Some results on generalized quaternions algebra with generalized Fibonacci quaternions

Rachid Chaker · Abdelkarim Boua

Abstract Recently, the most general form of the quaternion algebra on 3 parameters (3PGQ) has been introduced, which prompted us to look for some properties associated with this algebra, which will be called \( k_{\lambda_1, \lambda_2, \lambda_3} \) in this article. This paper consists of two main parts: the first part focuses on derivations, while the second part deals with the Fibonacci sequence, and the properties of both parts are related to the algebra \( k_{\lambda_1, \lambda_2, \lambda_3} \). First we determine the derivations of the algebra \( k_{\lambda_1, \lambda_2, \lambda_3} \) and get the algebra \( \text{Der}(k_{\lambda_1, \lambda_2, \lambda_3}) \) of derivations of \( k_{\lambda_1, \lambda_2, \lambda_3} \). Then we were able to obtain generalized derivations that have been studied by mathematicians in the context of algebras of certain normed spaces, and semi-prime and prime rings. In the second part, we introduce some properties of the Fibonacci quaternions in the generalized quaternion algebra \( k_{\lambda_1, \lambda_2, \lambda_3} \), which allow us to assume that the algebra \( k_{\lambda_1, \lambda_2, \lambda_3} \) is not always a division algebra.

Keywords derivations · Fibonacci quaternions · quaternions algebra

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1 Introduction

From 1805 to 1865, Sir William Rowan Hamilton studied quaternions as a mathematical extension of complex numbers. Quaternion algebra occupies an important place at the intersection of many mathematical topics. You cite the key features of non-commutative ring theory, group theory, Lie theory, arithmetic, geometry, number theory, representation theory, functions of a complex variable. For these considerations, the main purpose of this paper is to enrich the repertoire of the quaternion algebra \( k_{\lambda_1, \lambda_2, \lambda_3} \) with new properties. Many mathematicians have long been interested in discovering and exploring the derivations of various algebraic structures, such as Lie algebras, rings, near-rings, quaternion algebras, and so on, because they provide us with useful indicators for studying their structures algebraically. For more information on the use of derivations in a theoretical context, see ([1], [2], [3], [4], [8]) which offered a simple computational approach to
obtaining explicit derivations of Lie algebras. In the first part of this article, we consider the derivations of the 3-parameter generalized quaternion algebra as a class of Lie algebras, and then we compute the derivations in their matrix form, and then we consolidate these results by computing the generalized derivations. At the end of this part we compare our results with those of the article [9]. In the second part of this article, we introduce some properties of the Fibonacci sequence of quaternions in the generalized quaternion algebra with three parameters, which allows us to decide that the $k_{\lambda_1,\lambda_2,\lambda_3}$ algebra is not always a division algebra. The following is the organization of this paper:

Section 2 contains preliminary results for the algebra of quaternions, the Fibonacci sequence of quaternions, and derivations of the algebra. In Section 3, we detail the results obtained for the algebra of $k_{\lambda_1,\lambda_2,\lambda_3}$ derivation. We strengthen these results by computing the generalized derivations in Section 4. Finally, Section 5 presents the results obtained for the properties of the Fibonacci sequence of quaternions.

2 Definitions and notations

In this section, we state the definitions and the main results concerning the quaternions algebra has been introduced depending on 3 parameters (3PGQ), derivation of an algebra and the Fibonacci sequence of quaternions.

2.1 3-Parameter generalized quaternions

In the following, we’ll go over some key concepts and notations that will help us understand and expand on this topic. In [10], the set of generalized quaternions with 3 parameters (3PGQ) was introduced by Şentürk and Ünal as follows:

$$k_{\lambda_1,\lambda_2,\lambda_3} = \{a + be_1 + ce_2 + de_3 \mid a, b, c, d, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, e_1^2 = -\lambda_1\lambda_2, e_2^2 = -\lambda_1\lambda_3, e_3^2 = -\lambda_2\lambda_3, e_1e_2e_3 = -\lambda_1\lambda_2\lambda_3\}.$$

