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On certain principal directions in lightlike hypersurfaces of an indefinite Sasakian manifold

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Abstract In this paper, the geometry of some principal directions on a tangential lightlike hypersurface of an indefinite Sasakian manifold is investigated. In particular, we have proved, under some geometric conditions, that if the total space has constant holomorphic sectional curvature then such curvature is -3. Moreover, it is shown that the underlying lightlike hypersurface is, locally, foliated by totally geodesic lightlike hypersurfaces.

Keywords principal directions · screen integrable lightlike hypersurfaces · totally umbilic hypersurfaces · Hopf lightlike hypersurfaces

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1 Introduction

K.L. Duggal and A. Bejancu, in [3], initiated the study of lightlike submanifolds of semi-Riemannian manifolds. Their work was later updated by K.L. Duggal and B. Sahin in the book [4] and K.L. Duggal and D.H. Jin in the book [5]. In the above books, the authors laid a foundation for research on lightlike geometry by constructing their structural equations, among other results. In fact, they introduced a non-degenerate screen distribution to construct the lightlike transversal vector bundle, which has trivial intersection with the tangent bundle and developed local geometry of lightlike curves, hypersurfaces and in general, the submanifolds of arbitrary codimension. Other pioneering works on the theory include that of D. N. Kupeli [13]– whose approach is purely intrinsic compared to that of [3–5]. Since then, many researchers including but not limited to [1, 6–8, 10, 12], have researched on lightlike submanifolds and many interesting results have been obtained. Lightlike hypersurfaces appear in general relativity as models of different types of Black Hole horizons (see [3, 4] for details) and their theory is quite fundamental to modern mathematical physics.

In this paper, we extend the study on the geometry of lightlike hypersurfaces of indefinite Sasakian manifolds. In particular, we introduce the notion of principal directions. The paper is arranged as follows: in Section 2, we quote some basic notions necessary for the entire paper, Section 3 is devoted to some basic results on tangential lightlike hypersurfaces, Section 4 is on the geometry of particular principal directions and finally, Section 5 is focussed on the geometry of a certain principal distributions.

2 Preliminaries

A $(2n+1)$-dimensional differentiable manifold $(\bar{M}, \bar{g})$ is said to be an indefinite Sasakian manifold if it admits an almost contact structure $(\bar{\phi}, \zeta, \eta)$, that is $\bar{\phi}$ is a $(1,1)$-tensor of rank $2n$, ζ is a smooth vector field, and η is a 1-form satisfying

$$\bar{\phi}^2 = -I + \eta \otimes \zeta, \;\; \eta(\zeta) = 1, \;\; \bar{\phi}\zeta = 0, \;\; \eta(X) = \bar{g}(\zeta, X), \;\; \eta \circ \bar{\phi} = 0, \tag{2.1}$$

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = \bar{g}(X,Y) - \eta(X)\eta(Y), \; (\bar{\nabla}_X \bar{\phi})(Y) = \bar{g}(X,Y)\zeta - \eta(Y)X, \tag{2.2}$$

$$\bar{\nabla}_X \zeta = -\bar{\phi}X, \quad (\bar{\nabla}_X \eta)(Y) = \bar{g}(\bar{\phi}X, Y), \tag{2.3}$$

for all X and Y tangent to $\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\bar{g}$. Replacing Y with $\bar{\phi}Y$ in the first relation of (2.2), and then using (2.1), we get

$$\bar{g}(\bar{\phi}X, Y) = -\bar{g}(X, \bar{\phi}Y), \tag{2.4}$$

for any X and Y tangent to $\bar{M}$.

Let $p \in \bar{M}$, a plane section π in $T_p\bar{M}$ of a Sasakian manifold $\bar{M}$ is called a $\bar{\phi}$-section if it is spanned by a unit vector X orthogonal to ζ and $\bar{\phi}X$, where X is a non-null vector field on $\bar{M}$. The sectional curvature $K(X, \bar{\phi}X)$ of a $\bar{\phi}$-section is called a $\bar{\phi}$-sectional curvature. If $\bar{M}$ has a $\bar{\phi}$-sectional curvature c which does not depend on the $\bar{\phi}$-section at each point, then, c is constant in $\bar{M}$ and $\bar{M}$ is called a *Sasakian space form*, denoted by $\bar{M}(c)$. Moreover, the curvature tensor $\bar{R}$ of $\bar{M}$ satisfies (see [4, p. 307] for more details):

$$\begin{aligned}\bar{R}(X,Y)Z = {} & \frac{c+3}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ & + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \bar{g}(X,Z)\eta(Y)\zeta \\ & - \bar{g}(Y,Z)\eta(X)\zeta + \bar{g}(\bar{\phi}Y, Z)\bar{\phi}X - \bar{g}(\bar{\phi}X, Z)\bar{\phi}Y \\ & - 2\bar{g}(\bar{\phi}X, Y)\bar{\phi}Z\},\end{aligned} \tag{2.5}$$

for all X, Y and Z tangent to $\bar{M}$.

Let $(\bar{M}, \bar{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index q, where $0 < q < 2n+1$, and let (M, g) be a hypersurface of $\bar{M}$ with

$g = \bar{g}_{x|T_xM}$, $x \in M$. Then, M is a lightlike hypersurface of $\bar{M}$ if g is of constant rank $(2n-1)$ and the normal bundle $TM^{\perp}$ is a distribution of rank 1 on M [4]. A complementary distribution to $TM^{\perp}$ is a $(2n-1)$ non-degenerate distribution over M, called the *screen* distribution and denoted by $S(TM)$. This distribution is not unique, but it is known to canonically isomorphic to the factor bundle $TM\backslash TM^{\perp}$ discussed by D. N. Kupeli [13]. A lightlike hypersurface furnished with a given screen distribution id denoted by $(M, g, S(TM))$. Since $TM^{\perp}$ is a subbundle of TM, the following construction is crucial in understanding lightlike hypersurfaces.

Theorem 2.1 ([4], p. 44) *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. There exists a unique vector bundle $\mathrm{tr}(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighbourhood $\mathcal{U} \subset M$, there exists a unique section N of $\mathrm{tr}(TM)$ on $\mathcal{U}$ satisfying:*

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, W) = 0,$$

for all W tangent to $S(TM)_{|\mathcal{U}}$.

We call $\mathrm{tr}(TM)$ the lightlike transversal bundle, and N the lightlike transversal vector field. Throughout the paper, all manifolds are supposed to be paracompact and smooth. Also, by $\perp$ and $\oplus$ we mean the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 2.1 we may write down the following decompositions:

$$TM = TM^{\perp} \perp S(TM), \tag{2.6}$$

$$T\bar{M}|_M = TM \oplus \mathrm{tr}(TM) = S(TM) \perp \{TM^{\perp} \oplus \mathrm{tr}(TM)\}. \tag{2.7}$$

Let P be the projection morphism of TM onto $S(TM)$. Then the local Gauss-Weingarten equations of M and $S(TM)$ are the following (see [3, p. 82–85] for more details):

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N,$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \quad \tau(X) = \bar{g}(\bar{\nabla}_X N, \xi), \tag{2.8}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \quad \nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.9}$$

for all X and Y tangent to M, ξ tangent to $TM^{\perp}$ and N tangent to $\mathrm{tr}(TM)$. Here, ∇ and ∇^* are the induced linear connections on TM and $S(TM)$, respectively, B is the local second fundamental form of M and C is the local second fundamental form on $S(TM)$. Furthermore, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively, and τ is a differential one-form on TM. It is known that ∇^* is a metric connection on $S(TM)$ while ∇ is generally not a metric connection. In fact, from the fact $\bar{\nabla}\bar{g} = 0$, we get the expression of ∇g as

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \tag{2.10}$$

for all X, Y and Z tangent to M, where θ is a differential one-form on M dual to N, i.e. $\theta = \bar{g}(\cdot, N)$. Moreover, B is independent of $S(TM)$ and satisfies:

$$B(X, \xi) = 0, \tag{2.11}$$

for any X tangent to M. The local second fundamental forms B and C are related to their shape operators by the following equations:

$$g(A^*_\xi X, Y) = B(X, Y), \quad \bar{g}(A^*_\xi X, N) = 0, \tag{2.12}$$

$$g(A_N X, PY) = C(X, PY), \quad \bar{g}(A_N X, N) = 0, \tag{2.13}$$

for all X and Y tangent to M. From the relations (2.12) and (2.13), we notice that both A_N and A^*_ξ are screen-valued, and interrelate with their local second fundamental forms.

