An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXVIII, 2022, f. 2
https://doi.org/10.47743/anstim.2022.00016

Jordan homo-derivations on triangular matrix rings

Nadeem ur Rehman · Hafedh M. Alnoghashi

Abstract In this article, our aims to establish conditions on triangular matrix rings so that a map ψ satisfying

$$\psi(A \circ B) = \psi(A) \circ B + A \circ \psi(B) + \psi(A) \circ \psi(B)$$

for all A, B in a triangular matrix ring becomes additive.

Keywords additivity · Jordan homo-derivation · triangular matrix ring · nest algebras

Mathematics Subject Classification (2010) 16W20 · 16W25 · 16W10

1 Introduction

Let R be an associative ring. As usual, the Jordan product (or anti-commutator) of elements $a, b \in R$ is denoted by $a \circ b$ (i.e., $a \circ b = ab + ba$). An additive mapping $d : R \to R$ is called a derivation if $d(ab) = d(a)b + ad(b)$ holds for all $a, b \in R$. It is called a Jordan derivation if $d(a \circ b) = d(a) \circ b + a \circ d(b)$ for all $a, b \in R$. An additive mapping ψ from ring R into itself is said to be a homo-derivation (resp. Jordan homoderivation) if for any $a, b \in R$, $\psi(ab) = \psi(a)b + a\psi(b) + \psi(a)\psi(b)$ (resp. $\psi(a \circ b) = \psi(a) \circ b + a \circ \psi(b) + \psi(a) \circ \psi(b)$). Let $\psi : R \to R$ be a mapping (without the additivity assumption). We say ψ is a nonlinear Jordan homo-derivation if $\psi(a \circ b) = \psi(a) \circ b + a \circ \psi(b) + \psi(a) \circ \psi(b)$ for any $a, b \in R$. Let us assume that R is 2-torsion free, i.e. $2a = 0$ implies $a = 0$. Setting $a = b$ in the we get $\psi(a^2) = \psi(a)a + a\psi(a) + \psi(a)^2$, holds for all $a \in R$. Above expression it is clear that every homo-derivation is a Jordan homo-derivation, but the converse need not be true in general, see Example 3.1.

In the theory of associative rings, the question of when a particular application between two rings is additive has sparked considerable interest.

Martindale III gave one of the first results ever published, which in his condition necessarily requires the ring's idempotents, see [7]. Further, in [3], for multiplicative derivations, Daif considered the same problem. Recently, Yu Wang in [9], [10] and Bruno L. M. Ferreira in [5] discusses certain map's additivity. Rings and other algebraic structures have been used to analyze this problem since then (see [1], [6], [8]).

Although these authors use a familiar argument to support their findings, the methodology used is relevant to this research line. Motivated by the preceding research, it poses the following question: When is a nonlinear Jordan homo-derivation additive in a triangular matrix ring? In the present article, this question has been investigated, and it is shown that under certain appropriate restrictions, every nonlinear Jordan homo-derivation is additive. We present an application of our findings to nest algebras at the end of the paper.

We will use Yu Wang's [10] definition of triangular algebra, which makes our findings more general than Wai-Shun Cheung's [2] definition of triangular algebra.

Definition 1.1 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two rings ($\mathfrak{A}, \mathfrak{B}$ need not have an identity element) and let $\mathfrak{M}$ ($\mathfrak{A}, \mathfrak{B}$)-bimodule such that*

(i) *$\mathfrak{M}$ is faithful as a left $\mathfrak{A}$-module and faithful as a right $\mathfrak{B}$-module,*
(ii) *if $m \in \mathfrak{M}$ is such that $\mathfrak{A}m\mathfrak{B} = 0$ then $m = 0$.*

The ring

$$\mathfrak{T} = \{\begin{pmatrix} a & m \\ & b \end{pmatrix} : a \in \mathfrak{A}, b \in \mathfrak{B}, m \in \mathfrak{M}\}$$

is called a triangular matrix ring.

Set

$$\mathfrak{T}_{11} = \{\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix} : a \in \mathfrak{A}\}, \mathfrak{T}_{12} = \{\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} : m \in \mathfrak{M}\}$$

and

$$\mathfrak{T}_{22} = \{\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix} : b \in \mathfrak{B}\}.$$

Then we can write $\mathfrak{T} = \mathfrak{T}_{11} \oplus \mathfrak{T}_{12} \oplus \mathfrak{T}_{22}$. Henceforth, the element A_{ij} belongs $\mathfrak{T}_{ij}$ and the corresponding elements are in $\mathfrak{A}, \mathfrak{B}$ or $\mathfrak{M}$. By a direct calculation $A_{ij}A_{kl} = 0$ if $j \neq k$ where $i, j, k, l \in \{1, 2\}$.

Definition 1.2 *A ring R is called k-torsion free if $ka = 0$ implies $a = 0$, for any $a \in R$, where $k \in \mathbb{Z}$, $k > 0$.*

Definition 1.3 *Let $S \subseteq R$.*

(i) *A mapping $f : R \to R$ preserves S if $f(S) \subseteq S$.*
(ii) *A mapping $f : R \to R$ is said to be zero-power valued on S if f preserves S and if for each $x \in S$, there exists a positive integer $n(x) > 1$ such that $f^{n(x)} = 0$.*

2 The main result

The main result of this paper is the following.

Theorem 2.1 (Main Theorem) *Let $\mathfrak{T}$ be a triangular matrix ring satisfying the following conditions:*

(i) If $ax + xa = 0$ for all $x \in \mathfrak{A}$, then $a = 0$,
(ii) If $by + yb = 0$ for all $y \in \mathfrak{B}$, then $b = 0$.

