, where $Irr(G)$ denotes the set of irreducible complex characters of $G$. We say that an element $g$ of $G$ is a vanishing element if there exists an irreducible complex character $\chi$ of $G$ such that $\chi(g) = 0$. Denote $Van(G)$ the set $\{g \in G : \chi(g) = 0$ for some $\chi \in Irr(G)\}$, $Vo(G)$ the set $\{o(g) : g \in Van(G)\}$ consisting of the orders of the elements in $Van(G)$. The prime graph $\prod(Vo(G))$ is denoted by $\Gamma(G)$ and is called the vanishing prime graph of $G$. Note that for every finite group $G$, $\Gamma(G)$ is a subgraph of $GK(G)$. There is a strong relation between the structure of a group $G$ and the set $Vo(G)$. But not all finite simple groups are characterizable by the set of orders of their vanishing elements. For example, $Vo(PSL(3,5)) = Vo(Aut(PSL(3,5)))$, but $PSL(3,5) \ncong Aut(PSL(3,5))$. In [12], it is shown that if $G$ is a finite group such that $Vo(G) = Vo(A_5)$, then $G \cong A_5$. In [13], it is proved that if $G$ is a finite group such that $Vo(G) = Vo(Sz(2^{2m+1}))$, where $m \geq 1$, then $G \cong Sz(2^{2m+1})$.

The following conjecture is one of the important problem:

**CONJECTURE.** Let $G$ be a finite group and $M$ a finite nonabelian simple group such that $Vo(G) = Vo(M)$ and $|G| = |M|$. Then $G \cong M$.

The above conjecture was proved for simple groups $PSL(2,q)$, where $q \in \{5,7,8,9,17\}$, $PSL(3,4)$, $A_7$, $Sz(8)$ and $Sz(32)$. Then in [5], it is proved that Sporadic simple groups, Alternating groups, Projective special linear groups $PSL(2,p)$, where $p$ is an odd prime, and finite simple $K_n$- groups where $n \in \{3,4\}$, satisfying this conjecture. Now, we prove this conjecture for some finite simple groups as follows:

**Main Theorem.** If $G$ is a finite group such that $Vo(G) = Vo(PSL(3,p))$ and $|G| = |PSL(3,p)|$, where $p$ is prime number, Then $G \cong PSL(3,p)$.

## 2 Preliminaries

**Definition 2.1** [4]. *Let $a$ and $n$ be integers greater than 1. Then a Zsigmondy prime of $a^n - 1$ is a prime $l$ such that $l \mid (a^n - 1)$ but $l \nmid (a^i - 1)$ for $1 \leq i < n$. Put*

$$Z_n(a) = \{l : l \text{ is a Zsigmondy prime of } a^n - 1\}.$$

**Lemma 2.2** [1] *Let $G$ be a Frobenius group of even order with kernel $K$ and complement $H$. Then $t(G) = 2$, the prime graph components of $G$ are $\pi(H)$ and $\pi(K)$ and the following assertions hold:*

(1) *$K$ is nilpotent;*
(2) $|K| \equiv 1 \pmod{|H|}$.

**Lemma 2.3** [1] *Let $G$ be a 2-Frobenius group, i.e., $G$ is a finite group and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K$ and $G/H$ are Frobenius groups with kernels $H$ and $K/H$, respectively. Then:*

(a) *$t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;*
(b) *$G/K$ and $K/H$ are cyclic, $|G/K| \mid (|K/H|-1)$ and $G/K \leq \mathrm{Aut}(K/H)$.*

**Lemma 2.4** [11] *If $G$ is a finite group such that $t(G) \geq 2$, then $G$ has one of the following structures:*
*(a) $G$ is a Frobenius group or 2-Frobenius group;*
*(b) $G$ has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ and $K/H$ is a non-abelian simple group. In particular, $H$ is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of $G$ are the odd order components of $K/H$.*

**Lemma 2.5** [10] *If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers $p_i$ such that $(n+1)/2 < p_i < n$. Here*

$$
\begin{aligned}
s(n) &= 1, \ for\ 6 \leq n \leq 13; \\
s(n) &= 2, \ for\ 14 \leq n \leq 17; \\
s(n) &= 3, \ for\ 18 \leq n \leq 37; \\
s(n) &= 4, \ for\ 38 \leq n \leq 41; \\
s(n) &= 5, \ for\ 42 \leq n \leq 47; \\
s(n) &= 6, \ for\ n \geq 48.
\end{aligned}
$$