Let us denote by $\bar{R}$ the curvature tensors of $\bar{M}$. Using the Gauss-Weingarten formulae (2.8) and (2.9), we have the following equations (see details in [3, 4]).

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)Z, \xi) &= (\nabla_X B)(Y, Z) \\ &\quad - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z),\end{aligned} \tag{2.14}$$

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) \\ &\quad - (\nabla_Y C)(X, PZ) - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ),\end{aligned} \tag{2.15}$$

where ∇B and ∇C are given by

$$(\nabla_X B)(Y, Z) = X \cdot B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z), \tag{2.16}$$

$$(\nabla_X C)(Y, PZ) = X \cdot C(Y, PZ) - C(\nabla_X Y, PZ) - C(Y, \nabla^*_X PZ), \tag{2.17}$$

for all X, Y and Z tangent to M.

3 Some basic results

Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $\bar{M} = (\bar{M}, \bar{\phi}, \zeta, \eta)$, which is *tangent* to the structure vector field ζ, i.e. $\zeta \in \Gamma(TM)$. In this case, the authors in [4, Lemma 7.3.1., p. 318] (also see [2]) have proved that ζ is not a vector field of $\operatorname{Rad} TM$. With such a result in mind, ζ has been assumed to belong to the screen distribution $S(TM)$ in most studies on lightlike geometry in indefinite Sasakian manifolds. In this paper, we follow the same assumptions regarding the tangency of M to ζ. For convenience, a lightlike hypersurface which is tangent to ζ will be called a *tangential lightlike hypersurface*.

Let ξ and N be the lightlike normal and lightlike transversal sections, respectively. Then, in view of the first relation of (2.2) and the assumption ζ belongs to $S(TM)$, we have

$$\bar{g}(\bar{\phi}\xi, \bar{\phi}\xi) = \bar{g}(\xi, \xi) - \eta(\xi)\eta(\xi) = 0, \tag{3.1}$$

$$\bar{g}(\bar{\phi}N, \bar{\phi}N) = \bar{g}(N, N) - \eta(N)\eta(N) = 0. \tag{3.2}$$

It follows from (3.1) and (3.2) that $\bar{\phi}\xi$ and $\bar{\phi}N$ are lightlike vector fields too. Moreover, by relation (2.4), we have $\bar{g}(\bar{\phi}\xi,\xi) = -\bar{g}(\xi,\bar{\phi}\xi)$. It follows that

$$\bar{g}(\bar{\phi}\xi,\xi) = 0. \tag{3.3}$$

We can see, from (3.3), that $\bar{\phi}\xi$ is tangent to TM, as its transversal component vanishes.

Next, let us consider a screen distribution $S(TM)$ containing $\bar{\phi}TM^{\perp}$ as a vector subbundle. Then, in view of (2.7), one can easily see that N is orthogonal to $\bar{\phi}\xi$. Moreover, by relation (2.4), we have

$$\bar{g}(\bar{\phi}N,\xi) = -\bar{g}(N,\bar{\phi}\xi) = 0, \quad \bar{g}(\bar{\phi}N,N) = 0. \tag{3.4}$$

The relations in (3.4) indicate that $\bar{\phi}N$ is tangent to TM and in particular, it belongs to $S(TM)$. Thus, $\bar{\phi}\mathrm{tr}(TM)$ is also a vector subbundle of $S(TM)$. In view of the first relation in (2.2) and Theorem 2.1, we have

$$\bar{g}(\bar{\phi}\xi,\bar{\phi}N) = \bar{g}(\xi,N) - \eta(\xi)\eta(N) = 1. \tag{3.5}$$

It is then easy to see, from (3.5), that $\bar{\phi}TM^{\perp}\oplus\bar{\phi}\mathrm{tr}(TM)$ is a non-degenerate vector subbundle of $S(TM)$of rank 2.

Since ζ is tangent to M, and that $\bar{g}(\bar{\phi}\xi,\zeta) = \bar{g}(\bar{\phi}N,\zeta) = 0$, then there exists a non-degenerate distribution D_0 on TM such that

$$S(TM) = \{\bar{\phi}TM^{\perp}\oplus\bar{\phi}\mathrm{tr}(TM)\}\perp D_0\perp\langle\zeta\rangle_{\mathbb{R}}. \tag{3.6}$$

Here, $\langle\zeta\rangle_{\mathbb{R}}$ denotes the line bundle spanned by ζ. It is easy to check that D_0 is an almost complex distribution with respect to $\bar{\phi}$, i.e. $\bar{\phi}D_0 = D_0$. Throughout this paper, we assume that D_0 is nontrivial, i.e. $D_0 \neq \{0\}$.

Then, TM decomposes as

$$TM = TM^{\perp}\perp\{\overline{\phi}TM^{\perp}\oplus\overline{\phi}\mathrm{tr}(TM)\}\perp D_0\perp\langle\zeta\rangle_{\mathbb{R}}. \tag{3.7}$$

If we set $D := TM^{\perp}\perp\bar{\phi}TM^{\perp}\perp D_0$ and $D' = \bar{\phi}\mathrm{tr}(TM)$, then

$$TM = D\oplus D'\perp\langle\zeta\rangle_{\mathbb{R}}. \tag{3.8}$$

Here, D is an almost complex distribution and D' is carried by $\bar{\phi}$ into the transversal bundle. Let us consider the two lightlike sections:

$$U := -\bar{\phi}N, \quad V := -\bar{\phi}\xi. \tag{3.9}$$

Then, from (3.8), any X tangent to M is written as

$$X = RX + u(X)U + \eta(X)\zeta, \tag{3.10}$$

where R is a projection morphism of TM onto D and u is a differential 1-form defined on M by

$$u(X) = g(V,X), \tag{3.11}$$

for any X tangent to M. Applying $\bar{\phi}$ to (3.10) and using (2.1) and (3.9), we get

$$\bar{\phi}X = \phi X + u(X)N, \tag{3.12}$$

where ϕ is a (1,1)-tensor field defined on M by $\phi X = \bar{\phi}RX$. Furthermore, applying $\bar{\phi}$ to (3.12), we have

$$\phi^2 X = -X + \eta(X)\zeta + u(X)U, \quad u(U) = 1, \tag{3.13}$$

for all X tangent to M. It is easy to show that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y), \tag{3.14}$$

where v is a differential one-form locally defined on M by

$$v(X) = g(X, U), \tag{3.15}$$

for any X) tangent to M. Note that

$$g(\phi X, Y) = -g(X, \phi Y) - u(X)\theta(Y) - u(Y)\theta(X), \tag{3.16}$$

for all X and Y tangent to M.

Remark 3.1 In view of relation (3.16), ϕ is skew-symmetric with respect to g on $S(TM)$.

In what follows, we construct a lightlike hypersurface of an indefinite Sasakian manifold, tangent to ζ. To that end, $(\mathbb{R}_q^{2n+1}, \bar{\phi}_0, \zeta, \eta, \bar{g})$ will denote the manifold $\mathbb{R}_q^{2n+1}$ with its usual Sasakian structure given by [4]:

$$\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{n} y^i dx^i\right), \quad \zeta = 2\partial z,$$

$$\bar{g} = \eta\otimes\eta + \frac{1}{4}\left(-\sum_{i=1}^{q/2} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{n} dx^i \otimes dx^i + dy^i \otimes dy^i\right),$$

$$\bar{\phi}_0\left(\sum_{i=1}^{n}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z\right) = \sum_{i=1}^{n}(Y_i\partial x^i - X_i\partial y^i) + \sum_{i=1}^{n} Y_i y^i \partial z,$$

where $(x^i; y^i; z)$ are the Cartesian coordinates.

