Cauchy problem for Jordan-Moore-Gibson-Thompson equations of nonlinear acoustics with fractional operators

Rabah Djemiat · Bilal Basti · Noureddine Benhamidouche

Abstract In this paper, we examine the existence and uniqueness of solutions under the traveling wave forms for a free boundary Cauchy problem of space-fractional Jordan-Moore-Gibson-Thompson equations of nonlinear acoustics, which describe sound propagation in thermo-viscous elastic terms. It does so by applying the properties of Schauder’s and Banach’s fixed point theorems, while Caputo’s fractional derivative is used as the differential operator. For application purposes, some examples of explicit solutions are provided to demonstrate the usefulness of our main results.

Keywords traveling wave solutions · nonlinear acoustics · Cauchy problem · Jordan-Moore-Gibson-Thompson equations · space-fractional order · existence and uniqueness

Mathematics Subject Classification (2010) 35R11 · 35A01 · 34A08 · 35C06 · 34K37

1 Introduction and statement of results

Nonlinear fractional partial differential equations (PDEs) have been used to model many phenomena in various fields such as mathematics, physics, and the evolution phenomena in different scientific areas. The property of the fractional derivative operators plays an especially crucial role in applied mathematics and physics. They arise, for example, in the study of heat conduction processes, propagation of sound, medicine, and control theory, etc. For further reading on their use, readers can refer to the following books (Samko et al. 1993 [28], Podlubny 1999 [25], Kilbas et al. 2006 [20], Diethelm 2010 [13]).

Exact solutions are often used for second and higher-order nonlinear fractional-order’s PDEs to denote a particular solution. Furthermore, exact solutions of fractional equations are used to mathematically formulate and, thus, aid in defining the solution of physical and other problems, including functions of several variables such as the propagation of heat or sound, etc. (see [14,15,23,26]).
Several mathematical models are used to describe nonlinear acoustics phenomena. For example, in this work, we shall give a fractional model of nonlinear acoustics that is named the space-fractional Jordan-Moore-Gibson-Thompson (JMGT) equation. This equation results from modeling high-frequency ultrasound waves, and is written for $1 < \alpha \leq 2$ as follows

$$
\tau \psi_{ttt} + \mu \psi_{tt} - \kappa^2 \partial_x^2 \psi - \eta \partial_x^\alpha \psi_t = F(x, t, \psi, \psi_x, \psi_{xx}, (\psi_t)_{xx}),
$$

(1.1)

with

$$
\partial_x^\alpha \psi = \begin{cases} 
\partial_x^2 \psi, & \alpha = 2, \\
\frac{1}{\Gamma(2-\alpha)} \int_0^x (x-\tau)^{1-\alpha} \partial_x^2 \psi (\tau, t) d\tau, & 1 < \alpha < 2,
\end{cases}
$$

where the unknown scalar function $\psi = \psi(x, t)$ of a space and time variables $(x, t) \in \Omega$ with

$$
\Omega = \{(x, t) \in \mathbb{R} \times [0, T]; \kappa t \leq x \leq \ell\}, \text{ for } T > 0 \text{ and } \ell \geq \kappa T,
$$
denotes an acoustic velocity.

The fractional JMGT model (1.1) exhibits a variety of dynamical behaviors for solutions, which heavily depend on the positive physical parameters in the equation. To be specific concerning model (1.1), $\kappa$ stands for the speed of sound, and $\tau$ denotes the thermal relaxation in the view of the physical context of acoustic waves. Moreover, the parameter $\delta$ concerns the diffusivity of the sound carrying. See the works of Moore, Gibson and Thompson [21], and Jordan [18], for a detailed insight into their derivation and physical background, and [9, 19, 24] for a selection of results that account for their mathematical analysis.

The space-fractional equation (1.1) appears as a generalization of the Kuznetsov equation (1.2) (see [12]), for $\mu = 1$, $\tau = 0$ and $\alpha = 2$,

$$
\psi_{tt} - \kappa^2 \partial_x^2 \psi - \delta \partial_x^2 \psi_t = F(x, t, \psi, \psi_x, \psi_{xx}, (\psi_t)_{xx}).
$$

(1.2)

Both equations (1.1) and (1.2) are used as models in what is called nonlinear acoustics, and that deals with finite-amplitude wave propagation in fluids and solids and related phenomena. See the books of Beyer [8] or Rudenko and Soluyan [27].

Note that for $F \equiv 0$ and $\alpha = 2$, the PDE (1.1) represents the Moore-Gibson-Thompson equation:

$$
\tau \psi_{tt} + \mu \psi_{tt} - \kappa^2 \partial_x^2 \psi - \delta \partial_x^2 \psi_t = 0,
$$

which have recently been approached from various points of view. The study of the controllability properties of Moore-Gibson-Thompson type equations can be found for instance in [10, 21].

We define the Cauchy problem for $(x, t) \in \Omega$ and $1 < \alpha \leq 2$ as follows

$$
\begin{cases} 
\tau \psi_{tt} + \mu \psi_{tt} - \kappa^2 \partial_x^2 \psi - \delta \partial_x^\alpha \psi_t = F(x, t, \psi, \psi_x, \psi_{xx}, (\psi_t)_{xx}), \\
\psi (\kappa t, t) = u_0 \exp \left( -\frac{\kappa^2}{8} t \right), \ \psi_x (\kappa t, t) = (\psi_t)_x (\kappa t, t) = 0,
\end{cases}
$$

(1.3)
where $\tau, \mu, \kappa, \delta \in \mathbb{R}_+^*$, $u_0 \in \mathbb{C}$, also $F : \Omega \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is a nonlinear function.

The major goal of this work is to determine the existence and uniqueness of the fractional-order’s partial differential equation (1.1), under the traveling wave form

$$
\psi(x,t) = \exp \left( -\frac{\kappa^2}{\delta} t \right) u(x-\kappa t), \text{ with } \kappa, \delta \in \mathbb{R}_+^*.
$$

(1.4)

The basic profile $u$ is not known in advance and is to be identified.

This method permits us to reduce the fractional-order’s PDE (1.1) to a fractional differential equation; the idea is well illustrated with examples in our chapter. This approach (1.4) is promising and can also bring new results for other applications in FPDEs.

For the forthcoming analysis, we impose the following assumptions

(A1) $F$ is a continuous function that is invariant by the change of scale (1.4). It gives us

$$
F(x,t,\psi,\psi_t,\psi_{xx},(\psi_t)_{xx}) = \exp \left( -\frac{\kappa^2}{\delta} t \right) \times
\left( \delta \kappa f(\eta,u,u',u'') - \kappa^3 \tau u''' \right),
$$

(1.5)

where $\eta = x - \kappa t$ and $f : [0,\ell] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function.