**Lemma 2.6** [2,3] *(i) If $G$ is a finite nonabelian simple group except $A_7$, then $GK(G) = \Gamma(G)$.*
*(ii) If $G$ is a solvable group, then $\Gamma(G)$ has at most 2 connected components.*

**Lemma 2.7** [2] *Let $G$ be a finite nonsolvable group. If $\Gamma(G)$ is disconnected, then $G$ has a unique nonabelian composition factor $S$, and $t(S)$ is greater than or equal to the number of connected components of $\Gamma(G)$, unless $G$ is isomorphic to $A_7$.*

The following lemma is an easy consequence of [9], Corollary 22.26.

**Lemma 2.8** *If $\chi \in Irr(G)$ vanishes on a $p$- element for some prime $p$, then $p \mid \chi(1)$.*

Let $p$ be a prime number. A character $\chi \in Irr(G)$ is said to be of $p$ defect Zero, if $p \nmid |G|/\chi(1)$. Also, if $\chi \in Irr(G)$ is of $p$ defect Zero, then for every element $g \in G$ such that $p \mid o(g)$, we have $\chi(g) = 0$ ([6], theorem 8.17)

**Lemma 2.9** [7,8] *Let $S = PSL(3,q)$ and $OC_2 = \frac{q^2+q+1}{d}$, where $d = (3, q-1)$.*

- (i) *Let $q$ be an odd prime power. If $x \in \pi_1(S)$, $x^\alpha \mid |S|$ and $x^\alpha - 1 \equiv 0 \pmod{(OC_2)}$, then $x^\alpha = q^3$ or $q = 5$ and $x^\alpha = 2^5$.*
- (ii) *Let $q$ be an odd prime power. If $x \in \pi_1(S)$, $x^\alpha \mid |S|$, then $x^\alpha + 1 \not\equiv 0 \pmod{(OC_2)}$, for every positive integer $\alpha$.*
- (iii) *Let $q = 2^m$, where $m \geq 1$. If $x \in \pi_1(S)$, $x^\alpha \mid |S|$ and $x^\alpha - 1 \equiv 0 \pmod{(OC_2)}$, then $x = 2$ and $\alpha = 3m$, hence $x^\alpha = q^3$.*
- (iv) *Let $q = 2^m$, where $m \geq 1$. If $3 \mid q-1$ and $\frac{q^2+q-2}{6} \mid (q-1)^2(q+1)$, then $q = 16$.*

### 3 Main results

**Main Theorem.** Let $G$ be a group. Then $G \cong PSL(3,p)$ if and only if $|G| = |PSL(3,p)|$ and $Vo(G) = Vo(PSL(3,p))$, where $p$ is prime number.
**Proof.** If $G \cong PSL(3,p)$, then the conclusion is evident. Now we assume that $Vo(G) = Vo(PSL(3,p))$ and

$$|G| = |PSL(3,p)| = \frac{p^2+p+1}{(3,p-1)}p^3(p-1)^2(p+1),$$

where $r = \dfrac{p^2+p+1}{(3,p-1)}$ is prime number and $d = (3,p-1)$. We are going to prove the main theorem in the following steps.
**Step 1.** The prime graph of $G$ is disconnected.
*Proof.* According to Lemma 2.6, we have

$$\Gamma(G) = \Gamma(PSL(3,p)) = GK(PSL(3,p)).$$

Hence, by [11], $\Gamma(G)$ has two connected components and $r$ is isolated vertex in $\Gamma(G)$. So $G$ has an $r$-element $g$ such that $\chi(g) = 0$ for some irreducible complex character $\chi$ of $G$. Now, Lemma 2.8, implies that $r$ divides $\chi(1)$. Since $r|||PSL(3,p)|$ and $|G| = |PSL(3,p)|$, $\chi$ is an irreducible character of $r$-defect zero of $G$. So for every element $h \in G$ such that $r|o(h)$, we conclude that $\chi(h) = 0$. So by the fact that $r$ is an isolated vertex in $\Gamma(G)$, we conclude that $r$ is an isolated vertex in $GK(G)$. Hence $t(G) \geq 2$.