Example 3.1 Let $\bar{M} = (\mathbb{R}_2^7, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,+,+,-,+,+,+)$ with respect to canonical basis

$$\{\partial x^1, \partial x^2, \partial x^3, \partial y^1, \partial y^2, \partial y^3, \partial z\}.$$

Suppose M is a submanifold of $\mathbb{R}^7_2$ defined by $x^1 = y^3$. It is easy to see that a local frame of TM is given by

$$\xi = 2(\partial x^1 + \partial y^3 + y^1\partial z), \quad Z_1 = 2(\partial x^3 - \partial y^1 + y^3\partial z),$$
$$Z_2 = 2(\partial x^2 + y^2\partial z), \quad Z_3 = 2\partial y^2, \quad Z_4 = \partial x^3 + \partial y^1 + y^3\partial z,$$
$$Z_6 = \zeta = 2\partial z.$$

Hence, $TM^\perp = \mathrm{Span}\{\xi\}$, $\bar{\phi}_0 TM^\perp = \mathrm{Span}\{Z_1\}$. Note that $TM^\perp \cap \bar{\phi}_0 TM^\perp = \{0\}$. Next, $\bar{\phi}_0 Z_2 = -Z_3$, which implies that $D_0 = \mathrm{Span}\{Z_2, Z_3\}$ is invariant with respect to $\bar{\phi}_0$. By a direct calculation we see that $\mathrm{tr}(TM)$ is spanned by

$$N = -\partial x^1 + \partial y^3 - y^1\partial z,$$

such that $\bar{\phi}_0 N = Z_4$. It is easy to see that the vector fields U, V of (3.9) are given by $U = -Z_4$ and $V = -Z_1$. Hence, M is a lightlike hypersurface tangent to ζ.

Lemma 3.1 *On any lightlike hypersurface M of an indefinite Sasakian manifold $\bar{M}$, tangent to ζ, the following relations holds:*

$$\nabla_X \zeta = -\phi X, \tag{3.17}$$
$$B(X, \zeta) = -u(X), \tag{3.18}$$
$$C(X, \zeta) = -v(X), \tag{3.19}$$
$$B(X, U) = C(X, V), \tag{3.20}$$
$$(\nabla_X \phi)(Y) = g(X, Y)\zeta - \eta(Y)X - B(X, Y)U + u(Y)A_N X, \tag{3.21}$$

for all X and Y tangent to M.

Proof. Follows by straightforward calculations, while considering the relations (2.2), (2.3), (2.8) and (2.9).

□

As an immediate consequence of Lemma 3.1, we have the following.

Proposition 3.2 *There is no tangential lightlike hypersurface of in an indefinite Sasakian manifold such that*

1. *C is parallel, i.e. $\nabla C = 0$;*
2. *ϕ is parallel with respect to ∇, i.e. $\nabla\phi = 0$.*

Proof. Setting $X = PZ = \zeta$ and $Y = \xi$ in (2.17), we get

$$(\nabla_\zeta C)(\xi, \zeta) = \zeta \cdot C(\xi, \zeta) - C(\nabla_\zeta \xi, \zeta) - C(\xi, \nabla^*_\zeta \zeta). \tag{3.22}$$

But, in view of (2.9), (3.17), (3.18) and (3.19), we have

$$C(\xi, \zeta) = -v(\xi) = -g(\xi, U) = 0. \tag{3.23}$$
$$C(\nabla_\zeta \xi, \zeta) = -v(\nabla_\zeta \xi) = v(A^*_\xi \zeta) = B(U, \zeta) = -u(U) = -1. \tag{3.24}$$
$$C(\xi, \nabla^*_\zeta \zeta) = g(A_N \xi, \nabla_\zeta \zeta) = -g(A_N \xi, \phi\zeta) = 0. \tag{3.25}$$

Replacing (3.23), (3.24) and (3.25) in (3.22), we get

$$(\nabla_\zeta C)(\xi, \zeta) = 1. \tag{3.26}$$

It is clear from (3.26) that C is never parallel. Next, setting $X = \xi$ and $Y = \zeta$ in (3.21), and noting that $u(\zeta) = 0$, we get

$$(\nabla_\xi \phi)(\zeta) = -\xi,$$

which clearly shows that ϕ is never parallel.

□

We also have the following important lemma.

Lemma 3.3 *U and V, in (3.9), satisfy:*

$$\nabla_X U = \phi A_N X - \theta(X)\zeta + \tau(X)U, \tag{3.27}$$
$$\nabla_X V = \phi A^*_\xi X - \tau(X)V, \tag{3.28}$$

for any X tangent to M.

Proof. Letting $Y = U$ in (3.21) and using the fact $\phi U = 0$, we get

$$-\phi\nabla_X U = v(X)\zeta - B(X, U)U + A_N X, \tag{3.29}$$

for all X tangent to M. Applying ϕ to (3.29) and using the facts $\phi\zeta = 0$ and $\phi U = 0$, we have

$$-\phi^2\nabla_X U = \phi A_N X. \tag{3.30}$$

Then, in view of (3.13) and (3.30), we see that

$$\nabla_X U = \phi A_N X + \eta(\nabla_X U)\zeta + u(\nabla_X U)U. \tag{3.31}$$

But, a direct calculation reveals that

$$\eta(\nabla_X U) = g(\nabla_X U, \zeta) = -\bar{g}(U, \bar{\nabla}_X \zeta) = \bar{g}(U, \bar{\phi}X) = -\theta(X), \tag{3.32}$$

in which we have used (2.3) and (3.9). On the other hand,

$$u(\nabla_X U) = \bar{g}(\bar{\nabla}_X \bar{\phi}N, \bar{\phi}\xi) = \bar{g}(\bar{\phi}\,\bar{\nabla}_X N, \bar{\phi}\xi) = \bar{g}(\bar{\nabla}_X N, \xi) = \tau(X), \tag{3.33}$$

in which we have used the second relation in (2.2), and relation (2.8) and (3.9). Thus, putting (3.32) and (3.33) in (3.31) proves (3.27) of the lemma. A proof of (3.28) also follows easily from (3.21) by putting $Y = V$; equivalently, just set $Y = \xi$ and use the fact $\phi\xi = -V$.

□

In reference to Lemma 3.3, we prove the following result.

Corollary 3.4 *There is no tangential lightlike hypersurface $(M, g, S(TM))$ of an indefinite Sasakian manifold $\bar{M}$, such that*

1. U is parallel with respect to ∇;

2. V is parallel with respect to ∇.

Proof. Assume that U is parallel with respect to ∇, then $\nabla_X U = 0$, for all X tangent to M. With this condition, (3.27) gives

$$\phi A_N X - \theta(X)\zeta + \tau(X)U = 0. \tag{3.34}$$

As ζ is orthogonal to U, we take the inner product of (3.34) with respect to ζ to get

$$\theta(X) = g(\phi A_N X, \zeta) = -g(A_N X, \phi\zeta) = 0, \tag{3.35}$$

in which we have used (3.16), $\phi\zeta = 0$ along with the fact that A_N is a screen-valued operator. As ζ is a unit spacelike vector field, it follows from (3.35) that $\theta(X) = 0$, for any X tangent to M, which is a contradiction to the fact $\theta(\xi) = \bar{g}(\xi, N) = 1$.

Next, assume that V is parallel with respect to ∇, then $\nabla_X V = 0$, for any X tangent to M. It then follows from (3.28) that

$$\phi A^*_\xi X = \tau(X)V, \tag{3.36}$$

for all X tangent to M. Since N is orthogonal to V, (3.36) and (3.12) leads to

$$0 = \bar{g}(\phi A^*_\xi X, N) = -g(A^*_\xi X, \bar{\phi}N) = g(A^*_\xi X, U) = B(X, U), \tag{3.37}$$

for any X tangent to M. Replacing X with ζ to (3.37) gives $B(U, \zeta) = 0$, which is a contradiction to (3.18), which says $B(U, \zeta) = -u(U) = -1$. □

Remark 3.2 The assumptions U and V being parallel have been assumed in [10, Proposition 2, p. 1375], which were later used in [4, Lemma 7.2.1, p. 315]. With such assumptions, the authors of [10] and [4] deduce some more results on the type numbers of M and $S(TM)$ see, for instance, Corollary 3 in [10] and Corollary 7.2.2 in [4]. We, however, stress that the results reported therein may not be true since U and V are *never* parallel with respect to ∇, by Corollary 3.4 above.

