Well-posedness and asymptotic behavior of thermoelastic system with infinite memory

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Abstract In this work, we consider the thermoelastic system subject to infinite memory. For general assumptions about the memory kernel, we establish well-posedness using the semi group theory and decay result using Lyapunov method.

Keywords thermoelastic system · Lyapunov functional · infinite memory · asymptotic behavior

Mathematics Subject Classification (2010) 35B40 · 37L05 · 35B35

1 Introduction

In this paper, we study thermoelastic system with infinite memory:

$$
\begin{aligned}
&u_{tt}(x,t) - au_{xx}(x,t) + bv_x(x,t) + \int_0^\infty h(s)u_{xx}(x,t-s)ds = 0, & (x,t) \in (0,L) \times \mathbb{R}_+,
&v_t(x,t) - dv_{xx}(x,t) + bu_{xx}(x,t) = 0, & (x,t) \in (0,L) \times \mathbb{R}_+,
&u(0,t) = u(L,t) = 0, & v(0,t) = v(L,t) = 0, & t \in \mathbb{R}_+,
&u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & v(x,0) = v_0(x), & x \in (0,L),
\end{aligned}
$$

(1.1)

where $u$ and $v$ are the displacement and temperature, respectively, subscripts mean partial derivatives. $a$, $b$, $d$ and $L$ are some positive constants, $u_0$, $u_1$ and $v_0$ are the initial data and $h$ is the memory kernel, which satisfy the following hypothesis

(H1): Let $h \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ is a decreasing function such that

$$
h(0) > 0, \quad a - \int_0^\infty h(t)dt = a - \bar{h} = l > 0.
$$
There exists a non-increasing differentiable function \( \zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) satisfying
\[
h'(t) \leq -\zeta(t)h(t), \quad t \geq 0.
\]
There are many kernel functions that fulfill the (H1) and (H2) hypothesis, for example see [11,22].

Several authors have addressed the problem of stability of classical thermoelastic systems and many results have been established, for example Muñoz Rivera [18] considered one dimensional thermoelastic systems
\[
\begin{align*}
\left\{ u_{tt}(x, t) - u_{xx}(x, t) + \alpha v_x(x, t) &= 0, \quad (0, L) \times (0, \infty), \\
v_t(x, t) - v_{xx}(x, t) + \beta u_{xt}(x, t) &= 0, \quad (0, L) \times (0, \infty).
\end{align*}
\]
and proved well-posedness and exponential decay results, examined the effect of an internal delay term on the well-posedness and stability of a one-dimensional thermoelastic system modeled by the system :
\[
\begin{align*}
\left\{ u_{tt}(x, t) - au_{xx}(x, t - \tau) + bv_x(x, t) &= 0, \quad (0, L) \times (0, \infty), \\
v_t(x, t) - dv_{xx}(x, t) + bu_{xt}(x, t) &= 0, \quad (0, L) \times (0, \infty).
\end{align*}
\]
Through their analysis, they proved that the introduction of an internal time delay leads to ill-posedness and instability in the one-dimensional thermoelastic system, even when the delay term \( \tau \) is relatively small. In contrast, the system is shown to be exponentially stable in the absence of a delay term. A similar problem is considered in [17], but with the addition of Kelvin-Voigt damping \(-cax_{xx}(x, t)\) into the delayed equation. The authors established the well-posedness and exponential stability.

For works dealing with stability result in a one dimensional thermoelastic system with infinite memory in heat flow, see [5,6,23]. When dealing with multiple dimensions, the dissipation caused by heat conduction is generally not sufficient to generate a uniform decay rate. In a radially symmetric situations, Quintanilla and Racke [20] demonstrated an exponential decay in two or three dimensions, while in most two dimensional domains, the energy decays polynomially. Fabrizio et al. [4] investigated the asymptotic stability and exponential decay of a two dimensional thermoelastic system with an internal structure. For instance, Grasselli [7] investigated the two dimensional thermoplastic system, represented by the following thermoelastic plate with infinity memory:
\[
\begin{align*}
\left\{ u_{tt}(x, t) + \Delta(\Delta u(x, t) + v(x, t)) &= 0, \quad \Omega \times (0, \infty), \\
v_t(x, t) + \int_0^\infty h(s)[cv(x, t - s) - \Delta u(x, t - s)]ds &= 0, \quad \Omega \times (0, \infty).
\end{align*}
\]
The authors showed that a solution decays to zero under reasonably general assumptions on the memory kernel \( h \). However, the decay is not exponential.