□

**Step 2.** $G$ is not a Frobenius group.
*Proof.* Suppose on the contrary, $G$ is a Frobenius group with kernel $K$ and complement $H$. Since $r$ is an odd order component of $G$, by Lemma 2.2, $r \in \{\pi(H), \pi(K)\}$. If $2 \mid |H|$, then $|H| = p^3(p-1)^2(p+1)$ and $|K| = r$. Now, by Lemma 2.2, $|H|$ divides $|K|-1$, which is a contradiction. If $2 \mid |K|$, then $|K| = p^3(p-1)^2(p+1)$ and $|H| = r$. Let $t$ be a prime dividing $|K|$, $s \nmid p$ and $K_s$ be a Sylow $s$-subgroup of $K$. Then $s$ divides either $(p-1)^2$ or $p+1$. Let $s$ divides $p+1$ . Since $K_s \rtimes H$ is a Frobenius group with kernel $K_s$ and complement $H$, thus $|H| \mid |K_s| - 1$ and hence, $r \mid |K_s| - 1 = |p+1|_t - 1$, which is a contradiction. If $s$ divides $(p-1)^2$, then similar to the above, $r \mid |K_s| - 1 = |(p-1)^2|_s - 1$, which is impossible.

□

**Step 3.** $G$ is not a 2- Frobenius group.
*Proof.* If $G$ is a 2-Frobenius group, Lemma 2.3 implies that there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = r$ and $|G/K| \mid (|K/H|-1)$. Now, applying the previous argument for the Frobenius group $K$ with kernel $H$ and complement $K/H$ leads us to get a contradiction.

□

**Step 4.** There exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $H$ and $G/K$ are $\pi_1$-group, $K/H$ is a non abelian simple group with disconnected prime

graph, $H$ is a nilpotent group and $|G/H| \mid |Out(K/H)|$.
*Proof.* It follows from Lemma 2.4 and Steps 2 and 3.

□

**Step 5.** $K/H$ can not be an alternating group $\mathbb{A}_m$, where $m \geq 5$.
*Proof.* If $K/H \cong \mathbb{A}_m$, then since $r \in \pi(K/H)$, $m \geq r$. Also, since $p \geq 2$ is a prime number, $r \geq 7$. Thus by Lemma 2.5, there exists a prime number $u \in \pi(A_m)$ such that $(r+1)/2 < u < r$. Since $|G| = |PSL(3,p)|$, $u \mid p^3(p^2-1)(p-1)$. We can check at once that $u \nmid p$ and $u \nmid p-1$, so $u \in Z_2(p)$. It follows that $u = r-2$, where $r = 7$ and $p = 2$. In this case, $|PSL(3,2)| = 2^3.3.7$. Since $m \geq 7$ and $\mathbb{A}_m \mid |G|$, we get a contradiction.

□

**Step 6.** $K/H$ is not an sporadic simple group.
*Proof.* Suppose that $K/H$ is an sporadic simple group. Thus

$$r \in \{5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71\}.$$

Let $r = 7$, thus $K/H \in \{M_{22}, J_1, J_2, HS\}$. Since $r = 7$, we conclude that $p = 2$. But $5 \mid |K/H|$ and $5 \nmid |G|$, which is impossible. By the same method, we can consider the other possibilities for r.

□

**Step 7.** $K/H = PSL(3,p)$.
*Proof.* By Steps 5 and 6, and the classification theorem of finite simple groups, $K/H$ is a simple group of Lie type such that $t(K/H) \geq 2$ and $r \in OC(K/H)$. Thus $K/H$ is isomorphic to one of the finite simple groups. We consider each possibility for $K/H$, separately:

**Case 1.** Suppose that $K/H \cong A_s(q')$, where $(q'-1) \mid (s+1)$ and $s$ is an odd prime, then $r = \frac{q'^s-1}{q'-1}$ and $q'^{\frac{s(s+1)}{2}}(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^i-1) \mid p^3(p-1)^2(p+1)$. On the other hand, $r^5 = \frac{(q'^s-1)^5}{(q'-1)^5} \leq q'^{5s}$ and $q'^{s(s+1)-s} < q'^{\frac{s(s+1)}{2}}(q'^{s+1} - 1)\prod_{i=1}^{s-1}(q'^i-1) \leq p^3(p-1)^2(p+1) < r^5 \leq q'^{5s}$, which implies that $s < 5$. Hence $s = 3$, so $q'^2 + q' + 1 = \frac{p^2+p+1}{d}$. Since $(q'-1) \mid (s+1)$, $q' \in \{2, 3, 5\}$, which implies that $K/H \cong PSL(4,2)$, $K/H \cong PSL(4,3)$, $K/H \cong PSL(4,5)$. Let $K/H \cong PSL(4,3)$, then $\frac{p^2+p+1}{d} = 13$. If $d = 1$, then $p = 3$ and $|PSL(3,3)| = 2^4.3^3.13$. On the other hand, $5 \mid |K/H|$, which is a contradiction. If $d = 3$, then $p(p+1) = 38$, which is impossible. The same reasoning rules out the case when $K/H \cong PSL(4,5)$ or $K/H \cong PSL(4,2)$.
**Case 2.** If $K/H \cong C_n(q')$, where $n = 2^u \geq 2$, then $\frac{q'^n+1}{(2,q'-1)} = \frac{p^2+p+1}{d}$. Now, if $(2, q'-1) = 1$, then $q'^n = \frac{p^2+p+1-d}{d}$ and hence, $(q', p) = 1$ and

$q'^{n+1} > \frac{p^2+p+1-d}{d}$. On the other hand, $(p'^{\alpha})^{n^2} = |K/H|_{p'} \leq |G|_{p'} = (p-1)^2|_{p'}|p+1|_{p'} < (\frac{p^2+p+1-d}{d})^2 < (p'^{\alpha})^{2(n+1)}$, thus $n < 2$, which is impossible. If $(2, q'-1) = 2$, then similar to the above, we get a contradiction. The same reasoning completes the proof in the case when either $K/H \cong B_n(q')$ or $K/H \cong {}^2D_n(q')$, where $n = 2^u \geq 4$.

**Case 3.** If $K/H \cong B_s(3)$, where $s$ is an odd prime, then $\frac{3^s-1}{2} = \frac{p^2+p+1}{d}$. So $3^s = \frac{2}{d}(p^2+p+\frac{d+2}{2})$ and hence, $(3,p) = 1$ and $3^{s+1} > p^2+p+\frac{d+2}{2}$. Since $3^{s^2} = |K/H|_3 \leq |G|_3 = |(p-1)^2|_3|p+1|_3 < (p^2+p+\frac{d+2}{2})^2 < 3^{2(s+1)}$, thus $s^2 < 2(s+1)$ and hence, $s < 3$, which is impossible. By the same method, we can prove that $K/H$ cannot be a simple group $C_s(3)$.

**Case 4.** If $K/H \cong C_s(2)$, where $s$ is an odd prime, then $2^s - 1 = \frac{p^2+p+1}{d}$ and hence, $2^s = \frac{p^2+p+1+d}{d}$. Now, if $p = 2$, then $\frac{p^2+p+1}{d} = 7$ and hence, $s = 3$. In these cases, $|PSL(3,2)| = 2^3.3.7$. On the other hand, $2^9 \mid |C_s(2)|$, which is a contradiction. If $p \neq 2$, then $(2,p) = 1$ and $2^{s+1} > \frac{p^2+p+1+d}{d}$. We know that $2^{s^2} = |K/H|_2 \leq |G|_2 = |(q-1)^2|_2|q+1|_2 < (\frac{p^2+p+1+d}{d})^2 < 2^{2(s+1)}$ and hence, $s^2 < 2(s+1)$, which implies that $s < 3$, which is impossible. By the same method, we can prove that $K/H$ cannot be a simple group $D_s(q')$, where $s \geq 5$ is prime and $q' = 2, 3, 5$, $D_{s+1}(q')$, where $q' \in \{2, 3\}$, ${}^2D_s(3)$, where $5 < s \neq 2^m + 1$ and ${}^2D_s(3)$, where $s = 2^t + 1 \geq 5$.

**Case 5.** If $K/H \cong E_6(q')$, then $r = \frac{q'^6+q'^3+1}{(3,q'-1)}$ and $q'^{36}(q'^{12}-1)(q'^8-1)(q'^6-1)(q'^5-1)(q'^3-1)(q'^2-1) \mid p^3(p-1)^2(p+1)$. On the other hand, $r^5 = \frac{(q'^6+q'^3+1)^5}{(3,q'-1)^5} \leq (q'^9-1)^5 \leq q'^{45}$ and $q'^{36}(q'^{12}-1)(q'^8-1)(q'^6-1)(q'^5-1)(q'^3-1)(q'^2-1) \leq p^3(p-1)^2(p+1) < r^5 < q'^{45}$, which is impossible. Similarly $K/H$ cannot be isomorphic to ${}^2E_6(q')$, ${}^3D_4(q')$, $F_4(q')$ and ${}^2F_4(q')$, where $q' = 2^{2t+1} \geq 2$.