Corollary 3.5 *There is no tangential lightlike hypersurface $(M, g, S(TM))$ of an indefinite Sasakian manifold $\bar{M}$, such that U or ξ are conformal vector fields.*

Proof. Suppose that U is conformal on M. Then, there exist some smooth function ψ on M such that

$$(\pounds_U g)(X, Y) = 2\psi g(X, Y), \tag{3.38}$$

for any X and Y tangent to M. Here, $\pounds_U$ denotes the Lie-derivative with respect to U. In particular, U is Killing if $\psi = 0$.

Now, expanding the left hand side of (3.38) leads to

$$(\nabla_U g)(X,Y) + g(\nabla_X U, Y) + g(X, \nabla_Y U) = 2\psi g(X,Y). \tag{3.39}$$

So applying (2.10) to (3.39) gives

$$\begin{aligned} B(U,Y)\theta(X) + B(U,X)\theta(Y) \\ + g(\nabla_X U, Y) + g(X, \nabla_Y U) = 2\psi g(X,Y). \end{aligned} \tag{3.40}$$

Replacing X by ξ to (3.40) and using (2.11) leads to

$$B(U,Y) + g(\nabla_\xi U, Y) = 0, \tag{3.41}$$

for all Y tangent to M. Then, applying (3.27) to (3.41) gives

$$B(U,Y) + g(\phi A_N \xi, Y) - \eta(Y) + \tau(\xi)v(Y) = 0. \tag{3.42}$$

Replacing Y with ζ in (3.42) leads to $B(U,\zeta) = 1$, which is a contradiction to (3.18), which says $B(U,\zeta) = -1$. On the other hand, when ξ is conformal, a direct calculation, using (2.9) and (2.11) leads to

$$B(X,Y) = -\psi g(X,Y), \tag{3.43}$$

for all X and Y tangent to M. Replacing X with U and Y with ζ in (3.43) gives $B(U,\zeta) = 0$, which is also a contradiction to $B(U,\zeta) = -1$.

□

We close this section by discussing the integrability of the distributions $D \perp \langle\zeta\rangle_{\mathbb{R}}$ and $\bar{\phi}\mathrm{tr}(TM) \perp D_0 \perp \langle\zeta\rangle_{\mathbb{R}}$, for later use in this paper.

Proposition 3.6 *Suppose that M is a tangential lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. Then $D \perp \langle\zeta\rangle_{\mathbb{R}}$ is integrable if, and only if,*

$$B(X,\phi Y) = B(Y,\phi X),$$

for any X and Y tangent to $D \perp \langle\zeta\rangle_{\mathbb{R}}$.

Proof. In view of the second relation in (2.2) and (2.4), we derive

$$\begin{aligned} g(\nabla_X Y, \bar{\phi}\xi) &= \bar{g}(\bar{\nabla}_X Y, \bar{\phi}\xi) \\ &= -\bar{g}(\bar{\phi}\bar{\nabla}_X Y, \xi) \\ &= -\bar{g}(\bar{\nabla}_X \bar{\phi} Y - g(X,Y)\zeta + \eta(Y)X, \xi) \\ &= -\bar{g}(\bar{\nabla}_X \bar{\phi} Y, \xi), \end{aligned} \tag{3.44}$$

for any X and Y tangent to M. In particular, if X and Y are tangent to $D \perp \langle\zeta\rangle_{\mathbb{R}}$, (3.44) and the first relation in (2.8) give

$$\bar{g}(\nabla_X Y, \bar{\phi}\xi) = -g(\nabla_X \phi Y, \xi) - B(X,\phi Y) = -B(X,\phi Y). \tag{3.45}$$

Inter-changing X and Y in (3.45), and subtracting the new relation from (3.45), we get

$$\bar{g}([X,Y],\bar{\phi}\xi) = B(Y,\phi X) - B(X,\phi Y), \tag{3.46}$$

for any X and Y are tangent to $D \perp \langle\zeta\rangle_{\mathbb{R}}$. Finally, our claim follows from relations (3.8) and (3.46).

□

In a similar way, we have the following:

Proposition 3.7 *Suppose that M is a tangential lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. Then, the distribution $\bar{\phi}\mathrm{tr}(TM) \perp D_0 \perp \langle\zeta\rangle_{\mathbb{R}}$ is integrable if, and only if,*

$$C(X,Y) = C(Y,X), \quad C(X,\phi Y) = C(Y,\phi X),$$

for any X and Y tangent to $\bar{\phi}\mathrm{tr}(TM) \perp D_0 \perp \langle\zeta\rangle_{\mathbb{R}}$.

4 U and V as principal directions

Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. A vector field X tangent to M is said to be a principal direction or principal vector field with respect to A^*_ξ (resp. A_N) if there exist a smooth function f on M such that $A^*_\xi X = fPX$ (resp. $A_N X = fPX$). A smooth distribution D on M is said to be a principal distribution with respect to A^*_ξ (resp. A_N) if every vector field of D is a principal vector field with respect to A^*_ξ (resp. A_N). That is, $A^*_\xi X = fPX$ (resp. $A_N X = fPX$), for all $X \in \Gamma(D)$, for a certain smooth function σ. It is easy to see that if $D \equiv TM$, then M becomes a totally umbilic [3, p. 106] (resp. totally screen umbilic [3, p. 110]) lightlike hypersurface.

We have the following result:

Theorem 4.1 *There is no tangential lightlike hypersurface $(M, g, S(TM))$ of an indefinite Sasakian manifold $\bar{M}$ such that*

1. *U and ζ are principal directions with respect to A^*_ξ;*
2. *V is a principal direction with respect to A_N. Moreover, if C is symmetric on $S(TM)$, then ζ is never a principal direction with respect to A_N.*

Proof. Assume that U is a principal direction with respect to A^*_ξ, then $A^*_\xi U = fU$ for some smooth function f. It then follows from (3.18), with $X = U$, that $-u(U) = f\eta(U) = 0$, that is $u(U) = 0$, which is a contradiction to $u(U) = 1$. On the other hand, if ζ is principal, (3.18) gives $-u(X) = f\eta(X)$, for all $X \in \Gamma(TM)$. Taking $X = U$, in this relation we get $-u(U) = f\eta(U) = 0$, another contradiction. The second case follows from (3.19) in a similar way by replacing X with V.

□

In case of a lightlike hypersurface of an indefinite Kaehler manifold, Jin [9, Definitions 5.1 and 5.8] defines *Hopf* and *quasi Hopf* lightlike hypersurfaces of an indefinite Kaehler manifolds according to U being a principal direction with respect to A^*_ξ and A_N, respectively. We have the following for a lightlike hypersurface of an indefinite Sasakian manifold:

Corollary 4.2 *There is no tangential Hopf lightlike hypersurface $(M, g, S(TM))$ of an indefinite Sasakian manifold $\bar{M}$.*

We also have the following regarding totally umbilic [3, p. 106] and totally screen umbilic [3, p. 110] lightlike hypersurfaces:

Corollary 4.3 *There is no any tangential totally umbilic or totally screen umbilic lightlike hypersurface in an indefinite Sasakian manifold $\bar{M}$.*

Next, we prove the following result in case U is principal with respect to A_N.