(A2) There exist three positive constants $\beta, \gamma, \lambda > 0$ so that the function $f$ given by (1.5) satisfies

$$
|f(\eta,u,v,w) - f(\eta,\bar{u},\bar{v},\bar{w})| \leq \beta |u - \bar{u}| + \gamma |v - \bar{v}| + \lambda |w - \bar{w}|, \quad \forall \beta, \gamma, \lambda > 0,
$$

for each $\eta \in [0,\ell]$, and any $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{C}$.

(A3) There exist four nonnegative functions $a, b, c, d \in C([0,\ell], \mathbb{R}_+)$, such that

$$
|f(\eta,u,v,w)| \leq a(\eta) + b(\eta) |u| + c(\eta) |v| + d(\eta) |w|, \quad \forall \eta \in [0,\ell],
$$

for any $u, v, w \in \mathbb{C}$ and $\eta \in [0,\ell]$.

We denote by $\varpi$ the positive constant defined by

$$
\varpi = \max \left\{ \frac{\ell (|q| + \gamma) + \alpha (|\theta| + \lambda)}{\ell^{1-\alpha} \Gamma(\alpha + 1)}, \frac{\ell (|q| + \gamma') + \alpha (|\theta'| + \lambda')}{\ell^{1-\alpha} \Gamma(\alpha + 1)} \right\},
$$

where $q = \frac{\kappa^2}{\delta^2} \left( \frac{3\kappa^2}{\delta} - 2\mu \right)$, $\theta = \frac{\kappa^2}{\delta} \left( \frac{3\kappa^2}{\delta} - \mu \right)$, and

$$
a^* = \sup_{\eta \in [0,\ell]} a(\eta), \quad b^* = \sup_{\eta \in [0,\ell]} b(\eta), \quad c^* = \sup_{\eta \in [0,\ell]} c(\eta), \quad \text{and} \quad d^* = \sup_{\eta \in [0,\ell]} d(\eta).
$$

Throughout the rest of this paper, we put $p = \frac{\kappa^2}{\delta} \left( \frac{3\kappa^2}{\delta} - \mu \right)$.

Now, we give the principal theorems of this work.
Theorem 1.1 Assume that the assumptions (A1) – (A3) hold. If we put \( \varpi \in (0, 1) \) and
\[
\ell^{\alpha+1} \left( \frac{k^3}{\delta^3} \left| \frac{\tau k^2}{\delta} - \mu \right| + b^* \right) < \Gamma (\alpha + 2) (1 - \varpi), \tag{1.6}
\]
then, there is at least one solution of the Cauchy problem (1.3) on \( \Omega \) in the traveling wave form (1.4).

Theorem 1.2 Assume that the assumptions (A1), (A2) hold. If we put \( \varpi \in (0, 1) \) and
\[
\ell^{\alpha+1} \left( \frac{k^3}{\delta^3} \left| \frac{\tau k^2}{\delta} - \mu \right| + \beta \right) < \Gamma (\alpha + 2) (1 - \varpi), \tag{1.7}
\]
then the Cauchy problem (1.3) admits a unique solution in the traveling wave form (1.4) on \( \Omega \).

2 Preliminary and necessary definitions

In this section, we present the necessary definitions from the fractional calculus theory. By \( C ([0, \ell], \mathbb{C}) \), we denote the Banach space of continuous functions from \([0, \ell]\) into \( \mathbb{C} \) with the norm
\[
\| u \|_\infty = \sup_{\eta \in [0, \ell]} | u (\eta) |.
\]
We start with the definitions introduced in [20] with a slight modification in the notation.

Definition 2.1 ([20]) The left-sided (arbitrary) fractional integral of order \( \alpha > 0 \) of a continuous function \( u : [0, \ell] \to \mathbb{C} \) is given by
\[
\mathcal{I}_0^{\alpha} u (\eta) = \frac{1}{\Gamma (\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} u (\xi) \, d\xi, \quad \eta \in [0, \ell].
\]
\( \Gamma (\alpha) = \int_0^\infty \xi^{\alpha-1} \exp (-\xi) \, d\xi \) is the Euler gamma function.

Definition 2.2 (Caputo fractional derivative [20]) The left-sided Caputo fractional derivative of order \( \alpha > 0 \) of a function \( u : [0, \ell] \to \mathbb{C} \) is given by
\[
\mathcal{C}D_0^{\alpha} u (\eta) = \begin{cases} 
\frac{d^m u (\eta)}{d\eta^m}, & \text{for } \alpha = m \in \mathbb{N}, \\
\left[ \frac{\Gamma (m-\alpha)}{\Gamma (m)} \right]^{\alpha-1} \int_0^\eta \frac{d^m u (\xi)}{d\xi^m} \, d\xi, & \text{for } m - 1 < \alpha < m \in \mathbb{N}^*.
\end{cases}
\]

Lemma 2.3 ([20]) Assume that \( \mathcal{C}D_0^{\alpha} u \in C ([0, \ell], \mathbb{C}) \), for all \( \alpha > 0 \), then
\[
\mathcal{I}_0^{\alpha} \mathcal{C}D_0^{\alpha} u (\eta) = u (\eta) - \sum_{k=0}^{m-1} \frac{u^{(k)} (0)}{k!} \eta^k, \quad m - 1 < \alpha \leq m \in \mathbb{N}^*.
\]
3 Basic-profile’s existence and uniqueness results

Our initial aim is to infer that the function $u$ in (1.4) satisfies an equation that is employed in the definition of traveling wave solutions.