We also want to acknowledge the contribution of Hao and Wang [10], who studied an abstract thermoelastic system with Kelvin-Voigt damping:
\[
\begin{align*}
\left\{ u_{tt} + Au + \int_0^\infty h(s)A u(s)ds + Bu_t = A^\alpha v, \quad (0, \infty), \\
v_t + k A^\beta v + A^\alpha u_t &= 0, \quad (0, \infty),
\end{align*}
\]
where $A$ and $B$ are linear operators and the kernel $h$ satisfy the hypothesis (H1)-(H2). They showed well-posedness and general decay rate. For systems that have adopted a finite memory damping, we refer to [1,2,12–15].

Building upon previous works [10,17], this study aims to achieve more general results with memory damping instead of Kelvin-Voigt damping, allowing for much weaker damping. Specifically, we investigate the behavior of solutions to problem (1.1) under general decay rates of $h$, including the usual exponential decay rates and polynomials as special cases.

To the best of our knowledge, there is no result in the literature that deals with system stability such as (1.1) with infinite memory damping, unlike Kelvin-Voigt damping which is stronger and more frequently used, it may not always be the optimal choice to dissipate energy in a certain systems. Additionally, we consider a larger class of relaxation functions.

To facilitate a clear and concise presentation, this paper is divided into two sections. Section 2 focuses on the well-posedness of the problem, while Section 3 offers proof of the general decay of the energy.

## 2 Well-posedness of the problem

In this section, now we will prove the well-posedness of system (1.1). Based on the approach of [3], we introduce a new variable

$$\begin{aligned}
&w(x,t,s) = u(x,t) - u(x,t-s), \quad (0,L) \times \mathbb{R}^2_+, \\
&w(x,0,s) = w_0(s), \quad (0,L) \times \mathbb{R}_+, \\
&w(x,t,0) = 0, \quad (0,L) \times \mathbb{R}_+.
\end{aligned}$$

It is clear that

$$w_t + w_s = u_t, \quad \forall (x,t,s) \in (0,L) \times \mathbb{R}^2_+,$$

Thus, system (1.1) becomes

$$\begin{aligned}
&u_{tt} - lu_{xx} + bv_x - \int_0^\infty h(s)w_{xx}(x,t,s)ds = 0, \quad (0,L) \times \mathbb{R}^2_+, \\
v_t - dv_{xx} + bu_{xt} = 0, \quad (0,L) \times \mathbb{R}_+, \\
w_t + w_s = u_t, \quad (0,L) \times \mathbb{R}^2_+, \\
u(0,t) = u(L,t) = 0 \quad v(0,t) = v(L,t) = 0 \quad (0,\infty), \\
w(x,0,s) = w_0(s), \quad (0,L) \times \mathbb{R}_+, \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad v(x,0) = v_0(x) \quad (0,L).
\end{aligned}$$

This system is equivalent to

$$\begin{aligned}
&V_t = AV(t), \quad t > 0, \\
&V(0) = V_0 = (u_0, u_1, v_0, w_0)^T.
\end{aligned} \quad (2.1)$$
where the linear operator $A$ is defined as

$$
A \begin{pmatrix} u \\ \varphi \\ v \\ w \end{pmatrix} = \begin{pmatrix} l u_{xx} - b v_x + \int_0^\infty h(s) w_{xx}(s) ds \\ \varphi \\ d v_{xx} - b \varphi_x \\ - w_s + \varphi \end{pmatrix}
$$

and

$$
D(A) = \{ V = (u, \varphi, v, w)^T \in H / A(V) \in H \},
$$
in which $H$ is the energy space, defined by

$$
H = \left\{ H_0^1(0, L) \times L^2(0, L) \times L^2(0, L) \times L^2_0(0, L) \right\}.
$$

We endow $H$ with the inner product

$$
\langle V, \bar{V} \rangle_H = l \int_0^L u_x \bar{u}_x dx + \int_0^L \varphi \bar{\varphi} dx + \int_0^L v \bar{v} dx + \int_0^L \int_0^{+\infty} h(s) w_x(s) \bar{w}_x(s) ds dx,
$$

where $V = (u, \varphi, v, w)^T, \bar{V} = (\bar{u}, \bar{\varphi}, \bar{v}, \bar{w})^T \in H$.