Theorem 4.4 *Let $(M, g, S(TM))$ be a tangential lightlike hypersurface of an indefinite Sasakian space form $\bar{M}(c)$, such that U is a principal vector field with respect to A_N, that is $A_N U = fU$, for some smooth function f. Then, $f = B(U, U)$ and*

$$\begin{aligned} Xf + f\tau(X) - U \cdot B(X,U) \\ - 2B(\phi A_N X, U) + B(\nabla_U X, U) - \frac{3c+5}{4}\theta(X) = 0, \end{aligned} \tag{4.1}$$

$$\begin{aligned} U \cdot C(X,U) - 2C(X,U)\tau(U) \\ - C(\zeta, U)\theta(X) - C(\nabla_U X, U) + C(\phi A_N X, U) = 0, \end{aligned} \tag{4.2}$$

for all X tangent to M.

Proof. The first assertion follows from (3.20) and $A_N U = fU$. On the other hand, using (2.5) and (2.14), we get

$$\begin{aligned} (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) \\ = \frac{c-1}{4}\{\bar{g}(\bar{\phi}Y, Z)u(X) - \bar{g}(\bar{\phi}X, Z)v(Y) - 2\bar{g}(\bar{\phi}X, Y)v(Z)\}, \end{aligned} \tag{4.3}$$

for all X, Y and Z tangent to M. Replacing Y and Z by U in (4.3), we get

$$\begin{aligned} (\nabla_X B)(U,U) - (\nabla_U B)(X,U) \\ + f\tau(X) - B(X,U)\tau(U) = \frac{3}{4}(c-1)\theta(X), \end{aligned} \tag{4.4}$$

for all X tangent to M. Then, by direct calculations while considering (3.27) and (3.18), we have

$$\begin{aligned} (\nabla_X B)(U,U) &= Xf - 2B(\nabla_X U, U) \\ &= Xf - 2B(\phi A_N X, U) - 2\theta(X) - 2f\tau(X). \end{aligned} \tag{4.5}$$

On the other hand,

$$\begin{aligned}(\nabla_U B)(X,U) &= U\cdot B(X,U) - B(\nabla_U X,U) - B(X,\nabla_U U)\\ &= U\cdot B(X,U) - B(\nabla_U X,U) - \tau(U)B(X,U),\end{aligned} \tag{4.6}$$

in which we have used (3.27) and the fact $\phi U = 0$. Replacing (4.5) and (4.6) in (4.4) proves (4.1). Next, using (2.5) and (2.15), we get

$$(\nabla_X C)(U,U) - (\nabla_U C)(X,U) + C(X,U)\tau(U) = 0, \tag{4.7}$$

for all X tangent to M. But, in view of (3.27) and the relations

$$C(U,U) = fg(U,U) = 0, \quad C(U,\nabla^*_X U) = fg(U,\nabla^*_X U) = 0,$$

we derive

$$(\nabla_X C)(U,U) = -C(\phi A_N X,U) + C(\zeta,U)\theta(X). \tag{4.8}$$

In a similar way, we have

$$(\nabla_U C)(X,U) = U\cdot C(X,U) - C(\nabla_U X,U) - C(X,U)\tau(U), \tag{4.9}$$

for all X tangent to M. Then, replacing (4.8) and (4.9) in (4.7), we get (4.2).

□

It is well-known that B is a symmetric $(0,2)$-tensor on M, and degenerates in the direction of ξ, i.e., $B(\xi,\cdot) = 0$. On the other hand, C is generally non-symmetric on M and not even degenerate in the direction of ξ. But we note that if some conditions are imposed on the lightlike hypersurface M then C may be forced to be symmetric and thus obtain some useful information about M. Such conditions include M being *screen integrable*, *contact totally screen umbilic*, screen conformal, parallel screen distribution. The last two in this list force C to be symmetric with an additional condition $C(\xi,P\cdot) = 0$. We, however, note that M being screen integrable secures the symmetry of C on $S(TM)$ but does not lead to $C(\xi,P\cdot) = 0$. In fact, if $S(TM)$ is integrable, we see that $\theta([X,Y]) = 0$, for all X and Y tangent to $S(TM)$. This tell us that $C(X,Y) = C(Y,X)$ on $S(TM)$. Thus, let M be a screen integrable lightlike hypersurface and set

$$\sigma(X) = C(\xi,PX), \tag{4.10}$$

for any X tangent to M.

Next, let us denote by $\tilde{P}$ the projection of TM onto $\langle\xi\rangle^{\perp}_{\mathbb{R}} = D\oplus D'$. Then, according to (3.8), any X tangent to M can be decomposed as

$$X = \tilde{P}X + \eta(X)\zeta. \tag{4.11}$$

With this decomposition, Theorem 4.4 leads to the following result.

Corollary 4.5 *Let $(M, g, S(TM))$ be a screen integrable tangential lightlike hypersurface of an indefinite Sasakian space form $\bar{M}(c)$, with U as a principal direction with respect to A_N. The relations of Theorem 4.4 give*

$$\xi f - f\tau(\xi) - 2C(V, P\phi A_N\xi) = B(\tilde{P}A^*_\xi U, U) + \frac{3}{4}(c+3), \tag{4.12}$$

$$U \cdot \sigma(U) - \sigma(U)\tau(U) + \sigma(U)^2 = -f^2. \tag{4.13}$$

In particular, if $\sigma = 0$ then $f = 0$ and

$$c < -3, \;\; c = -3 \;\; or \;\; c > -3, \tag{4.14}$$

according to $B(\tilde{P}A^*_\xi U, U) > 0$, $B(\tilde{P}A^*_\xi U, U) = 0$ or $B(\tilde{P}A^*_\xi U, U) < 0$, respectively.

Proof. The first relation follows from (4.1) by replacing X with ξ and noting that

$$\begin{aligned} B(\nabla_U\xi, U) &= -B(A^*_\xi U, U) = -B(\tilde{P}A^*_\xi U, U) - \eta(A^*_\xi U)B(\zeta, U) \\ &= -B(\tilde{P}A^*_\xi U, U) - \eta(A^*_\xi U)B(\zeta, U) \\ &= -B(\tilde{P}A^*_\xi U, U) - u(U)^2 = -B(\tilde{P}A^*_\xi U, U) - 1, \end{aligned}$$

in which we have used (2.9), the fact $A^*_\xi U = \tilde{P}A^*_\xi U + \eta(A^*_\xi U)\zeta$ by (4.11) and (3.18). The second relation follows from (4.2) by replacing X by ξ, and noting that

$$C(\zeta, U) = C(U, \zeta) = -v(U) = 0, \tag{4.15}$$

$$C(\nabla_U\xi, U) = -C(A^*_\xi U, U) - C(\xi, U)\tau(U) = -f^2 - \sigma(U)\tau(U), \tag{4.16}$$

$$C(\phi A_N\xi, U) = C(P\phi A_N\xi, U) + \theta(\phi A_N\xi)C(\xi, U) = \sigma(U)^2, \tag{4.17}$$

in which we have used the symmetry of C on $S(TM)$ and relations (3.19), (2.9) and the fact $\phi U = 0$. Finally, when $\sigma = 0$, we see from (4.10) that $A_N\xi = 0$. Also, from (4.13) we have $f = 0$. On the other hand, (4.12) gives

$$B(\tilde{P}A^*_\xi U, U) + \frac{3}{4}(c+3) = 0. \tag{4.18}$$

Finally, the relations in (4.14) follows from (4.18) accordingly.