**Theorem 3.1** If the assumption (A1) holds, then the transformation (1.4) reduces the partial differential equation problem of space-fractional order (1.3) to the fractional differential equation of the form

$$C D^{\alpha+1}_0 u(\eta) = g(\eta), \quad \eta \in [0, \ell], \quad (3.1)$$

where

$$g(\eta) = pu(\eta) + qu'(\eta) + \theta u''(\eta) + f(\eta, u(\eta), u'(\eta), u''(\eta)).$$

with the conditions

$$u(0) = u_0 \text{ and } u'(0) = u''(0) = 0. \quad (3.2)$$

**Proof.** The fractional equation resulting from the substitution of expression (1.4) in the original fractional-order’s PDE (1.3), should be reduced to the standard bilinear functional equation (check [5, 6, 11, 14, 15, 17, 22, 23, 26, 29]). First, for $\eta = x - \kappa t$, we get $\eta \in [0, \ell]$ and

$$\tau u'' + \mu u = -\exp\left(-\frac{\kappa^2 t}{\delta}\right) \left[\kappa \delta (pu(\eta) + qu'(\eta) + \theta u''(\eta)) + \kappa^3 \tau u'''(\eta)\right]. \quad (3.3)$$

On the other hand, for $\xi = \tau - \kappa t$, we get

$$\frac{\partial^{\alpha} \psi}{\partial x^\alpha} = \int_\kappa^x \frac{(x - \tau)^{1-\alpha}}{\Gamma(2 - \alpha)} \frac{\partial^2 \psi(\tau, t)}{\partial \tau^2} d\tau = \exp\left(-\frac{\kappa^2 t}{\delta}\right) \int_\kappa^x \frac{(x - \tau)^{1-\alpha}}{\Gamma(2 - \alpha)} \frac{d^2 u(\tau - \kappa t)}{d\tau^2} d\tau = \exp\left(-\frac{\kappa^2 t}{\delta}\right) \int_0^\eta \frac{(\eta - \xi)^{1-\alpha}}{\Gamma(2 - \alpha)} \frac{d^2 u(\xi)}{d\xi^2} d\xi = \exp\left(-\frac{\kappa^2 t}{\delta}\right) C D^{\alpha}_0 u(\eta). \quad (3.4)$$

and

$$\frac{\partial^{\alpha} \psi_t}{\partial x^\alpha} = \int_\kappa^x \frac{(x - \tau)^{1-\alpha}}{\Gamma(2 - \alpha)} \frac{\partial^2 \psi_t(\tau, t)}{\partial \tau^2} d\tau = -\exp\left(-\frac{\kappa^2 t}{\delta}\right) \left(\frac{\kappa^2}{\delta} C D^{\alpha}_0 u(\eta) + \kappa C D^{\alpha+1}_0 u(\eta)\right). \quad (3.5)$$
If we replace (1.5), (3.3), (3.4) and (3.5) in the first equation of (1.3), we get

\[ C^{\alpha+1}D_{0^+} u(\eta) = g(\eta). \]

From the conditions in (1.3), we find

\[ \psi (\kappa t, t) = \exp \left( -\frac{\kappa^2}{\delta^2} t \right) u (\kappa t - \kappa t) = u(0) \exp \left( -\frac{\kappa^2}{\delta^2} t \right), \]

also

\[ \psi_+ (\kappa t, t) = \exp \left( -\frac{\kappa^2}{\delta^2} t \right) u_+ (\kappa t - \kappa t) = u_+ (0) \exp \left( -\frac{\kappa^2}{\delta^2} t \right), \]

and

\[ (\psi)_+ (\kappa t, t) = - \left( \frac{\kappa^2}{\delta} u_+ (\kappa t - \kappa t) + \kappa u''_+ (\kappa t - \kappa t) \right) \exp \left( -\frac{\kappa^2}{\delta^2} t \right) \]
\[ = - \left( \frac{\kappa^2}{\delta} u_+(0) + \kappa u''_+(0) \right) \exp \left( -\frac{\kappa^2}{\delta^2} t \right), \]

which implies that

\[ u(0) = u_0 \text{ and } u_+(0) = u''_+(0) = 0. \]

The proof is complete. \( \square \)

**Lemma 3.2** Assume that \( f : [0, \ell] \times C \times C \times C \to C \) is a continuous function, then the problem (3.1)–(3.2) is equivalent to the integral equation

\[ u(\eta) = u_0 + \frac{1}{\Gamma(\alpha + 1)} \int_0^\eta (\eta - \xi)^\alpha g(\xi) d\xi, \quad \forall \eta \in [0, \ell], \]

where \( g \in C ([0, \ell], C) \) satisfies the functional equation

\[ g(\eta) = p \left( u_0 + T_{1+}^{\alpha+1} g(\eta) \right) + \omega(\eta, g(\eta)), \]

with \( \omega : [0, \ell] \times C \to C \) is a function satisfying

\[ \omega(\eta, g(\eta)) = q T_{0+}^{\alpha} g(\eta) + \theta T_{0+}^{\alpha-1} g(\eta) \]
\[ + f \left( \eta, u_0 + T_{0+}^{\alpha+1} g(\eta), T_{0+}^{\alpha} g(\eta), T_{0+}^{\alpha-1} g(\eta) \right). \]

**Proof.** Using Theorem 3.1, and applying \( T_{0+}^{\alpha+1} \) to the equation (3.1), we obtain

\[ T_{0+}^{\alpha+1} C D_{0^+}^{\alpha+1} u(\eta) = T_{0+}^{\alpha+1} g(\eta). \]

From Lemma 2.3, we simply find

\[ T_{0+}^{\alpha+1} C D_{0^+}^{\alpha+1} u(\eta) = u(\eta) - u_0 - \eta u'(0) - \frac{1}{2} \eta^2 u''(0). \]
Substituting (3.2) gives us

\[ u(\eta) = u_0 + I^{\alpha+1}_0 g(\eta). \]  

(3.6)

As

\[ u'(\eta) = \frac{d}{d\eta} (u_0 + I^{\alpha+1}_0 g(\eta)) = I^{\alpha}_0 g(\eta) \]

and

\[ u''(\eta) = \frac{d^2}{d\eta^2} (u_0 + I^{\alpha+1}_0 g(\eta)) = I^{\alpha-1}_0 g(\eta), \]

then

\[ g(\eta) = pu(\eta) + qu'(\eta) + \theta u''(\eta) + f(\eta, u(\eta), u'(\eta), u''(\eta)) \]

\[ = p \left( u_0 + I^{\alpha+1}_0 g(\eta) \right) + q I^{\alpha}_0 g(\eta) + \theta I^{\alpha-1}_0 g(\eta) \]

\[ + f(\eta, u_0 + I^{\alpha+1}_0 g(\eta), I^{\alpha}_0 g(\eta), I^{\alpha-1}_0 g(\eta)) \]

\[ = p \left( u_0 + I^{\alpha+1}_0 g(\eta) \right) + \omega(\eta, g(\eta)). \]

Otherwise, starting by applying \( C^{\alpha+1}_0 \) on both sides of the equation (3.6) and using the linearity of Caputo’s derivative and the fact that \( C^{\alpha+1}_0 u_0 = 0 \), we find easily (3.1). Furthermore;

\[ u(0) = (u_0 + I^{\alpha+1}_0 g)(0) = u_0 \]

\[ u^{(k)}(0) = I^{\alpha-k+1}_0 g(0) = 0, \text{ for each } k = 1, 2. \]

The proof is complete. \( \square \)

**Theorem 3.3** Assume the assumptions \((A2), (A3)\) hold. If we put \( \varpi \in (0,1) \) and

\[ \frac{\ell^{\alpha+1}|p| + b^\ast}{\Gamma(\alpha + 2)(1 - \varpi)} < 1, \]

(3.7)

then the problem (3.1)–(3.2) has at least one solution on \([0, \ell]\).