Below, we present the result of the well-posedness of problem (2.1).

**Theorem 2.1** Assume that (H1) and (H2) hold, then for any $V_0 \in H$, then problem (2.1) has unique solution $V \in C([R_+, H])$.

Moreover, if $V_0 \in D(A)$, then

$$
V \in C([R_+, D(A))] \cap C^1([R_+, H]).
$$

**Proof.** In order to prove this theorem, we employ the semigroup approach.

$A$ is dissipative: Let $(u, \varphi, v, w)^T \in D(A)$, we have

$$
\langle AV, V \rangle_H = l \int_0^L u_x \varphi_x dx + \int_0^L l u_{xx} \varphi dx - b \int_0^L v_x \varphi dx + \int_0^L \varphi \int_0^{+\infty} h(s) w_{xx}(s) ds \varphi dx + \int_0^L (d v_{xx} - b \varphi_x) v dx + \int_0^L \int_0^{+\infty} h(s) w_x(s) (- w_s + \varphi) ds dx = -d \int_0^L |v_x|^2 dx + \frac{1}{2} \int_0^L \int_0^{+\infty} h(s) w_x(s)^2 ds dx.
$$
From assumption (H2), we conclude that
\[ \langle AV, V \rangle_H \leq 0. \]
Hence \( A \) is dissipative.

\textbf{A is maximal}: Let \( \lambda > 0 \) and \( F = (f_1, f_2, f_3, f_4)^T \in H \), we solve the system
\[ (\lambda I - A)V = F, \]
or
\[ \begin{aligned}
\lambda u - \varphi &= f_1, \\
\lambda \varphi - \lambda u_{xx} + bv_x - \int_0^\infty h(s)w_{xx}(s)ds &= f_2, \\
\lambda v - dv_{xx} + b\varphi_x &= f_3, \\
\lambda w + w_x - \varphi &= f_4. \\
\end{aligned} \tag{2.2} \]
Then, we have
\[ \varphi = \lambda u - f_1 \tag{2.3} \]
and as \( w(x, 0) = 0 \), we get
\[ w(x, s) = \left( \int_0^s e^{\lambda\sigma}(f_4(x, \sigma) + \varphi(x))dw \right)e^{-\lambda s} \]
\[ = \left( \int_0^s e^{\lambda\sigma}(f_4(x, \sigma) + \lambda u(x) - f_1(x))d\sigma \right)e^{-\lambda s}. \tag{2.4} \]
Using (2.2), (2.3) and (2.4), the functions \( u \) and \( v \) satisfy
\[ \begin{aligned}
\begin{cases}
\lambda^2 u - \tilde{l} u_{xx} + bv_x = \tilde{f}, \\
\lambda v - dv_{xx} + \lambda bu_x = f_3 + b(f_1)_x,
\end{cases} \tag{2.5} \\
\end{aligned} \]
where
\[ \tilde{l} = l + \lambda \int_0^\infty h(s)e^{-\lambda s}\left( \int_0^s e^{\lambda\sigma}d\sigma \right)ds \]
and
\[ \tilde{f} = \int_0^\infty h(s)e^{-\lambda s}\left( \int_0^s e^{\lambda\sigma}(f_4(x, \sigma) - f_1(x, \sigma))dw \right)ds + \lambda f_1 + f_2. \]
As a result, the corresponding variational formulation for equation (2.5) can be expressed as follows:
\[ A((u, v), (\omega, \theta)) = l(\omega, \theta), \tag{2.6} \]
where the bilinear form \( A : (H^1_0(\Omega))^4 \to \mathbb{R} \) is defined by
\[ A((u, v), (\omega, \theta)) = \int_0^L \left[ \lambda^2 u\omega + \tilde{l} u_x(\omega)_x \right]dx + \int_0^L \left[ \lambda v\theta + dv_x\theta_x \right]dx + b \int_0^L \left[ v_x\omega - \lambda u\theta_x \right]dx. \]
and the linear form \( l : (H^1_0(\Omega))^2 \to \mathbb{R} \) is given by
\[
l(\omega, \theta) = \int_0^L \tilde{f} \omega dx + \int_0^L (f_3 + b(f_1) x) \theta dx.
\]

Then, we can easily prove that form \( A \) is continuous and coercive, while form \( l \) is continuous. Applying the Lax-Milgram theorem, we can conclude that problem (2.6) has a unique solution \((u, v) \in (H^1_0(\Omega))^2\). Hence \( A \) is maximal.