□

Corollary 4.6 *Under the same assumptions of Corollary 4.5, there is no tangential lightlike hypersurface such that $A_N U = fU$ and*

$$U \cdot \sigma(U) - \sigma(U)\tau(U) + \sigma(U)^2 > 0.$$

A tangential lightlike hypersurface of an indefinite Sasakian manifold is said to be totally contact umbilic if there exists a smooth function ρ on M such that $B = \rho \otimes g$ on $\langle\zeta\rangle^{\perp}_{\mathbb{R}} := D \oplus D'$. That is;

$$B(\tilde{P}X, \tilde{P}Y) = \rho g(\tilde{P}X, \tilde{P}Y), \tag{4.19}$$

for all X and Y tangent to M. Using decomposition (4.11) and (3.18), relation (4.19) may be rewritten as

$$B(X,Y) = \rho\{g(X,Y) - \eta(X)\eta(Y)\} - \eta(X)u(Y) - \eta(Y)u(X), \tag{4.20}$$

for all X and Y tangent to M. In case $\rho = 0$, we say that M is totally contact geodesic. Similarly, M is called totally contact screen umbilic if

$$C(\tilde{P}X, \tilde{P}PY) = \mu g(\tilde{P}X, \tilde{P}PY), \tag{4.21}$$

for any X and Y tangent to M. Here, μ is also a smooth function on M. We may rewrite (4.21), using (4.11) and (3.19), as

$$\begin{aligned} C(X, PY) = \mu\{g(X,Y) - \eta(X)\eta(Y)\} \\ - \eta(Y)v(X) + \eta(X)C(\zeta, PY). \end{aligned} \tag{4.22}$$

So, if $S(TM)$ is integrable then C becomes symmetric on the entire $S(TM)$, and we see from (3.19) that

$$C(\zeta, PY) = C(PY, \zeta) = -v(PY). \tag{4.23}$$

Then, applying (4.22) and (4.23), we get

$$C(X, PY) = \mu\{g(X,Y) - \eta(X)\eta(Y)\} - \eta(Y)v(X) - \eta(X)v(Y), \tag{4.24}$$

for all X and Y tangent to M. In case $\mu = 0$, we say that M is totally contact screen geodesic.

Example 4.1 Consider the lightlike hypersurface in Example 3.1. By (2.3), we have $B(X, \zeta) = -u(X)$, for all X tangent to M. By a direct calculation, we have $[\xi, V] = -4\zeta$, and $[\xi, X] = 0$, for all other X tangent to M. Thus, using Koszul formula [11, p. 61], we have $\bar{g}(\bar{\nabla}_X Y, \xi) = 0$, for all X and Y tangent to $D \oplus D'$. Hence, $B = 0$ on $D \oplus D'$. On the other hand, $[N, U] = -\zeta$ and $[N, X] = 0$ for any other X tangent to M. Also, using Koszul formula [11, p. 61], we have $\bar{g}(\bar{\nabla}_X Y, N) = 0$, for any X and Y tangent to $D \oplus D'$. Thus, $C = 0$ on $D \oplus D'$. Note that, as $[\zeta, X] = 0$ for all X tangent to M, we have $C(\zeta, X) = C(X, \zeta) = -v(X)$. Clearly, M is totally contact geodesic, with a screen distribution which is also totally contact geodesic.

Corollary 4.7 *Under the assumptions of Corollary 4.5, if M is totally contact screen umbilic, then M is contact totally geodesic and $c = -3$.*

Proof. Assume that M is contact totally screen umbilic, then f becomes the umbilicity factor, that is $f = \mu$. Moreover, $\sigma(U) = C(\xi, U) = 0$, which means that $f = 0$, which means $\mu = 0$. On the other hand,

$$B(\tilde{P}A^*_\xi U, U) = C(\tilde{P}A^*_\xi U, V) = \mu g(\tilde{P}A^*_\xi U, V) = 0.$$

Thus, by (4.14), $c = -3$.

□

Next, we turn our attention to lightlike hypersurfaces in which $V := -\bar{J}\xi$ is a principal vector field with respect to A^*_ξ. That is, there exist a smooth function f such that $A^*_\xi V = fV$. It then follows from (3.20) that

$$C(V, V) = B(V, U) = fg(V, U) = f.$$

Proposition 4.8 *Let $(M, g, S(TM))$ be a tangential lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$, such that $A^*_\xi V = fV$. Then,*

$$f = -(\nabla_U B)(\zeta, \xi).$$

Proof. In view of (2.11) and (2.16), we have

$$\begin{aligned}(\nabla_U B)(\zeta, \xi) &= U \cdot B(\zeta, \xi) - B(\nabla_U \zeta, \xi) - B(\zeta, \nabla_U \xi) \\ &= -B(\zeta, \nabla_U \xi). \end{aligned} \tag{4.25}$$

On the other hand, by the second relation in (2.9) together with (3.18), we have

$$-B(\zeta, \nabla_U \xi) = u(\nabla_U \xi) = -u(A^*_\xi U) = -B(U, V) = -f. \tag{4.26}$$

Then, our claims follows from (4.25) and (4.26).

□

Corollary 4.9 *f vanishes whenever B is parallel.*

Proposition 4.10 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian space form $\bar{M}(c)$, such that $A^*_\xi V = fV$. Then f satisfies the relation*

$$\begin{aligned}&Xf + f\tau(X) + B(\nabla_V X, U) \\ &\qquad = V \cdot B(X, U) + B(\phi A^*_\xi X, U) - B(X, \phi A_N V), \end{aligned} \tag{4.27}$$

for all X tangent to M. In particular,

$$\xi f + f\tau(\xi) - f^2 = 0 \quad \textit{and} \quad \zeta f + f\tau(\zeta) = 0. \tag{4.28}$$

Proof. In view of (2.5) and (2.14), we have

$$(\nabla_X B)(V,U) - (\nabla_V B)(X,U) + f\tau(X) - \tau(V)B(X,U) = 0, \tag{4.29}$$

for all X tangent to M. On the other hand, using Lemma 3.3 we derive

$$(\nabla_X B)(V,U) = Xf - B(\phi A^*_\xi X, U), \tag{4.30}$$

$$\begin{aligned}(\nabla_V B)(X,U) = {} & V \cdot B(X,U) - B(\nabla_V X, U) \\ & - B(X, \phi A_N V) + \tau(V)B(X,U).\end{aligned} \tag{4.31}$$

Replacing (4.30) and (4.31) in (4.29) gives (4.27). Finally, the two relations in (4.28) follows easily from (4.27) by replacing X with ξ and ζ, respectively. □

We end this section by making the following general observations.

Theorem 4.11 *On any tangential lightlike hypersurface of an indefinite Sasakian manifold, the distributions*

1. $\langle U\rangle_{\mathbb{R}}$, $\langle\zeta\rangle_{\mathbb{R}}$ *and* $\langle U\rangle_{\mathbb{R}} \perp \langle\zeta\rangle_{\mathbb{R}}$ *are never principal with respect to* A^*_ξ*;*
2. $\langle V\rangle_{\mathbb{R}}$ *never principal with respect to* A_N*. Moreover, if* $S(TM)$ *is integrable, then* $\langle\zeta\rangle_{\mathbb{R}}$ *and* $\langle V\rangle_{\mathbb{R}} \perp \langle\zeta\rangle_{\mathbb{R}}$ *are never principal with respect to* A_N*.*

5 Two principal distributions on M

In this section, we discuss the geometry of the distributions $D_0 \perp \bar{\phi}TM^\perp$ and $\bar{\phi}\mathrm{tr}(TM) \perp D_0$ over a tangential lightlike hypersurface $(M, g, S(TM))$, showing that they contain several geometric information about M and the indefinite Sasakian manifold $\bar{M}$. Let us start by proving the following important lemma.

Lemma 5.1 *Let* $(M, g, S(TM))$ *be a tangential lightlike hypersurface of an indefinite Sasakian manifold* $\bar{M}(c)$*. Let* $D_0 \perp \bar{\phi}TM^\perp$ *be a principal distribution on* M *with respect to* A^*_ξ*, that is* $A^*_\xi X = fX$*, for any* X *tangent to* $D_0 \perp \bar{\phi}TM^\perp$*, then,* f *satisfies the differential equations*

$$Xf + f\tau(X) - f^2\theta(X) = 0, \tag{5.1}$$

$$\{Uf + f\tau(U)\}g(Y,Z) = fg(\nabla_Y Z, U) - B(\nabla_Y Z, U) + \frac{c-1}{4}\bar{g}(\bar{\phi}Y, Z), \tag{5.2}$$

for any X *tangent to* $D \perp \langle\zeta\rangle_{\mathbb{R}}$ *and* Y, Z *tangent to* $D_0 \perp \bar{\phi}TM^\perp$*.*