**Proof.** To begin the proof, we will transform the problem (3.1)–(3.2) into a fixed point problem. Let us define

\[ A\varphi(\eta) = u_0 + \frac{1}{\Gamma(\alpha + 1)} \int_0^\eta (\eta - \xi)^\alpha g(\xi) d\xi, \]

(3.8)

where

\[ g(\eta) = p\varphi(\eta) + \omega(\eta, g(\eta)), \eta \in [0, \ell], \]

with

\[ \omega(\eta, g(\eta)) = q I^\alpha_0 g(\eta) + \theta I^{\alpha-1}_0 g(\eta) + f(\eta, \varphi(\eta), I^\alpha_0 g(\eta), I^{\alpha-1}_0 g(\eta)). \]
As the assumptions (A2), (A3) hold, we notice that if \( g \in C([0, \ell], \mathbb{C}) \), then \( \mathcal{A}\varphi \) is indeed continuous (see the step 1 in this proof); therefore, it is an element of \( C([0, \ell], \mathbb{C}) \), and is equipped with the standard norm
\[
\|\mathcal{A}\varphi\|_{\infty} = \sup_{\eta \in [0, \ell]} |\mathcal{A}\varphi(\eta)|.
\]

Clearly, the fixed points of \( \mathcal{A} \) are solutions of the problem (3.1)–(3.2).

We demonstrate that \( \mathcal{A} \) satisfies the assumption of Schauder’s fixed point theorem (see [1–4, 7, 16]). This could be proved through three steps.

Step 1: \( \mathcal{A} \) is a continuous operator.

Let \( (\varphi_n)_{n \in \mathbb{N}} \) be a real sequence such that \( \lim_{n \to \infty} \varphi_n = \varphi \) in \( C([0, \ell], \mathbb{C}) \).

Then \( \forall \eta \in [0, \ell] \),
\[
|\mathcal{A}\varphi_n(\eta) - \mathcal{A}\varphi(\eta)| \leq \frac{1}{\Gamma(\alpha + 1)} \int_{0}^{\eta} (\eta - \xi)^{\alpha} |g_n(\xi) - g(\xi)| d\xi, \tag{3.9}
\]
where
\[
\begin{cases}
  g_n(\eta) = p\varphi_n(\eta) + \omega(\eta, g_n(\eta)), \\
  g(\eta) = p\varphi(\eta) + \omega(\eta, g(\eta)).
\end{cases}
\]

We have
\[
|g_n(\eta) - g(\eta)| = |p(\varphi_n(\eta) - \varphi(\eta)) + (\omega(\eta, g_n(\eta)) - \omega(\eta, g(\eta)))|
\leq \left( |p| + \beta \right) \|\varphi_n - \varphi\|_{\infty} + (|q| + \gamma) |\mathcal{I}_{0+}^{\alpha} (g_n(\eta) - g(\eta))|
\quad + (|\theta| + \lambda) |\mathcal{I}_{0+}^{\alpha-1} (g_n(\eta) - g(\eta))|.
\]

As
\[
|\mathcal{I}_{0+}^{\alpha} (g_n(\eta) - g(\eta))| \leq \frac{1}{\Gamma(\alpha)} \int_{0}^{\eta} (\eta - \xi)^{\alpha-1} |(g_n(\xi) - g(\xi))| d\xi
\leq \frac{\ell^{\alpha}}{\Gamma(\alpha + 1)} \|g_n - g\|_{\infty}
\]
and
\[
|\mathcal{I}_{0+}^{\alpha-1} (g_n(\eta) - g(\eta))| \leq \frac{\alpha \ell^{\alpha-1}}{\Gamma(\alpha + 1)} \|g_n - g\|_{\infty}.
\]

Then we get
\[
\|g_n - g\|_{\infty} \leq (|p| + \beta) \|\varphi_n - \varphi\|_{\infty} + \frac{\ell (|q| + \gamma) + \alpha (|\theta| + \lambda)}{\ell^{1-\alpha} \Gamma(\alpha + 1)} \|g_n - g\|_{\infty}
\leq (|p| + \beta) \|\varphi_n - \varphi\|_{\infty} + \varpi \|g_n - g\|_{\infty}.
\]

As \( \varpi \in (0, 1) \), thus
\[
\|g_n - g\|_{\infty} \leq \frac{|p| + \beta}{1 - \varpi} \|\varphi_n - \varphi\|_{\infty}.
\]
Since \( \varphi_n \to \varphi \), we get \( g_n \to g \) when \( n \to \infty \).

Now, let \( z > 0 \) be such that for each \( \eta \in [0, \ell] \), we get
\[
|g_n(\eta)| \leq z, \quad |g(\eta)| \leq z.
\]

Then, we have
\[
\frac{(\eta - \xi)^\alpha}{\Gamma(\alpha + 1)} |g_n(\eta) - g(\eta)| \leq \frac{(\eta - \xi)^\alpha}{\Gamma(\alpha + 1)} [||g_n(\eta)|| + |g(\eta)|] \\
\leq \frac{2z}{\Gamma(\alpha + 1)} (\eta - \xi)^\alpha.
\]

For each \( \eta \in [0, \ell] \), the function \( \xi \to \frac{2z}{\Gamma(\alpha + 1)} (\eta - \xi)^\alpha \) is integrable on \([0, \eta]\), then the Lebesgue dominated convergence theorem and (3.9) imply that
\[
|A\varphi_n(\eta) - A\varphi(\eta)| \to 0 \text{ as } n \to \infty,
\]
and hence
\[
\lim_{n \to \infty} \|A\varphi_n - A\varphi\|_\infty = 0.
\]

Consequently, \( A \) is continuous.