Since \( A \) is dissipative and maximal, we can deduce that \( A \) is the infinitesimal generator of a \( C^0 \)-semigroup of contractions on \( \mathcal{H} \). Therefore, according to Lummer-Phillips theorem [21], we can obtain the well-posedness result.

\[\square\]

3 Asymptotic behavior

In this section, we state and prove our stability result. We first define the energy of problem (1.1) by
\[
E(t) = \frac{1}{2} \int_0^L \left[ u_t^2 + (l)u_x^2 + v^2 + \int_0^\infty h(s)w_x^2 ds \right] dx.
\]  

The main result of this section is stated as follows:

**Theorem 3.1** Assume that (H1) and (H2) hold. Then, there exist three positive constants \( \alpha, A, B \) such that
\[
E(t) \leq Ae^{-\alpha \int_0^t \zeta(s) ds} + B \int_0^\infty h(s) ds.
\]  

Before proceeding with the proof, we need to state the following lemmas.

**Lemma 3.2** The energy \( E(t) \) satisfies
\[
E'(t) = -d \int_0^L \left[ v_x^2 + \frac{1}{2} \int_0^\infty h'(s)w_x^2 ds \right] dx
\]

**Proof.** Using (2) and integration by parts, we obtain (3.2).

\[\square\]

**Remark 3.1** From hypotheses (H1) and (H2) of the kernel function \( h \), we can conclude that the energy \( E(t) \) is dissipative.

**Lemma 3.3** Assume that \( h \) satisfy (H1) and (H2), then
\[
\int_0^L \left[ \int_0^\infty h(s)w(s) ds \right] dx \leq h \int_0^L \int_0^\infty h(s)w^2(s) ds dx,
\]
\[
\int_0^L \left[ \int_0^\infty h'(s)w(s) ds \right] dx \leq h(0) \int_0^L \int_0^\infty h'(s)w^2(s) ds dx.
\]
Lemma 3.4 Assume that \( (H1) \) and \( (H2) \) hold. Then the functional

\[
V_1(t) = \int_0^L uu_t dx
\]

satisfies the following estimate

\[
V_1'(t) \leq \int_0^L u_t^2 dx - \left( a - \frac{b}{\delta_1} - \left( 1 + \frac{\delta_2}{2} \right) \bar{h} \right) \int_0^L u_x^2 dx
\]

\[
+ b\delta_1 \int_0^L v^2 dx + \frac{1}{\delta_2} \int_0^L \int_0^\infty h(s)w_x^2 ds dx,
\]

(3.3)

where \( \delta_1 \) and \( \delta_1 \) are small positive constants.

Proof. By deriving \( V_1(t) \), using (1.1) and integrating by parts, we have

\[
V_1'(t) = \int_0^L u_t^2 dx - a \int_0^L u_x^2 dx + b \int_0^L vu_x dx
\]

\[
+ \int_0^L \int_0^\infty h(s)u_x(t-s)u_x(t) ds dx.
\]

(3.4)

By using Young’s inequality, Poincaré’s inequality and Lemma 3.3, we get

\[
b \int_0^L vu_x dx \leq b\delta_1 \int_0^L v^2 dx + \frac{b}{\delta_1} \int_0^L u_x^2 dx
\]

(3.5)

and

\[
\int_0^L \int_0^\infty h(s)u_x(t-s)u_x(t) ds dx
\]

\[
= - \int_0^L \int_0^\infty h(s)w_xu_x(t) ds dx + \bar{h} \int_0^L u_x^2 dx
\]

\[
\leq \frac{1}{\delta_2} \int_0^L \int_0^\infty h(s)w_x^2 ds dx + \left( 1 + \frac{\delta_2}{2} \right) \bar{h} \int_0^L u_x^2 dx,
\]

(3.6)

for \( \delta_1 \) and \( \delta_1 \) any small positive constants.