Proof. Take Y and Z tangent to $D_0 \perp \bar{\phi}TM^\perp$. Then, relations (2.5) and (2.14) give

$$\begin{aligned}(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + f\tau(X)g(Y,Z) \\ - f\tau(Y)g(X,Z) = \frac{c-1}{4}\bar{g}(\bar{\phi}Y, Z)u(X),\end{aligned} \tag{5.3}$$

for any X tangent to M. On the other hand, a direct calculation, while considering (2.10) and (2.16), lead to

$$(\nabla_X B)(Y,Z) = (Xf)g(Y,Z), \tag{5.4}$$

$$\begin{aligned}(\nabla_Y B)(X,Z) &= (Yf)g(X,Z) + f^2\theta(X)g(Y,Z)\\ &\quad + fg(\nabla_Y Z,X) - B(\nabla_Y Z,X),\end{aligned} \tag{5.5}$$

for all X tangent to M and Y,Z tangent to $D_0 \perp \bar{\phi}TM^{\perp}$. Replacing (5.4) and (5.5) in (5.3) give

$$\begin{aligned}&\{Xf + f\tau(X) - f^2\theta(X)\}g(Y,Z) - \{Yf + f\tau(Y)\}g(X,Z)\\ &\quad = fg(\nabla_Y Z,X) - B(\nabla_Y Z,X) + \frac{c-1}{4}\bar{g}(\bar{\phi}Y,Z)u(X),\end{aligned} \tag{5.6}$$

for all X tangent to M and Y,Z tangent to $D_0 \perp \bar{\phi}TM^{\perp}$.

Next, any X tangent to $D \perp \langle\zeta\rangle_{\mathbb{R}}$ can be written as

$$X = X' + \theta(X)\xi + \eta(X)\zeta, \tag{5.7}$$

where X' is the component of X along $D_0 \perp \bar{\phi}TM^{\perp}$. Then, applying (2.11), (3.18) and (5.7), we get

$$\begin{aligned}B(\nabla_Y Z,X) &= B(\nabla_Y Z,X') + \theta(X)B(\nabla_Y Z,\xi) + \eta(X)B(\nabla_Y Z,\zeta)\\ &= fg(\nabla_Y Z,X') - \eta(X)u(\nabla_Y Z)\\ &= fg(\nabla_Y Z,X') + \eta(X)g(\nabla_Y V,Z)\\ &= fg(\nabla_Y Z,X') + \eta(X)g(\phi A^*_{\xi}Y,Z)\\ &= fg(\nabla_Y Z,X') + f\eta(X)g(\phi Y,Z),\end{aligned} \tag{5.8}$$

for any X tangent to $D \perp \langle\zeta\rangle_{\mathbb{R}}$ and Y,Z tangent to $D_0 \perp \bar{\phi}TM^{\perp}$. On the other hand, applying (5.7), we have

$$\begin{aligned}fg(\nabla_Y Z,X) &= fg(\nabla_Y Z,X') + f\theta(X)g(\nabla_Y Z,\xi) + f\eta(X)g(\nabla_Y Z,\zeta)\\ &= fg(\nabla_Y Z,X') + f\eta(X)g(\nabla_Y Z,\zeta)\\ &= fg(\nabla_Y Z,X') - f\eta(X)g(\nabla_Y \zeta,Z)\\ &= fg(\nabla_Y Z,X') + f\eta(X)g(\phi Y,Z),\end{aligned} \tag{5.9}$$

for any X tangent to $D \perp \langle\zeta\rangle_{\mathbb{R}}$ and Y,Z tangent to $D_0 \perp \bar{\phi}TM^{\perp}$. Substituting (5.8) and (5.9) in (5.6), we get

$$\{Xf + f\tau(X) - f^2\theta(X)\}g(Y,Z) = \{Yf + f\tau(Y)\}g(X,Z),$$

for any X tangent to $D \perp \langle\zeta\rangle_{\mathbb{R}}$ and Y,Z tangent to $D_0 \perp \bar{\phi}TM^{\perp}$, from which (5.1) follows. Finally, (5.2) follows from (5.6) by replacing X with U. □

Theorem 5.2 *Under the same hypothesis as in Lemma 5.1, if the $\bar{\phi}$-invariant distribution $D \perp \langle\zeta\rangle_{\mathbb{R}}$ is integrable, then the following hold:*

1. *$f = 0$;*
2. *$c = -3$;*
3. *M is, locally, foliated by totally geodesic complex lightlike hypersurfaces Σ of $\bar{M}$, whose radical distribution, screen distribution and transversal bundle are given by $\operatorname{Rad} T\Sigma = \operatorname{Span}\{\xi, \bar{\phi}\xi\}$, $S(T\Sigma) = D_0 \perp \langle\zeta\rangle_{\mathbb{R}}$ and $\operatorname{tr}(T\Sigma) = \operatorname{Span}\{\bar{\phi}N, N\}$, respectively.*

Proof. Considering the relation in Proposition 3.6, we have

$$fg(X, \phi Y) = fg(Y, \phi X), \tag{5.10}$$

for any X and Y tangent to $D_0 \perp \bar{\phi}TM^{\perp}$. It follows from (3.16) and (5.10) that

$$2fg(\phi X, Y) = 0 \tag{5.11}$$

In particular, if X and Y are tangent to the non-degenerate D_0, relation (5.11) give $f = 0$.

As $f = 0$, relation (5.2) reduces to

$$B(\nabla_X Y, U) = \frac{c-1}{4}\bar{g}(\bar{\phi}X, Y), \tag{5.12}$$

for any X and Y tangent to $D_0 \perp \bar{\phi}TM^{\perp}$. Inter-changing X and Y in (5.12), and subtracting the new relation from (5.12), we get

$$B([X, Y], U) = \frac{c-1}{2}g(\bar{\phi}X, Y). \tag{5.13}$$

Since $D \perp \langle\zeta\rangle_{\mathbb{R}}$ is integrable, we see that $[X, Y]$ belongs to $D \perp \langle\zeta\rangle_{\mathbb{R}}$. Then, we may write $[X, Y]$ as

$$[X, Y] = [X, Y]' + \theta([X, Y])\xi + \eta([X, Y])\zeta, \tag{5.14}$$

where $[X, Y]'$ is tangent to $D_0 \perp \bar{\phi}TM^{\perp}$. Using (5.13), (5.14) and the fact that $f = 0$, we have

$$\eta([X, Y])B(U, \zeta) = \frac{c-1}{2}g(\bar{\phi}X, Y), \tag{5.15}$$

for any X and Y tangent to $D_0 \perp \bar{\phi}TM^{\perp}$. But

$$\begin{aligned} \eta([X, Y]) &= g(\nabla_X Y, \zeta) - g(\nabla_Y X, \zeta) \\ &= -g(Y, \nabla_X \zeta) + g(X, \nabla_Y \zeta) \\ &= g(\phi X, Y) - g(\phi Y, X) \\ &= 2g(\phi X, Y). \end{aligned} \tag{5.16}$$

As $B(U,\zeta) = -1$ (see relation (3.18)), relations (5.15) and (5.16) give

$$\frac{c+3}{2} g(\phi X, Y) = 0, \tag{5.17}$$

for any X and Y tangent to $D_0 \perp \bar{\phi} TM^{\perp}$. Since D_0 is non-degenerate, relation (5.17) give $c = -3$.