**Case 6.** If $K/H \cong G_2(q')$, where $2 < q' \equiv \epsilon \pmod 3$ and $\epsilon = \pm 1$, then $q'^2 - \epsilon q' + 1 = r$. We know that $|K/H| \mid |G|$. Since $|K/H| = q'^6(q'^2-1)(q'^6-1)$ and $q'^2 - \epsilon q' + 1 = r$, it follows that $|K/H| > |G|$, which is a contradiction. By the same method, we can prove that $K/H$ cannot be a simple group $G_2(q')$, where $q' \equiv 0 \pmod 3$ and ${}^2F_4(q')$, where $q' = 2^{2t+1} \geq 2$.

**Case 7.** If $K/H \cong {}^2A_s(q')$, where $(s, q') \neq (3,3)$, $(5,2)$, $(q'+1) \mid (s+1)$ and $s$ is an odd prime, then $r = \frac{q'^s+1}{q'+1}$ and $q'^{\frac{s(s+1)}{2}}(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^i - (-1)^i) \mid p^3(p-1)^2(p+1)$. Also, $r^5 = \frac{(q'^s+1)^5}{(q'+1)^5} < q'^{5s}$ and $q'^{\frac{s(s+1)}{2}}.q'^{\frac{s(s-1)}{2}+s} < q'^{\frac{s(s+1)}{2}}(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^i - (-1)^i) \leq q^3(q-1)^2(q+1) < r^5$, which implies that $s+1 < 5$ and hence, $s = 3$. In this state, $r = q'^2 - q' + 1$ and $q'^6(q'^4-1)(q'^2-1)(q'+1) \mid p^3(p-1)^2(p+1)$. But $q'^6(q'^4-1)(q'^2-1)(q'+1) > p^3(p-1)^2(p+1)$, which is a contradiction. By the same method, we can prove that $K/H$ cannot be a simple group ${}^2A_{s-1}(q')$.

**Case 8.** If $K/H \cong {}^2F_4(2)'$, then $|K/H| = 2^{11}\cdot 3^3\cdot 5^2\cdot 13$. Thus $\frac{p^2+p+1}{d} = 13$. If $d=1$, then $p=3$ and $|PSL(3,3)| = 2^4.3^3.13$. On the other hand, $5 \mid |K/H|$, which is a contradiction. If $d=3$, then $p(p+1)=38$, which is impossible. Similarly $K/H$ cannot be isomorphic to $E_7(2)$, $E_7(3)$, $A_2(4)$, ${}^2A_5(2)$ and ${}^2E_6(2)$.

**Case 9.** If $K/H \cong {}^2D_n(2)$, where $n = 2^m+1 \geq 5$, then $2^{n-1}+1 = \dfrac{p^2+q+1}{d}$. Thus $2^{n-1} = \frac{p^2+q+1-d}{d}$ and hence, $(2,p)=1$. We know that $2^{n(n-1)} = |K/H|_2 \leq |G|_2 = |(p-1)^2|_2|p+1|_2 < (\frac{p^2+q+1-d}{d})^3 < 2^{3n}$, so $n-1<3$, which is impossible. Similarly $K/H$ cannot be isomorphic to ${}^2D_n(3)$, where $9 \leq n = 2^m+1$.

**Case 10.** If $K/H \cong A_1(q')$, where $4 \mid q'$, then the odd order components of $K/H$ are $q'+1$ and $q'-1$. If $q'+1=r$, then $q' = r-1 = \frac{p^2+q+1}{d} - 1$ and hence, either $q' = p(p-1)$ or $q' = \frac{(p-1)(p+2)}{3}$, which are impossible. If $q'-1=r$, then by Lemma 2.9(i,iii), $q'=p^3$. Since $q'(q'-1)(q'+1) = |K/H| \mid |G| = r.p^3(p-1)^2(p+1)$, we can conclude that $(p^2-p+1) \mid (p^2-2p+1)$, which is a contradiction.