If a mapping $\psi : \mathfrak{T} \to \mathfrak{T}$ is zero-power valued on $\mathfrak{T}_{11}$, $\mathfrak{T}_{12}$ and $\mathfrak{T}_{22}$, and satisfies

$$\psi(A \circ B) = \psi(A) \circ B + A \circ \psi(B) + \psi(A) \circ \psi(B) \tag{2.1}$$

for all $A, B \in \mathfrak{T}$, then ψ is additive. In addition, if $\mathfrak{T}$ is 2-torsion free, then ψ is a Jordan homo-derivation.

3 Jordan homo-derivations on triangular matrix rings

Through this section, we always assume the conditions (i) and (ii) of Theorem 2.1 and (ii) in the Definition 1.3 that holds and satisfies equation (2.1). To prove our main theorem Theorem 2.1 we need some auxiliary lemmas.

Lemma 3.1 $\psi(0) = 0$.

Proof. Since $\psi(A \circ B) = \psi(A) \circ B + A \circ \psi(B) + \psi(A) \circ \psi(B)$ then $\psi(0) = 2\psi(0)^2$ and since ψ is nilpotent then $\psi(0) = 0$.

□

Lemma 3.2 *For any $A_{11} \in \mathfrak{T}_{11}$, $A_{22} \in \mathfrak{T}_{22}$, $A_{12} \in \mathfrak{T}_{12}$, we have*

1. $\psi(A_{11} + A_{12}) = \psi(A_{11}) + \psi(A_{12})$.
2. $\psi(A_{22} + A_{12}) = \psi(A_{22}) + \psi(A_{12})$.

Proof. We prove only (1) because the proof of (2) is similar. We only need to show that

$$[\psi(A_{11} + A_{12}) - \psi(A_{11}) - \psi(A_{12})] = 0.$$

For any $B_{22} \in \mathfrak{T}_{22}$ there exists a positive integer $n(B_{22}) > 1$ such that $\psi(B_{22})^{n(B_{22})} = 0$, for the simplification we will write $\psi(B_{22})^n = 0$, we compute

$$\begin{aligned}\psi[(A_{11} + A_{12})\psi(B_{22})^{n-1} + \psi(B_{22})^{n-1}(A_{11} + A_{12})] \\ = \psi(A_{11} + A_{12})\psi(B_{22})^{n-1} + (A_{11} + A_{12})\psi(B_{22})^n \\ + \psi(A_{11} + A_{12})\psi(B_{22})^n + \psi(B_{22})^n(A_{11} + A_{12}) \\ + \psi(B_{22})^{n-1}\psi(A_{11} + A_{12}) + \psi(B_{22})^n\psi(A_{11} + A_{12}).\end{aligned}$$

On the other hand,

$$\begin{aligned}
\psi[(A_{11}+A_{12})\psi(B_{22})^{n-1}+\psi(B_{22})^{n-1}(A_{11}+A_{12})]&\\
=&\psi(A_{12}\psi(B_{22})^{n-1}) = \psi(0) + \psi(A_{12}\psi(B_{22})^{n-1}+0)\\
=&\psi(A_{11}\psi(B_{22})^{n-1} + \psi(B_{22})^{n-1}A_{11})\\
&+ \psi(A_{12}\psi(B_{22})^{n-1} + \psi(B_{22})^{n-1}A_{12})\\
=&\psi(A_{11})\psi(B_{22})^{n-1}+A_{11}\psi(B_{22})^{n}+\psi(A_{11})\psi(B_{22})^{n}\\
&+\psi(B_{22})^{n}A_{11}+\psi(B_{22})^{n-1}\psi(A_{11})+\psi(B_{22})^{n}\psi(A_{11})\\
&+\psi(A_{12})\psi(B_{22})^{n-1}+A_{12}\psi(B_{22})^{n}+\psi(A_{12})\psi(B_{22})^{n}\\
&+\psi(B_{22})^{n}A_{12}+\psi(B_{22})^{n-1}\psi(A_{12})+\psi(B_{22})^{n}\psi(A_{12}).
\end{aligned}$$

It's worth noting that when we compare these two equalities, we get

$$\begin{aligned}
&[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})](\psi(B_{22})^{n-1}+\psi(B_{22})^{n})\\
&\quad+(\psi(B_{22})^{n-1}+\psi(B_{22})^{n})[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]=0.
\end{aligned}\tag{3.1}$$

Since $\psi(B_{22})^n = 0$, then

$$\begin{aligned}
&[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]\psi(B_{22})^{n-1}\\
&\quad+\psi(B_{22})^{n-1}[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]=0.
\end{aligned}\tag{3.2}$$

Now, if we compute

$$\psi[(A_{11}+A_{12})\psi(B_{22})^{n-2}+\psi(B_{22})^{n-2}(A_{11}+A_{12})]$$

then we get

$$\begin{aligned}
&[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})](\psi(B_{22})^{n-2}+\psi(B_{22})^{n-1})\\
&\quad+(\psi(B_{22})^{n-2}+\psi(B_{22})^{n-1})[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]=0.
\end{aligned}$$

By using (3.2) in last relation, we have

$$\begin{aligned}
&[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]\psi(B_{22})^{n-2}\\
&\quad+\psi(B_{22})^{n-2}[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]=0.
\end{aligned}\tag{3.3}$$

Now, as in (3.2) and (3.3), then by similarly we compute

$$\begin{aligned}
&[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]\psi(B_{22})\\
&\quad+\psi(B_{22})[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]=0.
\end{aligned}\tag{3.4}$$

Now, as above we can compute

$$\psi[(A_{11}+A_{12})B_{22}+B_{22}(A_{11}+A_{12})]$$

then we get

$$[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})](B_{22}+\psi(B_{22})) \\ +(B_{22}+\psi(B_{22}))[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]=0.$$