Each element $p = x_0 + x_1e_1 + x_2e_2 + x_3e_3$ of the set $k_{\lambda_1,\lambda_2,\lambda_3}$ is called a 3-parameters generalized quaternion (3PGQ). The real numbers $x_0, x_1, x_2, x_3$ are called components of $p$. Note that $\mathfrak{B}(k_{\lambda_1,\lambda_2,\lambda_3}) = \{e_0, e_1, e_2, e_3\}$ is the base of the $k_{\lambda_1,\lambda_2,\lambda_3}$, its vectors verify the following multiplication table:

<table>
<thead>
<tr>
<th>.</th>
<th>$e_0$</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_0$</td>
<td>1</td>
<td>$e_1$</td>
<td>$e_2$</td>
<td>$e_3$</td>
</tr>
<tr>
<td>$e_1$</td>
<td>$e_1$</td>
<td>$-\lambda_1\lambda_2$</td>
<td>$\lambda_1e_3$</td>
<td>$-\lambda_2e_2$</td>
</tr>
<tr>
<td>$e_2$</td>
<td>$e_2$</td>
<td>$-\lambda_1e_3$</td>
<td>$-\lambda_1\lambda_3$</td>
<td>$\lambda_3e_1$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>$e_3$</td>
<td>$\lambda_2e_2$</td>
<td>$-\lambda_3e_1$</td>
<td>$-\lambda_2\lambda_3$</td>
</tr>
</tbody>
</table>

Special cases:
Results about generalized quaternions algebra

(i) If $\lambda_1 = 1, \lambda_2 = \alpha, \lambda_3 = \beta$, then the algebra of 2PGQs is obtained.
(ii) If $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$, then the algebra of Hamilton quaternions is achieved.
(iii) If $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$, then gives us the algebra of split quaternions.
(iv) If $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 0$, then the algebra of semi-quaternions is attained.
(v) If $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$, then we get the algebra of split semiquaternions.

Any 3PGQ, $p = x_0 + x_1e_1 + x_2e_2 + x_3e_3$ consists of two parts, the scalar part and the vector part: $p = S(p) + V(p)$ such as:

$$S(p) = x_0 \text{ and } V(p) = x_1e_1 + x_2e_2 + x_3e_3.$$  

The rules of addition, scalar multiplication and multiplication are defined on $k$ as follows:

Let $p = x_0 + x_1e_1 + x_2e_2 + x_3e_3$, $q = y_0 + y_1e_1 + y_2e_2 + y_3e_3$ be 3PGQ, and $\alpha$ be a real number.

- **Addition:**
  \[ p + q = (S(p) + S(q)) + (V(p) + V(q)) = (x_0 + y_0) + (x_1 + y_1)e_1 + (x_2 + y_2)e_2 + (x_3 + y_3)e_3. \]

- **Multiplication by scalar:** $\alpha p = \alpha x_0 + \alpha x_1e_1 + \alpha x_2e_2 + \alpha x_3e_3$ for all $\alpha \in \mathbb{R}$.

- **Multiplication:** from the multiplication table we have

\[
pq = (x_0y_0 - \lambda_1\lambda_2x_1y_1 - \lambda_1\lambda_3x_2y_2 - \lambda_2\lambda_3x_3y_3) \\
+ e_1(x_0y_1 + y_0x_1 + \lambda_3x_2y_3 - \lambda_2x_3y_2) \\
+ e_2(x_0y_2 + y_0x_2 + \lambda_2x_3y_1 - \lambda_2x_1y_3) \\
+ e_3(x_0y_3 + y_0x_3 + \lambda_1x_1y_2 - \lambda_1x_2y_1)
\]

It’s worth noting that $e_0$ serves as an identity, which implies that $e_0.e_i = e_i.e_0 = e_i$ for every $i$ and so the center of $k_{\lambda_1,\lambda_2,\lambda_3}$, is $Z(k_{\lambda_1,\lambda_2,\lambda_3}) = \mathbb{R}.e_0 = \mathbb{R}$.

### 2.2 Derivation of an algebra

According to [9], we give the following definitions:

**Definition 2.1** We say that $D$ is a derivation of an algebra $A$ if $D$ is a linear map $D : A \rightarrow A$ such that

\[ D(xy) = D(x)y + xD(y) \text{ for all } x, y \in A. \]

We note that $Der(A)$ is the set of all derivations of an algebra $A$.