Step 2: Using (3.7), we put the positive real
\[
r \geq \left( |u_0| + \frac{a^* \ell^{\alpha+1}}{\Gamma(\alpha + 2) (1 - \varpi)} \right) \frac{\Gamma(\alpha + 2) (1 - \varpi)}{\Gamma(\alpha + 2) (1 - \varpi) - \ell^{\alpha + 1} (|p| + b^*)},
\]
and define the subset \( H \) as follows
\[
H = \{ \varphi \in C([0, \ell], \mathbb{C}) : \|\varphi\|_\infty \leq r \}.
\]

It is clear that \( H \) is bounded, closed and convex subset of \( C([0, \ell], \mathbb{C}) \).

Let \( A : H \to C([0, \ell], \mathbb{C}) \) be the integral operator defined by (3.8), then \( A(H) \subset H \).

Indeed, we have for each \( \eta \in [0, \ell] \)
\[
|g(\eta)| = |p\varphi(\eta) + \omega(\eta, g(\eta))| \\
\leq a^* + (|p| + b^*) |\varphi(\eta)| + \varpi \|g\|_\infty.
\]

Then, we get
\[
\|g\|_\infty \leq \frac{a^* + (|p| + b^*) r}{1 - \varpi}.
\]
Thus
\[
|\mathcal{A}\varphi(\eta)| \leq |u_0| + \frac{1}{\Gamma(\alpha + 1)} \int_0^\eta (\eta - \xi)^\alpha |g(\xi)| d\xi
\]
\[
\leq |u_0| + \frac{a^\alpha \ell^\alpha}{\Gamma(\alpha + 2)} \frac{(\alpha + 1)}{1 - \omega}
\]
\[
\leq |u_0| + \frac{a^\alpha \ell^\alpha + \frac{\ell}{\Gamma(\alpha + 2)}}{1 - (1 - \omega) \Gamma(\alpha + 2)} \frac{1}{1 - \omega}
\]
\[
\leq \left( |u_0| + \frac{a^\alpha \ell^\alpha}{(1 - \omega) \Gamma(\alpha + 2)} \left( \frac{\ell}{\Gamma(\alpha + 2)} (1 - \omega) \Gamma(\alpha + 2) \right) \right)
\]
\[
+ \frac{\ell}{\Gamma(\alpha + 2)} \frac{(\alpha + 1)}{1 - \omega}
\]
\[
\leq \frac{1}{\Gamma(\alpha + 1)} \int_0^\eta (\eta - \xi)^\alpha |g(\xi)| d\xi
\]

Then \( \mathcal{A}(H) \subset H \).

Step 3: \( \mathcal{A}(H) \) is relatively compact.

Let \( \eta_1, \eta_2 \in [0, \ell] \), \( \eta_1 < \eta_2 \), and \( \varphi \in H \). Then

\[
|\mathcal{A}\varphi(\eta_2) - \mathcal{A}\varphi(\eta_1)| = \frac{1}{\Gamma(\alpha + 1)} \int_0^{\eta_2} (\eta_2 - \xi)^\alpha g(\xi) d\xi
\]
\[
- \int_0^{\eta_1} (\eta_1 - \xi)^\alpha g(\xi) d\xi
\]
\[
\leq \frac{1}{\Gamma(\alpha + 1)} \int_0^\eta \left| ((\eta_2 - \xi)^\alpha - (\eta_1 - \xi)^\alpha) g(\xi) \right| d\xi
\]
\[
+ \frac{1}{\Gamma(\alpha + 1)} \int_{\eta_1}^{\eta_2} (\eta_2 - \xi)^\alpha |g(\xi)| d\xi
\]
\[
\leq \frac{a^\alpha + (|p| + b^*) r}{\Gamma(\alpha + 1)} \int_0^\eta (\eta_2 - \xi)^\alpha - (\eta_1 - \xi)^\alpha \right| d\xi
\]
\[
+ \int_{\eta_1}^{\eta_2} (\eta_2 - \xi)^{\alpha - 1} d\xi.
\]  

We have
\[
(\eta_2 - \xi)^\alpha - (\eta_1 - \xi)^\alpha = -\frac{1}{\alpha + 1} \frac{d}{d\xi} \left( (\eta_2 - \xi)^{\alpha + 1} - (\eta_1 - \xi)^{\alpha + 1} \right),
\]

then
\[
\int_0^\eta |(\eta_2 - \xi)^\alpha - (\eta_1 - \xi)^\alpha| d\xi \leq \frac{1}{\alpha + 1} \left[ (\eta_2 - \eta_1)^{\alpha + 1} + (\eta_2^{\alpha + 1} - \eta_1^{\alpha + 1}) \right],
\]

152
we also have
\[ \int_{\eta_1}^{\eta_2} (\eta_2 - \xi)^\alpha d\xi = -\frac{1}{\alpha + 1} \left[ (\eta_2 - \xi)^{\alpha+1} \right]_{\eta_1}^{\eta_2} \leq \frac{1}{\alpha + 1} (\eta_2 - \eta_1)^{\alpha+1}. \]
Thus (3.10) gives us
\[ |A\varphi(\eta_2) - A\varphi(\eta_1)| \leq \frac{2(\eta_2 - \eta_1)^{\alpha+1} + (\eta_2^{\alpha+1} - \eta_1^{\alpha+1})}{\Gamma(\alpha + 2) (1 - \varpi)} (a^* + (|p| + b^*) r). \]
The right-hand side of the latter inequality tends to zero when \( \eta_1 \to \eta_2 \).

As a consequence of steps 1 to 3, and through Ascoli-Arzelà theorem, we infer the continuity of \( A : H \to H \), its compact nature and its satisfaction of the assumption of Schauder’s fixed point theorem [16]. Therefore, \( A \) has a fixed point which solves the problem (3.1)–(3.2) on \([0, \ell]\).

\[ \square \]

Theorem 3.4 Assume the assumption (A2) holds. If we put
\[ \frac{\ell^{\alpha+1} (|p| + \beta)}{(1 - \varpi) \Gamma(\alpha + 2)} < 1, \quad (3.11) \]
then the problem (3.1)–(3.2) admits a unique solution on \([0, \ell]\).