By using (3.4) in last relation, we have

$$[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]B_{22} \\ +B_{22}[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]=0 \tag{3.5}$$

for all $B_{22} \in \mathfrak{T}_{22}$. Thus

$$[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]_{12}B_{22}=0$$

and

$$[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]_{22}B_{22} \\ +B_{22}[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]_{22}=0.$$

We have the conditions of the Definition 1.1 and the Theorem 2.1 (*ii*), respectively.

$$[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]_{12}=0$$

and

$$[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]_{22}=0.$$

To complete the proof, we now demonstrate that

$$[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]_{11}=0.$$

For any $B_{12} \in \mathfrak{T}_{12}$. note that

$$\begin{aligned}\psi[(A_{11}+A_{12})\psi(B_{12})^{n-1}&+\psi(B_{12})^{n-1}(A_{11}+A_{12})] \\ &=\psi(A_{11}\psi(B_{12})^{n-1}) \\ &=\psi(A_{11}\psi(B_{12})^{n-1}+\psi(B_{12})^{n-1}A_{11}) \\ &\quad+\psi(A_{12}\psi(B_{12})^{n-1}+\psi(B_{12})^{n-1}A_{12}).\end{aligned}$$

It follows that

$$[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})](\psi(B_{12})^{n-1}+\psi(B_{12})^{n}) \\ +(\psi(B_{12})^{n-1}+\psi(B_{12})^{n})[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]=0. \tag{3.6}$$

Now as in (3.1), then we have

$$[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]B_{12} \\ +B_{12}[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]=0.$$

Consequently,

$$[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]_{11}B_{12}=0.$$

It follows from Definition 1.1 that $[\psi(A_{11}+A_{12})-\psi(A_{11})-\psi(A_{12})]_{11}$=0, which completes the proof.

□

Lemma 3.3 *For any $A_{11} \in \mathfrak{T}_{11}$, $A_{22} \in \mathfrak{T}_{22}$ and $A_{12}, B_{12} \in \mathfrak{T}_{12}$, we have*

$$\psi(A_{11}A_{12} + B_{12}A_{22}) = \psi(A_{11}A_{12}) + \psi(B_{12}A_{22}).$$

Proof. Note that $A_{11}A_{12}+B_{12}A_{22} = (A_{11}+B_{12})(A_{12}+A_{22})+(A_{12}+A_{22})(A_{11}+B_{12})$. Thus by Lemma 3.2, we get

$$\begin{aligned}
\psi(A_{11}A_{12}+B_{12}A_{22}) =& \psi((A_{11}+B_{12})(A_{12}+A_{22})+(A_{12}+A_{22})(A_{11}+B_{12})) \\
=& \psi(A_{11}+B_{12})(A_{12}+A_{22})+(A_{11}+B_{12})\psi(A_{12}+A_{22}) \\
& +\psi(A_{11}+B_{12})\psi(A_{12}+A_{22})+\psi(A_{12}+A_{22})(A_{11}+B_{12}) \\
& +(A_{12}+A_{22})\psi(A_{11}+B_{12})+\psi(A_{12}+A_{22})\psi(A_{11}+B_{12}) \\
=& (\psi(A_{11}) + \psi(B_{12}))(A_{12} + A_{22}) \\
& + (A_{11} + B_{12})(\psi(A_{12}) + \psi(A_{22})) \\
& + (\psi(A_{11}) + \psi(B_{12}))(\psi(A_{12}) + \psi(A_{22})) + (\psi(A_{12}) \\
& + \psi(A_{22}))(A_{11}+B_{12})+(A_{12}+A_{22})(\psi(A_{11})+\psi(B_{12})) \\
& + (\psi(A_{12}) + \psi(A_{22}))(\psi(A_{11}) + \psi(B_{12})) \\
=& \psi(A_{11}A_{12} + A_{12}A_{11}) + \psi(B_{12}A_{22} + A_{22}B_{12}) \\
=& \psi(A_{11}A_{12}) + \psi(B_{12}A_{22}).
\end{aligned}$$

□

Lemma 3.4 *For any $A_{22} \in \mathfrak{T}_{22}$ and $A_{12}, B_{12} \in \mathfrak{T}_{12}$ we have $\psi(A_{12}A_{22} + B_{12}A_{22}) = \psi(A_{12}A_{22}) + \psi(B_{12}A_{22})$.*

Proof. For any $A_{11} \in \mathfrak{T}_{11}$, there exists a positive integer $n(A_{11}) > 1$ such that $\psi(A_{11})^{n(A_{11})} = 0$, for the simplification we will write $\psi(A_{11})^n = 0$, and $A_{22} \in \mathfrak{T}_{22}$. We need to show that

$$[\psi(A_{12}A_{22} + B_{12}A_{22}) - \psi(A_{12}A_{22}) - \psi(B_{12}A_{22})] = 0.$$

We calculate $\psi[A_{11}((A_{12} + B_{12})A_{22}) + ((A_{12} + B_{12})A_{22})A_{11}]$. On one hand,

$$\begin{aligned}
&\psi[\psi(A_{11})^{n-1}((A_{12} + B_{12})A_{22}) + ((A_{12} + B_{12})A_{22})\psi(A_{11})^{n-1}] = \\
&\psi(A_{11})^n((A_{12} + B_{12})A_{22}) + \psi(A_{11})^{n-1}\psi((A_{12} + B_{12})A_{22}) \\
&+\psi(A_{11})^n\psi((A_{12} + B_{12})A_{22}) + \psi((A_{12} + B_{12})A_{22})\psi(A_{11})^{n-1} \\
&+((A_{12} + B_{12})A_{22})\psi(A_{11})^n + \psi((A_{12} + B_{12})A_{22})\psi(A_{11})^{n-1}.
\end{aligned}$$