**Definition 2.2** A linear map $F$ of an algebra $A$ into itself is said to be a generalized derivation of $A$ if there exists a nonzero derivation $D \in Der(A)$ such that

\[ F(x \cdot y) = F(x) \cdot y + x \cdot F(y) \text{ for all } x, y \in A. \]  \hspace{1cm} (2.1)
Definition 2.3 Let $x \in A$. For every $y \in A$, the map
\[
D = ad(x) : A \rightarrow A
y \mapsto [x, y] = x.y - y.x
\]
is called an inner derivation of $A$.

$ad(k_{\lambda_1, \lambda_2, \lambda_3})$ denotes the set of all inner derivations of $k_{\lambda_1, \lambda_2, \lambda_3}$ as a subset of $\text{Der}(k_{\lambda_1, \lambda_2, \lambda_3})$. By bilinearity, $ad(k_{\lambda_1, \lambda_2, \lambda_3})$ can be generated by the maps $ad(e_i) : k_{\lambda_1, \lambda_2, \lambda_3} \rightarrow k_{\lambda_1, \lambda_2, \lambda_3}$, where $e_i \in \mathcal{B}(k_{\lambda_1, \lambda_2, \lambda_3})$. This means that any inner derivation $D = ad(x)$, $x \in k_{\lambda_1, \lambda_2, \lambda_3}$, is a linear combination of $ad(e_i)$. It should be noted that for any $e_i \in \mathcal{B}(k_{\lambda_1, \lambda_2, \lambda_3})$ the map $ad(e_i)$ is the zero map if and only if $e_i \in Z(k_{\lambda_1, \lambda_2, \lambda_3})$. This simply means that for any quaternion algebra we always obtain $ad(e_0) = 0$ since $e_0$ acts for all as a global identity. The set $ad(k_{\lambda_1, \lambda_2, \lambda_3})$ can be determined from the equation $ad(k_{\lambda_1, \lambda_2, \lambda_3}) \simeq \frac{\lambda_1 \lambda_2 \lambda_3}{Z(k_{\lambda_1, \lambda_2, \lambda_3})}$ and thus it is easy to determine its dimension. From $\text{Der}(k_{\lambda_1, \lambda_2, \lambda_3})/ad(k_{\lambda_1, \lambda_2, \lambda_3})$, we obtain outer (=noninner) derivations.

Definition 2.4 A derivation $D$ of an algebra $A$ into its center $Z(A)$ is called a central derivation.

Once we explicitly have all the derivations, we are able to determine (if any exist) the inner and/or central derivations.

2.3 Generalized Fibonacci quaternions

We present some relationships of Fibonacci numbers. The author of [7] generalized the Fibonacci numbers and provided many properties, including:

1. $h_n = h_{n-1} + h_{n-2}$ for all $n \geq 2$.
2. $h_{n+1} = pf_n + qf_{n+2}$, $h_0 = p$, and $h_1 = q$, where $p, q \in \mathbb{N}$, $n \geq 0$.
3. $f_{2n} = f_n^2 + 2f_{n-1}$ and $f_n^2 + f_{n-1}^2 = f_{2n-1}$, where $n \geq 1$.

In the algebra $k_{\lambda_1, \lambda_2, \lambda_3}$, we will calculate the norm of a Fibonacci quaternion and the norm of a generalized Fibonacci quaternion. Let $F_n = f_n, e_1 + f_{n+1}e_1 + f_{n+2}e_2 + f_{n+3}e_3$ be the $n$ th Fibonacci quaternion, according to [5] its norm is:

\[
N(F_n) = f_n^2 + \lambda_1 \lambda_2 f_{n+1}^2 + \lambda_1 \lambda_3 f_{n+2}^2 + \lambda_2 \lambda_3 f_{n+3}^2.
\]

3 Algebra of derivations for $k_{\lambda_1, \lambda_2, \lambda_3}$

Our goal in this theorem is to find a typical derivation of the 3-parameter generalized quaternions (3PGQ) in their matrix form, and we compare the result with [9, Theorem 3.5].
Theorem 3.1 The algebra $\text{Der}(k_{\lambda_1, \lambda_2, \lambda_3})$ of derivations for $k_{\lambda_1, \lambda_2, \lambda_3}$ is generated by the following matrices:

$$
D = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\frac{\lambda_3}{\lambda_2}a & -\frac{\lambda_3}{\lambda_2}b \\
0 & a & d & -\frac{\lambda_3}{\lambda_2}c \\
b & c & d & 0
\end{pmatrix} \in \text{Der}(k_{\lambda_1, \lambda_2, \lambda_3}),
$$