Proof. Theorem 3.3 states that (3.1)–(3.2) can be rendered a problem of a fixed point (3.8).
Let \( \varphi_1, \varphi_2 \in C([0, \ell], \mathbb{C}) \), then we get
\[ A\varphi_1(\eta) - A\varphi_2(\eta) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\eta (\eta - \xi)^\alpha (g_1(\xi) - g_2(\xi)) d\xi, \]
where
\[ g_i(\eta) = p\varphi_i(\eta) + \omega(\eta, g_i(\eta)), \]
\[ \omega(\eta, g_i(\eta)) = qI_0^\alpha g_i(\eta) + \theta I_{0^+}^{\alpha-1} g_i(\eta) + f(\eta, \varphi_i(\eta), I_0^\alpha g_i(\eta), I_{0^+}^{\alpha-1} g_i(\eta)), \quad \text{for } i = 1, 2. \]
Also
\[ |A\varphi_1(\eta) - A\varphi_2(\eta)| \leq \frac{1}{\Gamma(\alpha + 1)} \int_0^\eta |g_1(\xi) - g_2(\xi)| d\xi. \quad (3.12) \]
We have
\[ \|g_1 - g_2\|_\infty \leq \frac{|p| + \beta}{1 - \varpi} \|\varphi_1 - \varphi_2\|_\infty. \]
From (3.12) we find
\[ \|A\varphi_1 - A\varphi_2\|_\infty \leq \frac{\ell^{\alpha+1} (|p| + \beta)}{(1 - \varpi) \Gamma(\alpha + 2)} \|\varphi_1 - \varphi_2\|_\infty. \]
Thus, according to (3.11), \( A \) is considered a contraction operator.
Banach’s contraction principle (see [1–4,7,16]) helps us infer that \( A \) has only one fixed point which is the unique solution of the problem (3.1)–(3.2) on \([0, \ell]\).
4 Main theorems’ proof

This section demonstrates the proof of the existence and uniqueness of solutions of the given Cauchy problem for a space-fractional JMGT equation of nonlinear acoustics, which is:

\[
\begin{align*}
\tau \psi_{ttt} + \mu \psi_{tt} - \kappa^2 \partial_x^\alpha \psi - \delta \partial_x^\alpha \psi_t &= F(x, t, \psi, \psi_t, \psi_{xx}, (\psi_t)_x) , \\
\psi(\kappa t, t) &= u_0 \exp\left(-\frac{\kappa^2 t}{\delta}\right) , \quad \psi_x(\kappa t, t) = (\psi_t)_x(\kappa t, t) = 0,
\end{align*}
\]

(4.1)

under the traveling wave form

\[
\psi(x, t) = \exp\left(-\frac{\kappa^2 t}{\delta}\right) u(x - \kappa t) , \quad \text{with} \quad \kappa, \delta \in \mathbb{R}_+^* .
\]

(4.2)

**Proof of Theorem 1.1**

Assume the assumptions (A1)–(A3) hold. Using transformation (4.2), the Cauchy problem (4.1) is reduced to fractional order’s ordinary differential equation of the form

\[
C^\alpha_{\delta}D_{\nu}^{\alpha+1} u(\eta) = g(\eta) , \quad \eta \in [0, \ell] ,
\]

(4.3)

where

\[
g(\eta) = pu(\eta) + qu'(\eta) + \theta u''(\eta) + f(\eta, u(\eta), u'(\eta), u''(\eta)) ,
\]

with

\[
p = \frac{\kappa^3}{\delta^2} \left( \frac{\tau \kappa^2}{\delta} - \mu \right) , \quad q = \frac{\kappa^2}{\delta} \left( 3\tau \kappa^2 - 2\mu \right) , \quad \theta = \frac{\kappa}{\delta} \left( 3\tau \kappa^2 - \mu \right) ,
\]

(4.4)

along with the conditions

\[
u(0) = u_0 \quad \text{and} \quad \nu'(0) = \nu''(0) = 0 .
\]

(4.5)

By using (4.4), the condition (1.6) is equivalent to (3.7), which is

\[
\frac{\ell^{\alpha+1} (|p| + b^*)}{\Gamma(\alpha + 2)(1 - \varpi)} < 1 , \quad \text{with} \quad \varpi \in (0, 1) .
\]

Therefore, after proving that problem (4.3)–(4.5) has a solution in Theorem 3.3 when (3.7) holds, we can similarly prove the existence of at least a solution of the Cauchy problem for the space-fractional JMGT equation of nonlinear acoustics (4.1) under the traveling wave form (4.2). This can be achieved if (1.6) holds. The proof is complete.

**Example 1.** If we choose \( \tau = 2, \mu = 4, \alpha = \frac{3}{2}, \delta = 2, \kappa = 1, T = \frac{1}{4} \) and \( \ell = \frac{1}{4} \), we get

\[
\Omega = \left\{ (x, t) \in \mathbb{R} \times \left[ 0, \frac{1}{4} \right] : t \leq x \leq \frac{1}{4} \right\} .
\]
Consequently, the considered problem will be stated as follows
\[
\begin{aligned}
  2\psi_{tt} + 4\psi_t - \partial_x^2 \psi - 2\partial_x^2 \psi_t &= F(x,t,\psi,\psi_x,\psi_{tt},\psi_{xx},(\psi_t)_{xx}), \\
  \psi(t,t) &= u_0 \exp\left(-\frac{1}{2}t\right), \quad \psi_x(t,t) = (\psi_t)_x(t,t) = 0,
\end{aligned}
\]
where
\[
F(x,t,\psi,\psi_x,\psi_{tt},\psi_{xx},(\psi_t)_{xx}) = \frac{\ln \left(4|\psi| + e - 1\right)}{2\exp \left(x - \frac{1}{2}t\right)} \times
\frac{2\exp \left(-\frac{1}{2}t\right) + |\psi| + |\psi_x| + |\psi_{xx}|}{\exp \left(-\frac{1}{2}t\right) + |\psi| + |\psi_x| + |\psi_{xx}|}
+ 2\psi_{tt} + 2(\psi_t)_{xx}.
\]