Now on the other hand, by Lemma 3.3

$$\begin{aligned}\psi[\psi(A_{11})^{n-1}&((A_{12}+B_{12})A_{22})+((A_{12}+B_{12})A_{22})\psi(A_{11})^{n-1}]\\ &=\psi(\psi(A_{11})^{n-1}(A_{12}A_{22})+(\psi(A_{11})^{n-1}B_{12})A_{22})\\ &=\psi(\psi(A_{11})^{n-1}A_{12}A_{22})+\psi(\psi(A_{11})^{n-1}B_{12}A_{22})\\ &=\psi(\psi(A_{11})^{n-1}(A_{12}A_{22})+(A_{12}A_{22})\psi(A_{11})^{n-1})\\ &\quad+\psi(\psi(A_{11})^{n-1}(B_{12}A_{22})+(B_{12}A_{22})\psi(A_{11})^{n-1})\\ &=\psi(A_{11})^{n}(A_{12}A_{22})+\psi(A_{11})^{n-1}\psi(A_{12}A_{22})\\ &\quad+\psi(A_{11})^{n}\psi(A_{12}A_{22})+\psi(A_{12}A_{22})\psi(A_{11})^{n-1}\\ &\quad+(A_{12}A_{22})\psi(A_{11})^{n}+\psi(A_{12}A_{22})\psi(A_{11})^{n}\\ &\quad+\psi(A_{11})^{n}(B_{12}A_{22})+\psi(A_{11})^{n-1}\psi(B_{12}A_{22})\\ &\quad+\psi(A_{11})^{n}\psi(B_{12}A_{22})+\psi(B_{12}A_{22})\psi(A_{11})^{n-1}\\ &\quad+(B_{12}A_{22})\psi(A_{11})^{n}+\psi(B_{12}A_{22})\psi(A_{11})^{n}.\end{aligned}$$

Thus

$$\begin{aligned}&[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})](\psi(A_{11})^{n-1}+\psi(A_{11})^{n})\\ &+(\psi(A_{11})^{n-1}+\psi(A_{11})^{n})[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]=0.\end{aligned}$$

Since $\psi(A_{11})^n = 0$, then

$$\begin{aligned}&[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]\psi(A_{11})^{n-1}\\ &\qquad+\psi(A_{11})^{n-1}[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]=0.\end{aligned}$$

By using the same argument in Lemma 3.2, Eq. (3.2), we have

$$\begin{aligned}&[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]A_{11}\\ &\qquad+A_{11}[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]=0\end{aligned}$$

for all $A_{11} \in \mathfrak{T}_{11}$. It follows that

$$\begin{aligned}&[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]_{11}A_{11}\\ &\qquad+A_{11}[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]_{11}=0\end{aligned}$$

and

$$A_{11}[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]_{12}=0.$$

By conditions of the Definition 1.1 and Theorem 2.1 (i) respectively, we have

$$[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]_{11}==0$$

and

$$[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]_{12}=0.$$

Let us show now that $[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]_{22}=0$. To this end, for any $C_{12}\in\mathfrak{T}_{12}$, there exists a positive integer $n(C_{12})>1$ such that $\psi(C_{12})^{n(C_{12})}=0$; for the simplification we will write $\psi(C_{12})^n=0$; we compute

$$\begin{aligned}
\psi((A_{12}+B_{12})A_{22})\psi(C_{12})^{n-1} &+ ((A_{12}+B_{12})A_{22})\psi(C_{12})^n \\
&+ \psi((A_{12}+B_{12})A_{22})\psi(C_{12})^n + \psi(C_{12})^n((A_{12}+B_{12})A_{22}) \\
&+ \psi(C_{12})^{n-1}\psi((A_{12}+B_{12})A_{22}) + \psi(C_{12})^n\psi((A_{12}+B_{12})A_{22}) \\
=&\psi[((A_{12}+B_{12})A_{22})\psi(C_{12})^{n-1} + \psi(C_{12})^{n-1}((A_{12}+B_{12})A_{22})] = 0 \\
=&\psi((A_{12}A_{22})\psi(C_{12})^{n-1} + \psi(C_{12})^{n-1}(A_{12}A_{22})) \\
&+ \psi((B_{12}A_{22})\psi(C_{12})^{n-1} + \psi(C_{12})^{n-1}(B_{12}A_{22})) \\
=&\psi(A_{12}A_{22})\psi(C_{12})^{n-1} + (A_{12}A_{22})\psi(C_{12})^n + \psi(A_{12}A_{22})\psi(C_{12})^n \\
&+ \psi(C_{12})^n(A_{12}A_{22}) + \psi(C_{12})^{n-1}\psi(A_{12}A_{22}) + \psi(C_{12})^n\psi(A_{12}A_{22}) \\
&+ \psi(B_{12}A_{22})\psi(C_{12})^{n-1} + (B_{12}A_{22})\psi(C_{12})^n + \psi(B_{12}A_{22})\psi(C_{12})^n \\
&+ \psi(C_{12})^n(B_{12}A_{22}) + \psi(C_{12})^{n-1}\psi(B_{12}A_{22}) + \psi(C_{12})^n\psi(B_{12}A_{22}).
\end{aligned}$$

Thus

$$\begin{aligned}
&[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})](\psi(C_{12})^{n-1}+\psi(C_{12})^n) \\
&+(\psi(C_{12})^{n-1}+\psi(C_{12})^n)[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]=0.
\end{aligned}$$

Since $\psi(C_{12})^n=0$, then

$$\begin{aligned}
&[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]\psi(C_{12})^{n-1} \\
&\qquad+\psi(C_{12})^{n-1}[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]=0.
\end{aligned}$$

By using the same argument in Lemma 3.2, Eq. (3.2), we have

$$\begin{aligned}
&[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]C_{12} \\
&\qquad+C_{12}[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]=0
\end{aligned}$$

for all $C_{12}\in\mathfrak{T}_{12}$. Therefore, we obtain

$$C_{12}[\psi(A_{12}A_{22}+B_{12}A_{22})-\psi(A_{12}A_{22})-\psi(B_{12}A_{22})]_{22}=0.$$

By condition (i) of the Definition 1.1, we get $t_{22}=0$, which completes the proof.