(3.1)

where $a, b, c, d \in \mathbb{R}$, $\lambda_1 \lambda_2 \neq 0$ such that

$$
d = d_{\lambda_1, \lambda_3} \neq 0, \quad \text{if } \lambda_1 \lambda_3 = 0
$$

and

$$
d = 0, \quad \text{otherwise}.
$$

Proof. Let $D$ be a derivation of $k(\lambda_1, \lambda_2, \lambda_3)$. The matrix representation with respect to the basis $\mathfrak{B}(k_{\lambda_1, \lambda_2, \lambda_3})$ is a $4 \times 4$ matrix $[D] = (m_{ij})^T$ such that

$$
D(e_{i-1}) = \sum_{j=1}^{4} m_{ij} e_{j-1}, \quad 1 \leq i \leq 4.
$$

To find $m_{ij}$, simply apply the definition of derivation for the products $e_i e_j$ with $1 \leq i \leq j \leq 3$. We start by presenting the remark that enables us to calculate the first column. Since $e_0$ is a central idempotent, it follows that

$$
D(e_0) = 0 \quad \text{for every } D \in \text{Der}(k_{\lambda_1, \lambda_2, \lambda_3}).
$$

Moreover, if we are given a derivation $D$, then the first column of $[D]$ consists of only zeros. In fact, we have

$$
D(e_0 \cdot e_i) = D(e_0) \cdot e_i + e_0 \cdot D(e_i) \quad \text{for all } i = 1, 2, 3,
$$

which implies that

$$
D(e_i) = D(e_0) \cdot e_i + D(e_i),
$$

so, we get

$$
D(e_0) \cdot e_i = 0 \quad \text{for all } i = 1, 2, 3.
$$

Hence, one obtains $m_{11} = m_{12} = m_{13} = m_{14} = 0$ only by evaluating, for instance, $D(e_0)e_0 = 0$. Applying the Leibnitz rule to quaternionic units, we get $D(\bar{e}_i^2) = D(e_1)e_1 + e_1 D(e_1)$, which implies that

$$
-\lambda_1 \lambda_2 D(e_0) = (m_{21} e_0 \cdot e_1 + m_{22} e_1^2 + m_{23} e_2 \cdot e_1 + m_{24} e_3 \cdot e_1)
\quad + (m_{21} e_1 \cdot e_0 + m_{22} e_1^2 + m_{23} e_1 \cdot e_2 + m_{24} e_1 \cdot e_3)
\quad = -2\lambda_1 \lambda_2 m_{22} e_0 + 2m_{21} e_1 + 0e_2 + 0e_3.
$$

So,

$$
-\lambda_1 \lambda_2 m_{11} e_0 - \lambda_1 \lambda_2 m_{12} e_1 - \lambda_1 \lambda_2 m_{13} e_2 - \lambda_1 \lambda_2 m_{14} e_3 = -2\lambda_1 \lambda_2 m_{22} e_0 + 2m_{21} e_1
$$