Substituting (3.5) and (3.6) into (3.4), we obtain (3.3).

\[\Box\]

Lemma 3.5 Assume that \( (H1) \) and \( (H2) \) hold. Then the functional

\[
V_2(t) = - \int_0^L u_t \int_0^\infty h(s)w(s) ds dx
\]
satisfies the following estimate

\[
V'_2(t) \leq - \left( \bar{h} - \frac{h(0)\delta_4}{2} \right) \int_0^L u_x^2 \, dx + \frac{(a + 1) \bar{h} \delta_3}{2} \int_0^L u_x^2 \, dx \\
+ b\bar{h} \frac{\delta_3}{2} \int_0^L v^2 \, dx + \frac{h(0)}{2\delta_4} \int_0^L \int_{0}^{\infty} h'(s)w_x^2(s) \, ds \, dx \\
+ \left( \frac{(a + b + 1) \bar{h}}{2\delta_3} + \bar{h} \right) \int_0^L \int_{0}^{\infty} h(s)w_x^2(s) \, ds \, dx,
\]

where \( \delta_3 \) and \( \delta_4 \) are small positive constants.

Proof. Differentiating \( V'_2(t) \), using (2), and integrating by parts, we get

\[
V'_2(t) = a \int_0^L \int_0^{\infty} h(s)w_x(s)u_x(t) \, ds \, dx - b \int_0^L v \int_0^{\infty} h(s)w_x(s) \, ds \, dx \\
- \int_0^L \int_0^{\infty} h(s)u_x(t-s) \, ds \, \int_0^{\infty} h(s)w_x(s) \, ds \, dx \\
- \bar{h} \int_0^L u_t^2 \, dx - \int_0^L u_t \int_0^{\infty} h'(s)w \, ds \, dx.
\]

We can use Cauchy-Schwarz, Young inequalities and Lemma 3.3 to estimate the terms on the right-hand side of the relation (3.8) as follows

\[
a \int_0^L \int_0^{\infty} h(s)w_x(s)u_x(t) \, ds \, dx \leq \frac{\bar{a} \delta_3}{2} \int_0^L u_x^2 \, dx \\
+ \frac{\bar{a} \bar{h}}{2\delta_3} \int_0^L \int_0^{\infty} h(s)w_x^2(s) \, ds \, dx,
\]

\[
b \int_0^L v \int_0^{\infty} h(s)w_x(s) \, ds \, dx \leq \frac{\bar{b} \delta_3}{2} \int_0^L v^2 \, dx \\
+ \frac{\bar{b} \bar{h}}{2\delta_3} \int_0^L \int_0^{\infty} h(s)w_x^2(s) \, ds \, dx,
\]

\[
- \int_0^L \int_0^{\infty} h(s)u_x(t-s) \, ds \, \int_0^{\infty} h(s)w_x(s) \, ds \, dx \\
= \int_0^L \left( \int_0^{\infty} h(s)w_x(s) \, ds \right)^2 \, dx - \bar{h} \int_0^L u_x \int_0^{\infty} h(s)w_x(s) \, ds \, dx \\
\leq \bar{h} \frac{\delta_3}{2} \int_0^L u_x^2 \, dx + \left( \bar{h} + \bar{h} \frac{\delta_3}{2} \right) \int_0^L \int_0^{\infty} h(s)w_x^2(s) \, ds \, dx.
\]
and
\[- \int_0^L u_t \int_0^\infty h'(s) w(s) ds dx \leq h(0) \frac{\delta_4}{2} \int_0^L u_t^2 dx + \frac{h(0)}{2\delta_4} \int_0^L \int_0^\infty h'(s) w_x^2(s) ds dx, \quad (3.12)\]

for \(\delta_3\) and \(\delta_4\) any small positive constants.

Substituting (3.9)–(3.12) into (3.8), we obtain (3.7).

\[\Box\]

Proof of Theorem 3.1. We define the Lyapunov functional
\[L(t) = E(t) + k_1 V_1(t) + k_2 V_2(t), \quad (3.13)\]

where \(k_1\) and \(k_2\) are positive constants.