□

Lemma 3.5 *For any $A_{12},B_{12}\in\mathfrak{T}_{12}$, we have $\psi(A_{12}+B_{12})=\psi(A_{12})+\psi(B_{12})$.*

Proof. For any $A_{22} \in \mathfrak{T}_{22}$, there exists a positive integer $n(A_{22}) > 1$ such that $\psi(A_{22})^{n(A_{22})} = 0$, for the simplification we will write $\psi(A_{22})^n = 0$. We need to show that $[\psi(A_{12} + B_{12}) - \psi(A_{12}) - \psi(B_{12})] = 0$. We calculate $\psi[(A_{12} + B_{12})\psi(A_{22})^{n-1} + \psi(A_{22})^{n-1}(A_{12} + B_{12})]$. On one hand,

$$\begin{aligned}\psi[(A_{12}+B_{12})\psi(A_{22})^{n-1}&+\psi(A_{22})^{n-1}(A_{12}+B_{12})]\\ =&\psi(A_{12}+B_{12})\psi(A_{22})^{n-1}+(A_{12}+B_{12})\psi(A_{22})^{n}+\psi(A_{12}+B_{12})\psi(A_{22})^{n}\\ &+\psi(A_{22})^{n}(A_{12}+B_{12})+\psi(A_{22})^{n-1}\psi(A_{12}+B_{12})+\psi(A_{22})^{n}\psi(A_{12}+B_{12}).\end{aligned}$$

Now on the other hand, by Lemma 3.4

$$\begin{aligned}\psi[(A_{12} + B_{12})\psi(A_{22})^{n-1}+\psi(A_{22})^{n-1}(A_{12} + B_{12})]&\\ =&\psi(A_{12}\psi(A_{22})^{n-1} + B_{12}\psi(A_{22})^{n-1})\\ =&\psi(A_{12}\psi(A_{22})^{n-1}) + \psi(B_{12}\psi(A_{22})^{n-1})\\ =&\psi(A_{12}\psi(A_{22})^{n-1} + \psi(A_{22})^{n-1}A_{12})\\ &+ \psi(B_{12}\psi(A_{22})^{n-1} + \psi(A_{22})^{n-1}B_{12})\\ =&\psi(A_{12})\psi(A_{22})^{n-1} + A_{12}\psi(A_{22})^{n}\\ &+ \psi(A_{12})\psi(A_{22})^{n} + \psi(A_{22})^{n}A_{12}\\ &+ \psi(A_{22})^{n-1}\psi(A_{12}) + \psi(A_{22})^{n}\psi(A_{12})\\ &+ \psi(B_{12})\psi(A_{22})^{n-1} + B_{12}\psi(A_{22})^{n}\\ &+ \psi(B_{12})\psi(A_{22})^{n} + \psi(A_{22})^{n}B_{12}\\ &+ \psi(A_{22})^{n-1}\psi(B_{12}) + \psi(A_{22})^{n}\psi(B_{12}).\end{aligned}$$

Thus

$$\begin{aligned}[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})](\psi(A_{22})^{n-1}+\psi(A_{22})^{n})&\\ +(\psi(A_{22})^{n-1}+\psi(A_{22})^{n})[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]&=0\end{aligned}$$

Since $\psi(A_{22})^n = 0$, then

$$\begin{aligned}[\psi(A_{12}+B_{12}) - \psi(A_{12})-\psi(B_{12})]\psi(A_{22})^{n-1}&\\ +\psi(A_{22})^{n-1}[\psi(A_{12}+B_{12}) - \psi(A_{12}) - \psi(B_{12})]&=0.\end{aligned}$$

By using the same argument in Lemma 3.2, Eq. (3.2), we have

$$\begin{aligned}[\psi(A_{12} + B_{12}) - \psi(A_{12})-\psi(B_{12})]A_{22}&\\ +A_{22}[\psi(A_{12} + B_{12}) - \psi(A_{12}) - \psi(B_{12})] &= 0\end{aligned}$$

for all $A_{22} \in \mathfrak{T}_{22}$. It follows that

$$[\psi(A_{12} + B_{12}) - \psi(A_{12}) - \psi(B_{12})]_{12}A_{22} = 0$$

and

$$[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]_{22}A_{22} \\ +A_{22}[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]_{22}=0.$$

By conditions of the Definition 1.1 and Theorem 2.1 (ii) respectively, we have

$$[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]_{12}=0$$

and

$$[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]_{22}=0.$$

Let us show now that $[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]_{11}=0$. To this end, for any $C_{12}\in\mathfrak{T}_{12}$ there exists a positive integer $n(C_{12})>1$ such that $\psi(C_{12})^{n(C_{12})}=0$, for the simplification we will write $\psi(C_{12})^n=0$. We compute