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which gives
\[ -\lambda_1\lambda_2 m_{11} = -2\lambda_1\lambda_2 m_{22} = 0; -\lambda_1\lambda_2 m_{12} = 2m_{21} \]
and hence \( m_{21} = 0 \), since we already know \( m_{12} = 0 \), we get \( m_{11} = m_{12} = m_{13} = m_{14} = m_{22} = m_{21} = 0 \). If we continue going that way, we obtain
\[ D(e_2) = D(e_2) e_2 + e_2 D(e_2), \]
which implies that
\[ -\lambda_1\lambda_3 D(e_0) = (m_{31} e_0 \cdot e_2 + m_{32} e_1 \cdot e_2 + m_{33} e_2^2 + m_{34} e_3 \cdot e_2) + (m_{31} e_2 \cdot e_0 + m_{32} e_2 \cdot e_1 + m_{33} e_2^2 + m_{34} e_2 \cdot e_3) = -2\lambda_1\lambda_3 m_{33} e_0 + 0 e_1 + 2m_{31} e_2 + 0 e_3, \]
and hence \( -\lambda_1\lambda_3 m_{11} = -2\lambda_1\lambda_3 m_{33}, -\lambda_1\lambda_3 m_{13} = 2m_{31}, m_{31} = 0 \)
and
\[ m_{33} = \begin{cases} 0, & \text{if } \lambda_1\lambda_3 \neq 0 \\ \text{otherwise,} & \text{if } \lambda_1\lambda_3 = 0. \end{cases} \]
Also,
\[ D(e_3) = D(e_3) e_3 + e_3 D(e_3), \]
it follows that
\[ -\lambda_2\lambda_3 D(e_0) = (m_{41} e_0 \cdot e_3 + m_{42} e_1 \cdot e_3 + m_{43} e_2 \cdot e_3 + m_{44} e_3^2) + (m_{41} e_3 \cdot e_0 + m_{42} e_3 \cdot e_1 + m_{43} e_3 \cdot e_2 + m_{44} e_3^2) = -2\lambda_2\lambda_3 m_{44} e_0 + 0 e_1 + 0 e_2 + 2m_{41} e_3, \]
which gives \( -\lambda_2\lambda_3 m_{11} = -2\lambda_2\lambda_3 m_{44} \) and \( -\lambda_2\lambda_3 m_{14} = 2m_{41} \). Since \( m_{14} = 0 \), we conclude that, \( m_{41} = 0 \) and
\[ m_{44} = \begin{cases} 0, & \text{if } \lambda_2\lambda_3 \neq 0 \\ \text{otherwise,} & \text{if } \lambda_2\lambda_3 = 0. \end{cases} \]
Now we apply the same procedure for \( e_1 e_2 = -e_2 e_1 : \)
\[ D(e_1) = D(e_1) e_2 + e_1 \cdot D(e_2), \]
\[ D(e_1 e_3) = (m_{21} e_0 \cdot e_2 + m_{22} e_1 \cdot e_2 + m_{23} e_2^2 + m_{24} e_3 \cdot e_2) + (m_{31} e_1 \cdot e_0 + m_{32} e_2 \cdot e_3 + m_{33} e_2^2 + m_{34} e_3 \cdot e_2) \]
\[ \lambda_1 D(e_3) = m_{21} e_2 + \lambda_1 m_{22} e_3 - \lambda_1\lambda_3 m_{23} e_0 - \lambda_3 m_{24} e_1 + m_{31} e_1 - \lambda_1\lambda_3 m_{32} e_0 + m_{33} e_1 e_3 - \lambda_2 m_{34} e_2 \]
\[ = (-\lambda_1\lambda_3 m_{23} - \lambda_1\lambda_3 m_{32}) e_0 + (m_{31} - \lambda_3 m_{24}) e_1 + (m_{21} - \lambda_2 m_{34}) e_2 + (\lambda_1 m_{22} + \lambda_1 m_{33}) e_3. \]
Thus, from \( \lambda_1 D(e_3) = \lambda_1 m_{41} e_0 + \lambda_1 m_{42} e_1 + \lambda_1 m_{43} e_2 + \lambda_1 m_{44} e_3, \) we obtain
\[ \lambda_1 m_{41} = -\lambda_1\lambda_3 m_{23} - \lambda_1\lambda_2 m_{32}, \lambda_1 m_{42} = m_{31} - \lambda_3 m_{24}, \lambda_1 m_{43} = m_{21} - \lambda_2 m_{34}, \]
\[ \lambda_1 m_{44} = 0. \]
and \( \lambda_1 m_{44} = \lambda_1 m_{22} + m_{33} \lambda_1 \). Therefore, we have
\[
\begin{align*}
m_{32} &= -\frac{\lambda_3}{\lambda_2} m_{23}, \
m_{42} &= -\frac{\lambda_3}{\lambda_1} m_{24}, \
m_{43} &= -\frac{\lambda_2}{\lambda_1} m_{34}, \quad \text{and} \
m_{33} &= m_{44},
\end{align*}
\]
where \( \lambda_1, \lambda_2 \neq 0 \).