Next, as $D \perp \langle \zeta \rangle_{\mathbb{R}}$ is integrable, we let Σ be its leaf and ∇^{Σ} the connection on such a leaf. Note that Σ is 2-lightlike submanifold of $\bar{M}$, whose radical and screen distributions are given by $\mathrm{Rad}\, T\Sigma = \mathrm{Span}\{\xi, \bar{\phi}\xi\}$ and $S(T\Sigma) = D_0 \perp \langle \zeta \rangle_{\mathbb{R}}$. Moreover, the transversal bundle of Σ is given by $\mathrm{tr}(T\Sigma) = \mathrm{Span}\{\bar{\phi}N, N\}$. It follows from [4, p. 246] that Σ is a complex lightlike hypersurface of $\bar{M}$. Furthermore, we have the Gauss equation of Σ as

$$\bar{\nabla}_X Y = \nabla^{\Sigma}_X Y + \bar{g}(\bar{\nabla}_X Y, \bar{\phi}\xi)\bar{\phi}N + B(X,Y)N, \tag{5.18}$$

for any X and Y tangent to Σ. From (5.18), we see that the second fundamental form II of Σ is

$$\begin{aligned} II(X,Y) &= \bar{g}(\bar{\nabla}_X Y, \bar{\phi}\xi)\bar{\phi}N + B(X,Y)N & (5.19)\\ &= B(X,\phi Y)U + B(X,Y)N. & (5.20) \end{aligned}$$

Next, we show that $II = 0$. First, any X tangent to Σ can be written as

$$X = X' + \theta(X)\xi + \eta(X)\zeta, \tag{5.21}$$

where X' is tangent to $D_0 \perp \bar{\phi} TM^{\perp}$. Using (5.21), (3.18) and $f = 0$, we have

$$\begin{aligned} B(X,\phi Y) &= B(X',\phi Y) + \eta(X)B(\phi Y,\zeta)\\ &= fg(X',\phi Y) - \eta(X)u(\phi Y) = 0. \end{aligned} \tag{5.22}$$

In the same way, we have

$$\begin{aligned} B(X,Y) &= B(X',Y) + \eta(X)B(Y,\zeta)\\ &= fg(X',Y) - \eta(X)u(Y) = 0, \end{aligned} \tag{5.23}$$

for any X and Y tangent to Σ. It follows from (5.19), (5.22) and (5.23) that $II = 0$, and Σ is totally geodesic in $\bar{M}$.

□

Next, we turn to the distribution $\bar{\phi}\mathrm{tr}(TM) \perp D_0$. In that angle, we first prove the following lemma:

Lemma 5.3 *Let $(M, g, S(TM)$ be a screen integrable tangential lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}(c)$, such that $C(\xi, \cdot) = 0$. Let $\bar{\phi}\mathrm{tr}(TM) \perp D_0$ be a principal distribution on M with respect to A_N, that is $A_N X = fX$, for any X tangent to $\bar{\phi}\mathrm{tr}(TM) \perp D_0$, then*

$$\left\{\xi f - f\tau(\xi) - \frac{c+3}{4}\right\} g(X,Y) = fB(X,Y),$$

for any X and Y tangent to $\bar{\phi}\mathrm{tr}(TM) \perp D_0$.

Proof. From (2.5) and (2.15), we have

$$\begin{aligned}(\nabla_X C)(Y,Z) - (\nabla_Y C)(X,Z) + f\tau(Y)g(X,Z) - f\tau(X)g(Y,Z)\\ = \frac{c+3}{4} g(Y,Z)\theta(X) + \frac{c-1}{4}\bar{g}(\bar{\phi}Y, Z)v(X), \quad (5.24)\end{aligned}$$

for all X tangent to M and Y, Z tangent to $\bar{\phi}\mathrm{tr}(TM) \perp D_0$. On the other hand, using (2.10), the assumption $C(\xi, \cdot) = 0$ and decomposition $X = PX + \theta(X)\xi$, for any X tangent to M, we derive

$$(\nabla_X C)(Y,Z) = (Xf)g(Y,Z) \quad (5.25)$$

$$\begin{aligned}(\nabla_Y C)(X,Z) = (Yf)g(X,Z)\\ + fB(Y,Z)\theta(X) + fg(\nabla^*_Y Z, X) - C(\nabla^*_Y Z, PX), \quad (5.26)\end{aligned}$$

Replacing (5.25) and (5.26) in (5.24) give

$$\begin{aligned}\left\{Xf - f\tau(X) - \frac{c+3}{4}\theta(X)\right\} g(Y,Z) - fB(Y,Z)\theta(X)\\ - \{Yf - f\tau(Y)\}g(X,Z) + C(\nabla^*_Y Z, PX) - fg(\nabla^*_Y Z, X)\\ = \frac{c-1}{4}\bar{g}(\bar{\phi}Y, Z)v(X), \quad (5.27)\end{aligned}$$

for all X tangent to M and Y, Z tangent to $\bar{\phi}\mathrm{tr}(TM) \perp D_0$. Finally, our result follows easily by replacing X with ξ in (5.27).

□

Next, we prove the following result with the help of Lemma 5.3.

Theorem 5.4 *Under the same hypothesis as in Lemma 5.3, if the distribution $\bar{\phi}\mathrm{tr}(TM) \perp D_0 \perp \langle\zeta\rangle_{\mathbb{R}}$ is integrable, then the following hold:*

1. *$f = 0$;*
2. *$c = -3$;*
3. *M is, locally, foliated by totally geodesic lightlike hypersurfaces $\Sigma$of $S(TM)$, whose radical distribution, screen distribution and transversal bundle are given by $\mathrm{Rad}\, T\Sigma = \mathrm{Span}\{\bar{\phi}N\}$, $S(T\Sigma) = D_0 \perp \langle\zeta\rangle_{\mathbb{R}}$ and $\mathrm{tr}(T\Sigma) = \mathrm{Span}\{\bar{\phi}\xi\}$, respectively.*

Proof. Since $\bar{\phi}\mathrm{tr}(TM) \perp D_0 \perp \langle\zeta\rangle_{\mathbb{R}}$ is integrable, Proposition 3.7 lead to

$$fg(X,\phi Y) = fg(Y,\phi X), \tag{5.28}$$

for any X and Y tangent to $\bar{\phi}\mathrm{tr}(TM) \perp D_0 \perp \langle\zeta\rangle_{\mathbb{R}}$. From (5.28), we deduce that $f = 0$ since D_0 is non-degenerate.

Then, Lemma 5.3 lead to

$$\frac{c+3}{4}g(X,Y) = 0, \tag{5.29}$$

for any X and Y tangent to $\bar{\phi}\mathrm{tr}(TM) \perp D_0$. Since D_0 is non-degenerate, relation (5.29) give $c = -3$.

Next, denote by Σ a leaf of $\bar{\phi}\mathrm{tr}(TM) \perp D_0 \perp \langle\zeta\rangle_{\mathbb{R}}$. Note that Σ is lightlike hypersurface of $S(TM)$ such that $T\Sigma^{\perp} = \mathrm{Span}\{\bar{\phi}N\}$ and $S(T\Sigma) = D_0 \perp \langle\zeta\rangle_{\mathbb{R}}$. Moreover, $\mathrm{tr}(T\Sigma) = \mathrm{Span}\{\bar{\phi}\xi\}$. Denote by II the second fundamental form of Σ in $S(TM)$. Then, the Gauss equation of Σ is

$$\begin{aligned}\nabla^*_X Y &= \nabla^{\Sigma}_X Y + II(X,Y)\\ &= \nabla^{\Sigma}_X Y + g(\nabla^*_X Y, U)V,\end{aligned} \tag{5.30}$$

for any X and Y tangent to Σ, where ∇^{Σ} is the connection on Σ. But in view of (3.27), we derive

$$II(X,Y) = g(\nabla^*_X Y, U)V = -g(Y,\nabla_X U)V = C(X,\phi Y)V. \tag{5.31}$$

Note that any X tangent to Σ can be written as $X = X' + \eta(X)\zeta$, where X' is tangent to $\bar{\phi}\mathrm{tr}(TM) \perp D_0$. It, then, follows from (5.31) that

$$\begin{aligned}II(X,Y) &= C(X,\phi Y)V\\ &= C(X',\phi Y) + \eta(X)C(\zeta,\phi Y)\\ &= fg(X',\phi Y) + \eta(X)C(Y,\phi\zeta)\\ &= 0,\end{aligned}$$

in which we have used the second relation in Proposition 3.7, $f = 0$ and $\phi\zeta = 0$. Hence, Σ is totally geodesic in $S(TM)$, which completes the proof. □

In view of relation (5.1.39) of [4, p. 199] and the fact each Σ is totally geodesic, we have the following result:

Corollary 5.5 *Each of the lightlike hypersurfaces Σ mentioned in Theorems 5.2 and 5.4 admit an induced metric connection ∇^{Σ}.*

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