$$\begin{aligned}
&\psi(A_{12}+B_{12})\psi(C_{12})^{n-1}+(A_{12}+B_{12})\psi(C_{12})^n+\psi(A_{12}+B_{12})\psi(C_{12})^n\\
&+\psi(C_{12})^n(A_{12}+B_{12})+\psi(C_{12})^{n-1}\psi(A_{12}+B_{12})+\psi(C_{12})^n\psi(A_{12}+B_{12})\\
&=\psi[(A_{12}+B_{12})\psi(C_{12})^{n-1}+\psi(C_{12})^{n-1}(A_{12}+B_{12})]=0\\
&=\psi(A_{12}\psi(C_{12})^{n-1}+\psi(C_{12})^{n-1}A_{12})+\psi(B_{12}\psi(C_{12})^{n-1}+\psi(C_{12})^{n-1}B_{12})\\
&=\psi(A_{12})\psi(C_{12})^{n-1}+A_{12}\psi(C_{12})^n+\psi(A_{12})\psi(C_{12})^n+\psi(C_{12})^nA_{12}\\
&\quad+\psi(C_{12})^{n-1}\psi(A_{12})+\psi(C_{12})^n\psi(A_{12})+\psi(B_{12})\psi(C_{12})^{n-1}+B_{12}\psi(C_{12})^n\\
&\quad+\psi(B_{12})\psi(C_{12})^n+\psi(C_{12})^nB_{12}+\psi(C_{12})^{n-1}\psi(B_{12})+\psi(C_{12})^n\psi(B_{12}).
\end{aligned}$$

This yields that

$$[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})](\psi(C_{12})^{n-1}+\psi(C_{12})^n) \\ +(\psi(C_{12})^{n-1}+\psi(C_{12})^n)[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]=0.$$

Since $\psi(C_{12})^n=0$, then

$$[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]\psi(C_{12})^{n-1} \\ +\psi(C_{12})^{n-1}[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]=0.$$

By using the same argument in Lemma 3.2, Eq. (3.2), we have

$$[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]C_{12} \\ +C_{12}[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]=0.$$

Therefore, we have $[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]_{11}C_{12}=0$ for all $C_{12}\in\mathfrak{T}_{12}$. By condition of the Definition 1.1, we get $[\psi(A_{12}+B_{12})-\psi(A_{12})-\psi(B_{12})]_{11}=0$, which completes the proof.

□

Lemma 3.6 *For any $A_{11},B_{11}\in\mathfrak{T}_{11}$, $A_{22},B_{22}\in\mathfrak{T}_{22}$, we have*

(1) $\psi(A_{11}+B_{11}) = \psi(A_{11}) + \psi(B_{11})$.
(2) $\psi(A_{22}+B_{22}) = \psi(A_{22}) + \psi(B_{22})$.

Proof. We only prove (1). For any $C_{22} \in \mathfrak{T}_{22}$ there exists a positive integer $n(C_{22}) > 1$ such that $\psi(C_{22})^{n(C_{22})} = 0$, for the simplification we will write $\psi(C_{22})^n = 0$. We need to show that $[\psi(A_{11}+B_{11}) - \psi(A_{11}) - \psi(B_{11})] = 0$, we get

$$\begin{aligned}
&\psi(A_{11}+B_{11})\psi(C_{22})^{n-1}+(A_{11}+B_{11})\psi(C_{22})^n+\psi(A_{11}+B_{11})\psi(C_{22})^n \\
&\quad + \psi(C_{22})^n(A_{11}+B_{11})+\psi(C_{22})^{n-1}\psi(A_{11}+B_{11})+\psi(C_{22})^n\psi(A_{11}+B_{11}) \\
=&\psi[(A_{11}+B_{11})\psi(C_{22})^{n-1} + \psi(C_{22})^{n-1}(A_{11}+B_{11})] = 0 \\
=&\psi(A_{11}\psi(C_{22})^{n-1} + \psi(C_{22})^{n-1}A_{11}) + \psi(B_{11}\psi(C_{22})^{n-1} + \psi(C_{22})^{n-1}B_{11}) \\
=&\psi(A_{11})\psi(C_{22})^{n-1} + A_{11}\psi(C_{22})^n + \psi(A_{11})\psi(C_{22})^n + \psi(C_{22})^n A_{11} \\
&\quad + \psi(C_{22})^{n-1}\psi(A_{11}) + \psi(C_{22})^n\psi(A_{11}) + \psi(B_{11})\psi(C_{22})^{n-1} \\
&\quad + B_{11}\psi(C_{22})^n + \psi(B_{11})\psi(C_{22})^n + \psi(C_{22})^n B_{11} + \psi(C_{22})^{n-1}\psi(B_{11}) \\
&\quad + \psi(C_{22})^n\psi(B_{11}).
\end{aligned}$$

This gives us

$$\begin{aligned}
&[\psi(A_{11}+B_{11})-\psi(A_{11})-\psi(B_{11})](\psi(C_{22})^{n-1}+\psi(C_{22})^n) \\
&\qquad +(\psi(C_{22})^{n-1}+\psi(C_{22})^n)[\psi(A_{11}+B_{11})-\psi(A_{11})-\psi(B_{11})]=0.
\end{aligned}$$

Since $\psi(C_{22})^n = 0$ then

$$\begin{aligned}
&[\psi(A_{11}+B_{11}) - \psi(A_{11})-\psi(B_{11})]\psi(C_{22})^{n-1} \\
&\qquad + \psi(C_{22})^{n-1}[\psi(A_{11}+B_{11}) - \psi(A_{11}) - \psi(B_{11})]=0.
\end{aligned}$$

By using the same argument in Lemma 3.2, Eq. (3.2), we have

$$\begin{aligned}
&[\psi(A_{11}+B_{11}) - \psi(A_{11})-\psi(B_{11})]C_{22} \\
&\qquad + C_{22}[\psi(A_{11}+B_{11}) - \psi(A_{11}) - \psi(B_{11})] = 0,
\end{aligned}$$

which implies that

$$[\psi(A_{11}+B_{11}) - \psi(A_{11}) - \psi(B_{11})]_{12} = 0$$

and

$$[\psi(A_{11}+B_{11}) - \psi(A_{11}) - \psi(B_{11})]_{22} = 0.$$

Similarly, by considering $(A_{11}+B_{11})A_{12}+A_{12}(A_{11}+B_{11})$ and using Lemma 3.5 one can deduce that $[\psi(A_{11}+B_{11}) - \psi(A_{11}) - \psi(B_{11})]_{11} = 0$. □