We can deduce from the previous theorem that the maximum dimension of \( \text{Der}(k\lambda_1,\lambda_2,\lambda_3) \) is 4. Furthermore, since \( Z(k\lambda_1,\lambda_2,\lambda_3) = \mathbb{R} \cdot e_0 \), there is only one noninner and no central derivation. From \( \text{ad}(k\lambda_1,\lambda_2,\lambda_3) \cong k\lambda_1,\lambda_2,\lambda_3/Z(k\lambda_1,\lambda_2,\lambda_3) \) we can conclude that the algebra \( \text{ad}(k\lambda_1,\lambda_2,\lambda_3) \) of inner derivations is generated by the following matrices:
\[
\begin{align*}
\text{ad}(e_1) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\lambda_2 \\ 0 & 0 & 2\lambda_1 & 0 \end{pmatrix}, \\
\text{ad}(e_2) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\lambda_3 \\ 0 & 0 & 0 & 0 \\ 0 & -2\lambda_1 & 0 & 0 \end{pmatrix}, \\
\text{ad}(e_3) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2\lambda_3 & 0 \\ 0 & 2\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

If \( a = b = c = \lambda_1 \lambda_3 = 0 \), then \( D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) is the only noninner derivation of \( k\lambda_1,\lambda_2,\lambda_3 \), i.e. \( D \in \text{Der}(k\lambda_1,\lambda_2,\lambda_3)/\text{ad}(k\lambda_1,\lambda_2,\lambda_3) \). It is clear that \( D \) does not commute with any inner derivation.

For \( \lambda_1 = 1 \) we find the following result [9, Theorem 3.5].

For \( \lambda_1 = \lambda_2 = 1, \lambda_3 = 0 \) (semiquaternions, we can easily find the result [9, Corollary 3.6].

For \( \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1 \) (split quaternions, we can quickly locate the result [9, Corollary 3.7].

For \( \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0 \) (split semiquaternions), it’s simple to find [9, Corollary 3.9].

4 Generalized derivations

In the following theorem, we aim to generalize the result obtained in theorem 3.1 to generalized derivations of the algebra of quaternions depending on 3-parameters (3PGQ).

**Theorem 4.1** Let \( F : k\lambda_1,\lambda_2,\lambda_3 \to k\lambda_1,\lambda_2,\lambda_3 \) be a generalized derivation of \( k\lambda_1,\lambda_2,\lambda_3 \) and \( \mathfrak{B}(k\lambda_1,\lambda_2,\lambda_3) = \{ e_0, e_1, e_2, e_3 \} \) denotes its standard base. Then
the matrix representation $[F]$ of $F$ is as follows:

$$[F] = \begin{pmatrix}
m_{11} - \lambda_1 \lambda_2 m_{12} & -\lambda_1 \lambda_3 m_{13} & -\lambda_2 \lambda_3 m_{14} \\
m_{12} & m_{11} - \lambda_3 m_{14} & \lambda_3 m_{13} \\
m_{13} & \lambda_2 m_{14} & m_{11} - \lambda_2 m_{12} \\
m_{14} & -\lambda_1 m_{13} & \lambda_1 m_{12} & m_{11}
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & -\frac{\lambda_2}{\lambda_1} a & -\frac{\lambda_3}{\lambda_1} b \\
0 & a & d & -\frac{\lambda_3}{\lambda_1} c \\
0 & b & c & d
\end{pmatrix},$$

where $m_{ij}, a, b, c, d \in \mathbb{R}$, $\lambda_1 \lambda_2 \neq 0$ such that

$$D = \begin{pmatrix}
d = d_{\lambda_1, \lambda_3} & 0, \\
0 & 0
\end{pmatrix},$$

Proof. Let $[F] = (m_{ij})^T$ be the $4 \times 4$ matrix whose entries are defined by the following equations:

$$F(e_0) = m_{11} e_0 + m_{12} e_1 + m_{13} e_2 + m_{14} e_3,$$

$$F(e_1) = m_{21} e_0 + m_{22} e_1 + m_{23} e_2 + m_{24} e_3,$$

$$F(e_2) = m_{31} e_0 + m_{32} e_1 + m_{33} e_2 + m_{34} e_3,$$

$$F(e_3) = m_{41} e_0 + m_{42} e_1 + m_{43} e_2 + m_{44} e_3.$$

We may calculate the $m_{ij}$ by replacing $x$ with $e_0$ and $y$ with $e_i$ in the equation (2.1) so:

$$F(e_i) = F(e_0 \cdot e_i) = F(e_0) \cdot e_i + e_0 \cdot D(e_i) \text{ for all } i = 1, 2, 3. \quad (4.1)$$