By taking the derivative of (3.13), estimates in Lemmas 3.2, 3.4, 3.5 and the fact that
\[\int_0^\infty h(s) w_x^2 ds dx \leq - \int_0^\infty \zeta(s) h'(s) w_x^2 ds dx \leq -\zeta(0) \int_0^\infty h'(s) w_x^2 ds dx,\]

we obtain
\[L'(t) \leq - \left[ \frac{d}{\delta_2} - \frac{b\delta_1 k_1 - b\delta_3 k_2}{2} \right] \int_0^L v^2 dx \]
\[- \left[ \left( a - \frac{b}{\delta_1} - \frac{2 + \delta_2}{2} \right) k_1 - \frac{(a + 1) \tilde{h} \delta_3}{2} k_2 \right] \int_0^L u_x^2 dx \]
\[- \left[ \left( \frac{\tilde{h} - h(0) \delta_4}{2} \right) k_2 - k_1 \right] \int_0^L u_t^2 dx \]
\[+ \left( \frac{k_1}{\delta_2} + \frac{(a + b + 1) \tilde{h}}{2\delta_3} + \tilde{h} \right) k_2 \int_0^L \int_0^\infty h(s) w_x^2(s) ds dx \]
\[+ \left[ \frac{h(0)}{2\delta_4} k_2 + \frac{1}{2} \right] \int_0^\infty h'(s) w_x^2 ds dx, \quad (3.14)\]

Now, we pick \(\delta_i, i = 1, 2, 3, 4\) \(k_1\) and \(k_2\) small enough such that coefficients of (3.14) are all strictly negative. Thus, there exist three positive constants \(\theta_1, \theta_2\) and \(\theta_3\), such that
\[L'(t) \leq - \theta_1 E(t) + \theta_2 \int_0^L \int_0^\infty h(s) w_x^2(s) ds dx \]
\[+ \theta_3 \int_0^L \int_0^\infty h'(s) w_x^2(s) ds dx. \quad (3.15)\]
From (H2) and (3.15), we have

\[(ζ(t)L(t))' \leq -θ_1ζ(t)E(t) + θ_2ζ(t)\int_0^L \int_0^∞ h(s)w_x^2(s)dsdx + θ_3ζ(t)\int_0^L \int_0^∞ h'(s)w_x^2(s)dsdx = -θ_1ζ(t)E(t) + θ_2ζ(t)\int_0^L \int_0^∞ h(s)w_x^2(s)dsdx + θ_3ζ(t)\int_0^L \int_0^∞ h'(s)w_x^2(s)dsdx. \quad (3.16)\]

By using the fact that

\[\int_0^L \int_0^t ζ(t)h(s)w_x^2(s)dsdx \leq -\int_0^L \int_0^t h'(s)w_x^2(s)dsdx \leq -\int_0^L \int_∞^t ζ(s)h'(s)w_x^2(s)dsdx \leq -2E'(t)\]

and

\[\int_0^L \int_t^∞ h(s)w_x^2(s)dsdx \leq \frac{3}{2} \int_0^L \int_t^∞ h(s)\left[u_x^2(t) + u_x^2(t-s)\right]dsdx ≤ \frac{3}{2} \left(\int_0^L u_x^2dx + \sup_{r≤0} \left|\int_r^L u_x^2(r)dx\right|\right) \int_1^∞ h(s)ds ≤ \frac{3}{2} \left(\frac{2E(0)}{a-h} + \sup_{r≤0} \left|\int_r^L u_x^2(r)dx\right|\right) \int_1^∞ h(s)ds \leq θ_4 \int_1^∞ h(s)ds,\]

The result (3.16) becomes as follows

\[(ζ(t)L(t))' \leq -θ_1ζ(t)E(t) + θ_2θ_4ζ(t)\int_t^∞ h(s)ds - 2θ_2E'(t) + θ_3ζ(t)\int_0^L \int_0^∞ h'(s)w_x^2(s)dsdx \leq -θ_1ζ(t)E(t) - 2θ_2E'(t) + θ_2θ_4ζ(t)\int_t^∞ h(s)ds. \quad (3.17)\]