Lemma 3.7 *For any $A_{11} \in \mathfrak{T}_{11}$, $A_{12} \in \mathfrak{T}_{12}$, $A_{22} \in \mathfrak{T}_{22}$, we have $\psi(A_{11} + A_{12} + A_{22}) = \psi(A_{11}) + \psi(A_{12}) + \psi(A_{22})$.*

Proof. For any $B_{11} \in \mathfrak{T}_{11}$ there exists a positive integer $n(B_{11}) > 1$ such that $\psi(B_{11})^{n(B_{11})} = 0$, for the simplification we will write $\psi(B_{11})^n = 0$. We need to show that $[\psi(A_{11} + A_{12} + A_{22}) - \psi(A_{11}) - \psi(A_{12}) - \psi(A_{22})] = 0$. By Lemma 3.2, we get

$$\begin{aligned}
\psi(A_{11}+A_{12}+A_{22})\psi(B_{11})^{n-1}&+(A_{11}+A_{12}+A_{22})\psi(B_{11})^n \\
&+ \psi(A_{11}+A_{12}+A_{22})\psi(B_{11})^n + \psi(B_{11})^n(A_{11} + A_{12} + A_{22}) \\
&+ \psi(B_{11})^{n-1}\psi(A_{11}+A_{12}+A_{22})+\psi(B_{11})^n\psi(A_{11}+A_{12}+A_{22}) \\
=&\psi[(A_{11} + A_{12}+A_{22})\psi(B_{11})^{n-1}+\psi(B_{11})^{n-1}(A_{11}+A_{12}+A_{22})] \\
=&\psi(A_{11}\psi(B_{11})^{n-1} + \psi(B_{11})^{n-1}A_{11} + \psi(B_{11})^{n-1}A_{12}) \\
=&\psi(A_{11}\psi(B_{11})^{n-1} + \psi(B_{11})^{n-1}A_{11}) + \psi(\psi(B_{11})^{n-1}A_{12}) \\
=&\psi(A_{11}\psi(B_{11})^{n-1} + \psi(B_{11})^{n-1}A_{11}) + \psi(\psi(B_{11})^{n-1}A_{12} \\
&+ A_{12}\psi(B_{11})^{n-1}) + \psi(A_{22}\psi(B_{11})^{n-1} + \psi(B_{11})^{n-1}A_{22}).
\end{aligned}$$

This gives us

$$\begin{aligned}
&[\psi(A_{11}+A_{12}+A_{22})-\psi(A_{11})-\psi(A_{12})-\psi(A_{22})](\psi(B_{11})^{n-1}+\psi(B_{11})^n) \\
&+(\psi(B_{11})^{n-1}+\psi(B_{11})^n)[\psi(A_{11}+A_{12}+A_{22})-\psi(A_{11})-\psi(A_{12})-\psi(A_{22})]=0.
\end{aligned}$$

Since $\psi(B_{11})^n = 0$ then

$$\begin{aligned}
&[\psi(A_{11}+A_{12}+A_{22})-\psi(A_{11})-\psi(A_{12})-\psi(A_{22})]\psi(B_{11})^{n-1} \\
&\qquad+\psi(B_{11})^{n-1}[\psi(A_{11}+A_{12}+A_{22})-\psi(A_{11})-\psi(A_{12})-\psi(A_{22})]=0.
\end{aligned}$$

By using the same argument in Lemma 3.2, Eq. (3.2), we have

$$\begin{aligned}
&[\psi(A_{11}+A_{12}+A_{22})-\psi(A_{11})-\psi(A_{12})-\psi(A_{22})]B_{11} \\
&\qquad+B_{11}[\psi(A_{11}+A_{12}+A_{22})-\psi(A_{11})-\psi(A_{12})-\psi(A_{22})]=0.
\end{aligned}$$

We can infer by conditions of the Theorem 2.1 and conditions of Definition 1.1 that

$$[\psi(A_{11} + A_{12} + A_{22}) - \psi(A_{11}) - \psi(A_{12}) - \psi(A_{22})]_{11} = 0$$

and

$$[\psi(A_{11} + A_{12} + A_{22}) - \psi(A_{11}) - \psi(A_{12}) - \psi(A_{22})]_{12} = 0.$$

Similarly, one can get $[\psi(A_{11}+A_{12}+A_{22})-\psi(A_{11})-\psi(A_{12})-\psi(A_{22})]_{22}=0$, which completes the proof.

□

Now we are ready to prove our main result.

Proof of the Theorem 2.1. For any $A, B \in \mathfrak{T}$ we write $A = A_{11}+A_{12}+A_{22}$ and $B = B_{11} + B_{12} + B_{22}$. Applying Lemmas 3.5-3.7, we have

$$\begin{aligned}
&\psi(A+B)\\
&\quad=\psi(A_{11} + A_{12} + A_{22} + B_{11} + B_{12} + B_{22})\\
&\quad=\psi[(A_{11} + B_{11}) + (A_{12} + B_{12}) + (A_{22} + B_{22})]\\
&\quad=\psi(A_{11} + B_{11}) + \psi(A_{12} + B_{12}) + \psi(A_{22} + B_{22})\\
&\quad=\psi(A_{11}) + \psi(B_{11}) + \psi(A_{12}) + \psi(B_{12}) + \psi(A_{22}) + \psi(B_{22})\\
&\quad=\psi(A_{11} + A_{12} + A_{22}) + \psi(B_{11} + B_{12} + B_{22})\\
&\quad=\psi(A) + \psi(B),
\end{aligned}$$

that is, ψ is additive. In addition, if $\mathfrak{T}$ is 2-torsion free, then for any $A \in \mathfrak{T}$, we have $2\psi(A^2) = \psi(2A^2) = \psi(AA + AA) = 2[\psi(A)A + A\psi(A) + \psi(A)^2]$. Therefore, ψ is a Jordan homo-derivation.