**Case 11.** If $K/H \cong A_1(q')$, where $4 \mid q'-1$, then $q'=r$ or $\frac{q'+1}{2} = r$. Now, we consider the following cases:

**(i)** Let $p=2$ then $q'=r=7$, but $4 \nmid q'-1$, which is a contradiction.

**(ii)** Let $p$ be an odd prime, $q'=r$ and $d=1$, then $q' = p^2+p+1$ and $|K/H| = (p^2+p+1)(p^2+p)(\frac{p^2+p+2}{2})$. Since $|K/H| \mid |G|$, we can conclude that $\frac{p^2+p+2}{2} \mid p^2(p-1)^2$, which implies that $p=5$ and hence, $q'=31$. But $4 \mid q'-1$, which is a contradiction. If $d=3$, then $q' = \frac{p^2+p+1}{3}$ and since $|K/H| \mid |G|$, $\frac{p^2+p-2}{6} \mid p^3(p-1)^2(p+1)$. On the other hand, $(p+2,p-1)=3$ and $\frac{p^2+p-2}{6} = \frac{(p-1)(p+2)}{6}$, which implies that $\frac{(p+2)}{6} \mid p-1$, which is a contradiction. If $\frac{q'+1}{2} = r$, then by 2.9(ii), we get a contradiction. The same reasoning rules out the case when $K/H \cong A_1(q')$, where $4 \mid q'+1$.

**Case 12.** If $K/H \cong {}^2G_2(q')$, where $q' = 3^{2t+1} > 3$, then $q' - \sqrt{3q'}+1 = \frac{p^2+p+1}{d}$ or $q'+\sqrt{3q'}+1 = \frac{p^2+p+1}{d}$. Let $(3,p)=1$. If $q'-\sqrt{3q'}+1 = \frac{p^2+p+1}{d}$, then $q' > \frac{p^2+p+1}{d}$. Also, $(3^{2t+1})^3 = |K/H|_3 \leq |G|_3 < (\frac{p^2+p+1}{d})^2 < (3^{2t+1})^3$, which is a contradiction. Let $q'+\sqrt{3q'}+1 = \frac{p^2+p+1}{d}$, and $d=1$, thus $3^{t+1}(3^t+1) = p(p+1)$. Now, since $(3,p)=1$, $3 \nmid q$ and hence, $|p+1|_3 = 3^{t+1}$. Thus $|G|_3 = |p+1|_3 < 3^{t+1}$. On the other hand, $3^{3(2t+1)} = |K/H|_3 \leq |G|_3 < 3^{t+1}$, which is a contradiction. If $d=3$, then $q'+\sqrt{3q'}+1 = \frac{p^2+p+1}{3}$ and hence, $3^{t+2}(3^t+1) = (p-1)(p+2)$. Thus either $3^{t+1} \mid (p-1)$ and $(p+2) \mid 3(3^t+1)$ or $3^{t+1} \mid (p+2)$ and $(p-1) \mid 3(3^t+1)$. This forces $p-1 = 3^{t+1}$ and $p+2 = 3(3^t+1)$. This guarantees that $|G|_3 < 3^{3t+3}$. Also, $3^{3(2t+1)} = |K/H|_3 \leq |G|_3 < 3^{3t+3}$, which is a contradiction. Assume that $(3,p) \neq 1$. So $d=1$ and $q' \pm \sqrt{3q'}+1 = p^2+p+1$, this forces $p = 3^{t+1}$ and