The transformation
\[
\psi(x,t) = \exp \left(-\frac{1}{2}t\right) \psi(\eta), \quad \text{with} \quad \eta = x - t,
\]
reduces the partial differential equation problem of space-fractional order (4.6) to the ordinary differential equation of fractional order of the form
\[
g(\eta) = -\frac{3}{8} u(\eta) - \frac{5}{4} u'(\eta) - \frac{1}{2} u''(\eta) + f(\eta,u,u',u''), \quad \eta \in \left[0,\frac{1}{4}\right],
\]
with the conditions
\[
u(0) = u_0 \quad \text{and} \quad u'(0) = u''(0) = 0,
\]
where
\[
p = -\frac{3}{8}, q = -\frac{5}{4}, \quad \text{and} \quad \theta = -\frac{1}{2},
\]
and
\[
f(\eta,u,u',u'') = \frac{\ln \left(4\eta + e - 1\right) \left[2 + |u(\eta)| + |u'(\eta)| + |u''(\eta)|\right]}{4\exp(\eta) \left[1 + |u(\eta)| + |u'(\eta)| + |u''(\eta)|\right]} + \frac{1}{4} u(\eta) + u'(\eta) + \frac{1}{2} u''(\eta).
\]
Because \(\ln \left(4\eta + e - 1\right), \exp(\eta)\) are nonnegative continuous functions for any \(\eta \in \left[0,\frac{1}{4}\right]\), the function \(f\) is jointly continuous. Then
\[
f(\eta,u,v) = \frac{\ln \left(4\eta + e - 1\right) \left[2 + |u| + |v| \right]}{4\exp(\eta) \left[1 + |u| + |v| \right]} + \frac{1}{4} u + v + \frac{1}{2} w.
\]
For any \(u,v,w,\tilde{u},\tilde{v},\tilde{w} \in \mathbb{C}\) and \(\eta \in \left[0,\frac{1}{4}\right]\), we get
\[
|f(\eta,u,v,w) - f(\eta,\tilde{u},\tilde{v},\tilde{w})| \leq \frac{1}{2} |u - \tilde{u}| + \frac{5}{4} |v - \tilde{v}| + \frac{3}{4} |w - \tilde{w}|.
\]
Therefore, the assumption \((A2)\) is satisfied with
\[
\beta = \frac{1}{2}, \quad \gamma = \frac{5}{4} \quad \text{and} \quad \lambda = \frac{3}{4}.
\]

Also, we have
\[
|f(\eta, u, v, w)| \leq \frac{\ln (4\eta + e - 1)}{4\exp(\eta)} (2 + |u| + |v| + |w|) + \frac{1}{4} |u| + |v| + \frac{1}{2} |w|.
\]
Thus, the assumption \((A3)\) is satisfied with
\[
\begin{align*}
a(\eta) &= \frac{\ln(4\eta + e - 1)}{2\exp(\eta)}, \\
b(\eta) &= \frac{\ln(4\eta + e - 1)}{4\exp(\eta)} + \frac{1}{4}, \\
c(\eta) &= \frac{\ln(4\eta + e - 1)}{4\exp(\eta)} + 1, \\
d(\eta) &= \frac{\ln(4\eta + e - 1)}{4\exp(\eta)} + \frac{1}{2}.
\end{align*}
\]
We also have
\[
a^* = \frac{1}{2}, \quad b^* = \frac{1}{2}, \quad c^* = \frac{5}{4}, \quad d^* = \frac{3}{4},
\]
with
\[
\varpi = \max \left\{ \frac{\ell(|q| + \gamma) + \alpha(|\theta| + \lambda)}{\ell^{\alpha+1}}, \frac{\ell(|q| + c^*) + \alpha(|\theta| + d^*)}{\ell^{\alpha+1}} \right\} = \frac{5}{3\sqrt{\pi}} < 1.
\]
And the condition \((1.6)\)
\[
\ell^{\alpha+1} \left( \frac{\kappa^3 \gamma^3}{\delta} - \mu \right) + b^* = \frac{7}{256}
\]
\[
< \Gamma(\alpha + 2) (1 - \varpi) \simeq 0.19835.
\]
It follows from theorem 1.1, that the Cauchy problem \((4.6)\) has at least one solution.

**Proof of Theorem 1.2**

Similarly to the steps that we followed during the proof of Theorem 1.1, the existence and uniqueness of a traveling wave solution to problem \((4.1)\) is demonstrated using Theorem 3.4, provided that the condition \((1.7)\) holds true. The proof is complete.

**Example 2.** If we put \(\tau = 3\pi, \mu = 6, \alpha = 1.9, \delta = 3\pi^2, \kappa = \pi, T = \frac{1}{60}\) and \(\ell = \frac{\pi}{60}\), we get
\[
\Omega = \left\{ (x, t) \in \mathbb{R} \times \left[ 0, \frac{1}{60} \right] ; \; \pi t \leq x \leq \frac{\pi}{60} \right\}.
\]
Thus, the studied problem will be written as follows
\[
\begin{align*}
3\pi \psi_{ttt} + 6\psi_{tt} - \pi^2 \psi_{t} + 3\pi^2 \psi_{x} + 3\pi^2 \psi_{x} + 3\pi^2 \psi_{xx} (\psi_{t})_{xx}, \\
\psi(\pi t, t) = u_0 \exp \left( -\frac{t}{3} \right), \quad \psi_x (\pi t, t) = (\psi_{t})_x = 0,
\end{align*}
\]
\[
(4.7)
\]
where
\[ F(x,t,\psi,\psi_x,\psi_{xx},(\psi_t)_{xx}) = \frac{3\pi^3 \exp\left(-\frac{2}{3}t\right) \cos(x-\pi t)}{\exp\left(-\frac{1}{3}t\right) + \left|\psi\right| + \left|\psi_x\right| + \left|\psi_{xx}\right|} + 3\pi^3 \psi_t + 3\pi^3 (\psi_t)_{xx}, \]

The transformation
\[ \psi(x,t) = \exp\left(-\frac{1}{3}t\right) u(\eta), \text{ with } \eta = x - \pi t, \]

reduces the partial differential equation problem of space-fractional order (4.6) to the ordinary differential equation of fractional order of the form
\[ g(\eta) = \frac{\pi - 6}{27\pi^3} u(\eta) + \frac{\pi - 4}{3\pi^2} u'(\eta) + \frac{\pi - 2}{\pi} u''(\eta) + f(\eta, u, u', u''), \]

with the conditions
\[ u(0) = u_0 \text{ and } u'(0) = u''(0) = 0, \]

where
\[ p = \frac{\pi - 6}{27\pi^3}, \quad q = \frac{\pi - 4}{3\pi^2}, \quad \text{and} \quad \theta = \frac{\pi - 2}{\pi}, \]

and
\[ f(\eta, u, u', u'') = \frac{\cos(\eta)}{1 + |u(\eta)| + |u'(\eta)| + |u''(\eta)|} + \frac{1}{9} u(\eta) + \frac{2\pi}{3} u'(\eta) + \left(\pi^2 - \frac{1}{3}\right) u''(\eta). \]

Because \( \cos(\eta) \) is nonnegative continuous function for any \( \eta \in \left[0, \frac{\pi}{60}\right] \), the function \( f \) is jointly continuous. Then
\[ f(\eta, u, v, w) = \frac{\cos(\eta)}{1 + |u| + |v| + |w|} + \frac{1}{9} u + \frac{2\pi}{3} v + \frac{3\pi^2 - 1}{3} w. \]