Now, we conclude this section with the following Example.

Example 3.1 Consider S be a ring such that $s^2 = 0$ for all $s \in S$, but the product of some elements of S is nonzero. Since $s^2 = 0$, so $(s+t)^2 = 0$ for all $s, t \in S$ this implies that $s \circ t = 0$ for all $s, t \in S$. Suppose $R = \{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} :$ $a, b, c \in S\}$. Let $J : R \to R$ such that $J\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then it is easy to see that J is a Jordan homo-derivation, but not a homo-derivation.

4 An application to nest algebras

A nest $\mathcal{N}$ is a totally ordered set of closed subspaces of a Hilbert space $\mathcal{H}$ such that $\{0\}$,$\mathcal{H} \in \mathcal{N}$, and $\mathcal{N}$ is closed under the taking of arbitrary intersections and closed linear spans of its elements. The nest algebra associated to $\mathcal{N}$ is the set $\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq N$ for all $N \in \mathcal{N}\}$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators over a complex Hilbert space $\mathcal{H}$. We recall the standard result ([2], Proposition 16), which says that we can view $\mathcal{T}(\mathcal{N})$ as triangular algebra $\begin{pmatrix} A & M \\ & B \end{pmatrix}$ where A, B are themselves nest algebras.

Proposition 4.1 *If $N \in \mathcal{N} - \{0, H\}$ and E is the orthonormal projection onto N, then $E\mathcal{N}E$ and $(1 - E)\mathcal{N}(1 - E)$ are nest, $\mathcal{T}(E\mathcal{N}E) = E\mathcal{T}(\mathcal{N})E$ and $\mathcal{T}((1 - E)\mathcal{N}(1 - E)) = (1 - E)\mathcal{T}(\mathcal{N})(1 - E)$. Furthermore*
$$\mathcal{T}(\mathcal{N}) = \begin{pmatrix} \mathcal{T}(E\mathcal{N}E) & E\mathcal{T}(\mathcal{N})(1 - E) \\ & \mathcal{T}((1 - E)\mathcal{N}(1 - E)) \end{pmatrix}$$

We refer the reader to [4] for the general theory of nest algebras.

Corollary 4.2 *Let P_n be an increasing sequence of finite dimensional subspaces such that their union is dense in $\mathcal{H}$. Consider $\mathcal{P} = \{\{0\}, P_n, n \geq 1, \mathcal{H}\}$ a nest and $\mathcal{T}(\mathcal{P})$ the set consists of all operators which have a block upper triangular matrix with respect to $\mathcal{P}$. If a mapping $\psi : \mathcal{T}(\mathcal{P}) \to \mathcal{T}(\mathcal{P})$ satisfies*

$$\psi(f \circ g) = \psi(f) \circ g + f \circ \psi(g) + \psi(f) \circ \psi(g)$$

for all $f, g \in \mathcal{T}(\mathcal{P})$, then ψ is additive.

Acknowledgements The authors are greatly indebted to the referee for his/her constructive comments and suggestion, which improves the quality of the paper a lot. For the first author, this research is supported by the Council of Scientific and Industrial Research (CSIR-HRDG), India, Grant No. 25(0306)/20/EMR-II.

References

1. Cheng, X.; Jing, W. – *Additivity of maps on triangular algebras*, Electron. J. Linear Algebra **17** (2008), 597-615.
2. Cheung, W.-S. – *Commuting maps of triangular algebras*, J. London Math. Soc.(2), **63** (2001), no. 1, 117-127.
3. Daif, M.N. – *When is a multiplicative derivation additive?*, Internat. J. Math. & Math. Sci., **14** (1991), no. 3, 615-618.
4. Davidson, K.R. – *Nest Algebras. Triangular forms for operator algebras on Hilbert space*, Pitman Research Notes in Mathematics Series, 191, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, (1988).
5. Ferreira, B.L.M. – *Multiplicative maps on triangular n-matrix rings*, International Journal of Mathematics, Game Theory and Algebra, **23** (2014), no. 2, 1-14.
6. Lu, F. – *Jordan maps on associative algebras*, Comm. Algebra, **31** (2003), no. 5, 2273-2286.
7. Martindale-III, W.S. – *When are multiplicative mappings additive?*, Proc. Amer. Math. Soc., **21** (1969), 695-698.
8. Molnár, L.; Šemrl, P. – *Elementary operators on standard operator algebras*, Linear Multilinear Algebra, **50** (2002), no. 4, 315-319.
9. Wang, Y. – *The additivity of multiplicative maps on rings*, Comm. Algebra, **37** (2009), no. 7, 2351-2356.
10. Wang, Y. – *Additivity of multiplicative maps on triangular rings*, Linear Algebra Appl., **434** (2011), no. 3, 625-635.

Received: 2.XII.2021 / Revised: 1.V.2022 / Accepted: 2.V.2022

Authors

Nadeem ur Rehman (Corresponding author),
Department of Mathematics,
Aligarh Muslim University,
202002 Aligarh, India,
E-mail: nu.rehman.mm@amu.ac.in,
E-mail: rehman100@gmail.com

Hafedh M. Alnoghashi,
Department of Mathematics,
Aligarh Muslim University,
202002 Aligarh, India,
E-mail: halnoghashi@gmail.com