$p+1=3^t\pm 1$, which is a contradiction.
**Case 13.** If $K/H\cong {}^2D_{s+1}(2)$, where $s=2^n-1$ and $n\geq 2$, then $2^s+1=\dfrac{p^2+p+1}{d}$ or $2^{s+1}+1=\dfrac{p^2+p+1}{d}$. If $2^s+1=r$, then $2^s=p(p+1)$ or $2^s=\dfrac{(p+1)(p-2)}{3}$, which is impossible. The same reasoning rules out the case when $2^{s+1}+1=r$.
**Case 14.** If $K/H\cong {}^2B_2(q')$, where $q'=2^{2t+1}$ and $t\geq 1$, then $r\in\{q'-1,q'\pm\sqrt{2q'}+1\}$. Let $q'-1=r$ and $d=1$. Thus $2(2^{2t}-1)=p(p+1)$. Now, if $p=2$, then $p+1=3$. It follows that $t=1$, $q'=8$ and $|K/H|=2^6.5.7.13$. On the other hand, $|PSL(3,2)|=2^3.3.7$. But $5\mid |K/H|$, which is a contradiction. Thus $|p+1|_2=2$ and since $|p-1|_2\leq 2^t$, $|G|_2\leq 2^{2t+1}$. Moreover, $2^{2(2t+1)}=|K/H|_2\leq |G|_2\leq 2^{2t+1}$, which is a contradiction. If $q'-1=r$ and $d=3$, then similar to the above we get a contradiction. Assume that $q'+\sqrt{2q'}+1=r$ and $d=3$. Thus $\frac{p^2+p-2}{3}=2^{t+1}(2^t+1)$ and hence, $(p-1)(p+2)=3.2^{t+1}(2^t+1)$. Since $3\mid p-1$, $p-1=3k$ for some positive integer $k$. Thus $3k(k+1)=2^{t+1}(2^t+1)$ and hence, $k(k+1)=2^{t+1}(\frac{2^t+1}{3})$. Now, if $2^{t+1}\mid k$, then $k+1<\frac{2^t+1}{3}$ and if $2^{t+1}\mid k+1$, then $k\leq\frac{2^t+1}{3}$, which are impossible. If $d=1$, then $p^2+p+1=q'+\sqrt{2q'}+1$ and $p(p+1)=2^{t+1}(2^t+1)$, which is impossible. The same reasoning rules out the case when $q'-\sqrt{2q'}+1=r$.
**Case 15.** If $K/H\cong E_8(q')$, then $r\in\{q'^8-q'^7+q'^5-q'^4+q'^3-q'+1, q'^8+q'^7-q'^5-q'^4-q'^3+q'+1, q'^8-q'^6+q'^4-q'^2+1, q'^8-q'^4+1\}$. If $q'^8-q'^7+q'^5-q'^4+q'^3-q'+1=r$, then $r<q'^9$. On the other hand, $r^5<q'^{45}$ and $|G|<r^5$. Since $q'^{120}\mid |K/H|$ and $|K/H|\mid |G|$, we get a contradiction. For other cases, similarly we get a contradiction.
**Case 16.** If $K/H\cong A_2(2)$ then $\frac{p^2+p+1}{d}\in\{3,7\}$. Since $p$ is a prime number, $\frac{p^2+p+1}{d}=7$, which implies that $K/H\cong PSL(3,2)$.
**Case 17.** Suppose that $K/H\cong A_{s-1}(q')$, where $(s,q')\neq(3,2)$, $(3,4)$ and $s$ is an odd prime, then $r=\frac{q'^s-1}{(s,q'-1)(q'-1)}$ and $q'^{\frac{s(s-1)}{2}}\prod_{i=1}^{s-1}(q'^i-1)\mid p^3(p-1)^2(p+1)$. On the other hand, $r^5=\frac{(q'^s-1)^5}{(s,q'-1)^5(q'-1)^5}\leq q'^{5s}$ and $q'^{s(s-1)-s}<q'^{\frac{s(s-1)}{2}}\prod_{i=1}^{s-1}(q'^i-1)\leq p^3(p-1)^2(p+1)<r^5$, which implies that $s(s-1)-s<5s$. Hence $s=3,5$. If $s=5$, then $\frac{p^2+p+1}{d}=\frac{q'^4+q'^3+q'^2+q'+1}{(5,q'-1)}$ and hence, $(q',p)=1$. Also, $q'^{10}(q'-1)(q'^2-1)(q'^3-1)(q'^4-1)\mid p^3(p-1)^2(p+1)$. But $q'^{10}\nmid p^3(p-1)^2(p+1)$, which is a contradiction. If $s=3$, then $\frac{q'^3-1}{(3,q'-1)(q'-1)}=\frac{p^2+p+1}{d}$ and hence, by Lemma 2.9(i,iii), $q'\in\{p,5\}$. Thus either $q'=p$ and $K/H\cong PSL(3,p)$ or $q'=5$ and $K/H\cong PSL(3,5)$.
**Step 8.** $G\cong PSL(3,p)$. *Proof.* By the above cases, we have $K/H\cong$

$PSL(3,p)$, i.e., $|K/H| = |PSL(3,p)| = |G|$, thus by case 3, $H = 1$ and $K = G$. Therefore $G \cong PSL(3,p)$.

$\square$

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