Now, we introduce

\[\mathcal{L}(t) = ζ(t)L(t) + 2θ_2E(t).\]
Else, we can easily prove the equivalence property
\[ \theta_5 E(t) \leq \mathcal{L}(t) \leq \theta_6 E(t). \] (3.18)

Combining (3.17) and (3.18), we deduce that
\[
\mathcal{L}'(t) \leq -\theta_1 \zeta(t) E(t) + \theta_2 \theta_4 \zeta(t) \int_t^\infty h(s) ds
\leq -\alpha \zeta(t) \mathcal{L}(t) + \beta \zeta(t) \int_t^\infty h(s) ds,
\]
where \( \alpha = \frac{\theta_1}{\theta_6} \) and \( \beta = \theta_2 \theta_4 \). Multiplying this inequality by \( e^{\alpha \int_0^t \zeta(s) ds} \) and integrating over \((0,t)\), we have
\[
\mathcal{L}(t) \leq e^{-\alpha \int_0^t \zeta(s) ds} \left( \mathcal{L}(0) + \beta \int_0^t e^{\alpha \int_0^s \zeta(r) ds} \zeta(r) \int_r^\infty h(s) ds dr \right). \] (3.19)

On the other hand, we have
\[
\alpha \int_0^t e^{\alpha \int_0^r \zeta(s) ds} \zeta(r) \int_r^\infty h(s) ds dr = \int_0^t \frac{d}{dr} \left( e^{\alpha \int_0^s \zeta(s) ds} \right) \int_r^\infty h(s) ds dr
= e^{\alpha \int_0^t \zeta(s) ds} \int_t^\infty h(s) ds - \int_0^\infty h(s) ds
+ \int_0^t h(r) e^{\alpha \int_0^r \zeta(s) ds} dr
\]
and
\[
\int_0^t h(r) e^{\alpha \int_0^s \zeta(s) ds} ds dr \leq \int_0^t (h(r))^{1-\alpha} \left( h(r) e^{\alpha \int_0^r \zeta(s) ds} \right)^\alpha dr
\leq h(0) \int_0^t (h(r))^{1-\alpha} dr
\leq h(0) \int_0^\infty (h(r))^{1-\alpha} dr = \theta_7.
\]

Consequently, combining with (3.19) given
\[
\mathcal{L}(t) \leq e^{-\alpha \int_0^t \zeta(s) ds} \left( \mathcal{L}(0) + \frac{\beta}{\alpha} \theta_7 \right) + \frac{\beta}{\alpha} \int_t^\infty h(s) ds. \] (3.20)

Finally using the relation (3.18), we get (3.2) with \( \alpha = \frac{\theta_1}{\theta_6} \), \( A = \frac{\mathcal{L}(0) + \beta \theta_7 / \alpha}{\theta_6} \) and \( B = \frac{\beta}{\alpha \theta_6} \).

\[ \square \]

\textbf{Example 3.1} Our result (3.2) provides a more general decay rate according to the relaxation function:
**Exponential decay.** Let \( h(t) = c e^{-(1+t)\theta} \) where \( 0 < \theta \leq 1 \) and \( c > 0 \) small enough so that (H1) holds, then \( \zeta(t) = \theta (1+t)^{\theta-1} \). The result (3.2) become
\[
E(t) \leq c_1 e^{-c_2(1+t)^\theta},
\]
where \( c_1 \) and \( c_2 \) are positives constant.

**Polynomial decay.** Let \( h(t) = \frac{c}{(1+t)^\eta} \) where \( \eta > 1 \) and \( c > 0 \) small enough so that (H1) holds, then \( \zeta(t) = \frac{\eta}{1+t} \). The result (3.2) become
\[
E(t) \leq \frac{c_1}{(1+t)^{c_2}},
\]
where \( c_1 \) and \( c_2 \) are positives constant.

**Logarithmic decay.** Let \( h(t) = \frac{c}{\ln(1+t)^\eta} \) where \( \eta > 1 \) and \( c > 0 \) small enough so that (H1) holds, then \( \zeta(t) = \frac{\eta}{1+t} \ln(1+t) \). The result (3.2) become
\[
E(t) \leq \frac{c_1}{\ln(1+t)^{c_2}},
\]
where \( c_1 \) and \( c_2 \) are positives constant.

Further information and additional examples can be found in the references [8,9,16,19].

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**References**


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