For any \( u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{C} \) and \( \eta \in \left[0, \frac{\pi}{60}\right] \), we have
\[ |f(\eta, u, v, w) - f(\eta, \bar{u}, \bar{v}, \bar{w})| \leq \frac{10}{9} |u - \bar{u}| + \frac{2\pi + 3}{3} |v - \bar{v}| + \frac{3\pi^2 + 2}{3} |w - \bar{w}|. \]

Therefore, the assumption (A2) is satisfied with
\[ \beta = \frac{10}{9}, \quad \gamma = \frac{2\pi + 3}{3} \quad \text{and} \quad \lambda = \frac{3\pi^2 + 2}{3}. \]
Also, we have
\[ \varpi = \frac{\ell (|q| + \gamma) + \alpha (|\theta| + \lambda)}{\ell^{1-\alpha} \Gamma (\alpha + 1)} \simeq 0.80326 < 1. \]

What remains is to show that the condition (1.7)
\[ \frac{\ell^{\alpha+1} \left( \frac{k_3}{2y} \left| \frac{\tau \kappa^2}{\delta^2} \right| - \mu \right) + \beta}{\Gamma (\alpha + 2) (1 - \varpi)} \simeq 2.061 \times 10^{-4} < 1, \]
is satisfied. It follows from theorem 1.2 that the Cauchy problem (4.7) has a unique solution.

5 Explicit solutions

In this section, we present some explicit solutions on the traveling wave form of the Cauchy problem (4.1)

**Solution 1:** Let \( p, q, \theta \in \mathbb{R} \) and \( \beta, \gamma, \lambda \in \mathbb{R}^*_+ \), for \( y > 1 \), we get
\[ u (\eta) = \eta^\varphi, \]
which represents a solution of the problem (4.3)–(4.5), where
\[ f(\eta, u(\eta), u'(\eta), u''(\eta)) = -\frac{1}{2} \frac{\tau \kappa^6}{\delta^3} \psi - \frac{3 \tau \kappa^5}{\delta^2} \psi_x + \mu \psi_{tt} - \frac{2 \tau \kappa^4}{\delta} \psi_{xx} + \kappa^2 \tau (\psi_t)_{xx} + \exp \left( -\frac{\kappa^2}{\delta} t \right) \frac{k \delta \Gamma (y + 1)}{\Gamma (y - \alpha)} (x - \kappa t)^{y-\alpha-1}. \]

**Solution 2:** Let \( p, q, \theta \in \mathbb{R} \) and \( \beta, \gamma, \lambda \in \mathbb{R}^*_+ \), for \( y \in \mathbb{C} \) we get
\[ u (\eta) = \sin (y \eta) - y \eta, \]
which represents a solution of the problem (4.3)–(4.5), where
\[ f(\eta, u(\eta), u'(\eta), u''(\eta)) = -\frac{1}{2} \frac{\tau \kappa^6}{\delta^3} \psi - \frac{3 \tau \kappa^5}{\delta^2} \psi_x + \mu \psi_{tt} - \frac{2 \tau \kappa^4}{\delta} \psi_{xx} + \kappa^2 \tau (\psi_t)_{xx} + \exp \left( -\frac{\kappa^2}{\delta} t \right) \frac{k \delta \Gamma (y + 1)}{\Gamma (y - \alpha)} (x - \kappa t)^{y-\alpha-1}. \]
Where $E_{\alpha,\beta}(\eta)$ is the function of Mittag-Leffler type. Then the solution of problem (4.1) is presented as follows

$$\psi(x,t) = \exp\left(-\frac{\kappa^2 t}{\delta}\right) \left[\sin(xy - y\kappa t) - y(x - \kappa t)\right].$$

Where

$$F(x,t,\psi,\psi_t,\psi_{xx},(\psi_t)_{xx}) = -\frac{1}{2}y^3 (x - \kappa t)^{3-\alpha} \exp\left(-\frac{\kappa^2 t}{\delta}\right) \times
E_{1,4-\alpha}(iy(x - \kappa t)) + \frac{\kappa^2 \tau}{\delta} \psi_t
\quad - y^3(x - \kappa t)^{3-\alpha} E_{1,4-\alpha}(-iy(x - \kappa t)) \times
\exp\left(-\frac{\kappa^2 t}{\delta}\right) + \left(\frac{\kappa^4 \mu}{\delta^2} - \frac{2\tau \kappa^6}{\delta^3}\right) \psi
\quad + \left(\frac{2\kappa^3 \mu}{\delta} - 5\kappa^5 \tau\right) \psi_{xx}
\quad - \delta \kappa \theta \psi_{xx} + \kappa^2 \tau (\psi_t)_{xx}. $$

**Solution 3:** Let $p, q, \theta \in \mathbb{R}$ and $\beta, \gamma, \lambda \in \mathbb{R}^*_+$, for $y \in \mathbb{C}$, we get

$$u(\eta) = 2 \cos(y\eta) + y^2 \eta^2,$$

which represents a solution of the problem (4.3)–(4.5), where

$$f(\eta, u(\eta), u'(\eta), u''(\eta)) = -\frac{i}{2}y^3 \eta^{3-\alpha} (E_{1,4-\alpha}(iy\eta) - E_{1,4-\alpha}(-iy\eta))
\quad - pu(\eta) - q u'(\eta) - \theta u''(\eta).$$

Then the solution of problem (4.1) is presented as follows

$$\psi(x,t) = 2 \exp\left(-\frac{\kappa^2 t}{\delta}\right) \cos(yx - y\kappa t) + y^2 \exp\left(-\frac{\kappa^2 t}{\delta}\right) (x - \kappa t)^2.$$
Acknowledgements This work has been supported by the General Direction of Scientific Research and Technological Development (DGRSTD)- Algeria.

References


160


Received: 23.IX.2022 / Revised: 15.I.2023 / Accepted: 18.I.2023

Authors

Rabah Djemiat,
Laboratory of Pure and Applied Mathematics,
Mohamed Boudiaf University of M’sila,
M’sila, 28000, Algeria,
E-mail: rabahdjemiat19@gmail.com

Bilal Basti (Corresponding author),
Department of Mathematics,
Ziane Achour University of Djelfa,
Djelfa, 17000, Algeria,
E-mail: bilabalbisti@gmail.com; b.basti@univ-djelfa.dz

Noureddine Benhamidouche,
Laboratory of Pure and Applied Mathematics,
Mohamed Boudiaf University of M’sila,
M’sila, 28000, Algeria,
E-mail: benhamidouche.noureddine@univ-msila.dz