For some derivation $D \in Der(k_{\lambda_1, \lambda_2, \lambda_3})$. As a result, we have $F(e_0) = F(e_0) + D(e_0)$, from which we cannot determine the items in the first column of $[F]$ directly. Nonetheless, we obtain from equation (4.1) with $i = 1, 2,$ and $3$ the following relations:

$$m_{21} = -\lambda_1 \lambda_2 m_{12}, m_{22} = m_{11}, m_{23} = \lambda_2 m_{14} + d_{23}, m_{24} = -\lambda_1 m_{13} + d_{24},$$

$$m_{31} = -\lambda_1 \lambda_3 m_{13}, m_{32} = -\lambda_3 m_{14} - \frac{\lambda_3}{\lambda_2} d_{23}, m_{33} = m_{11} + d_{33}, m_{34} = \lambda_1 m_{12} + d_{34},$$

$$m_{41} = -\lambda_2 \lambda_3 m_{14}, m_{42} = \lambda_3 m_{13} - \frac{\lambda_3}{\lambda_1} d_{24}, m_{43} = -\lambda_2 m_{12} - \frac{\lambda_2}{\lambda_1} d_{34}, m_{44} = m_{11} + d_{33}.$$  \hfill \Box

Remark 4.1 For $\lambda_1 = 1$, we can easily find [9, Theorem 4.2].

Example 4.1 Let $D$ denote the derivation of $k_{\lambda_1, \lambda_2, \lambda_3}$ such that $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $d_{24} = -d_{12} = 1$, and $d_{ij} = 0$ otherwise. Also we take $f_{11} = 1$, $f_{ij} = 0$ otherwise. Then $F(x) = x_0 e_0 + (x_1 - x_3) e_1 + x_2 e_2 + (x_0 + x_3) e_3$ is a generalized derivation of $k_{\lambda_1, \lambda_2, \lambda_3}$.
5 Generalized Fibonacci quaternions

In this section, we will study some properties of generalized Fibonacci quaternions in the generalized algebra of quaternions. The following result presents a generalization of [6, Theorem 2.4] in the generalized quaternion algebra \( (3PGQs) \).

**Theorem 5.1** The norm of the \( n \)th Fibonacci quaternion \( F_n \) in a generalized quaternion algebra \( \mathbb{k}(\lambda_1, \lambda_2, \lambda_3) \) is

\[
N(F_n) = h_{2n+2}^1 + 2\lambda_1\lambda_3,3\lambda_1\lambda_3 + h_{2n+3}^{(\lambda_1\lambda_2-1)+2\lambda_3(\lambda_2-\lambda_1),\lambda_3(\lambda_2-\lambda_1)} - 2(\lambda_1\lambda_2 - 1 + \lambda_3(\lambda_2 - \lambda_1))f_n f_{n+1}.
\]

**Proof.** We have

\[
N(F_n) = f_n^2 + \lambda_1\lambda_2 f_{n+1}^2 + \lambda_1\lambda_3 f_{n+2}^2 + \lambda_2\lambda_3 f_{n+3}^2
\]

\[
= f_n^2 + \lambda_1\lambda_2 f_{n+1}^2 + \lambda_3(\lambda_1 f_{n+2}^2 + \lambda_2 f_{n+3}^2)
\]

\[
= f_{2n+1} + (\lambda_1\lambda_2 - 1)f_{n+1}^2 + \lambda_3(\lambda_1 f_{2n+5}^2 - \lambda_1 f_{n+3}^2 + \lambda_2 f_{n+3}^2)
\]

\[
= f_{2n+1} + (\lambda_1\lambda_2 - 1)f_{n+1}^2 + \lambda_3(\lambda_1 f_{2n+5} + \lambda_2 f_{n+3})
\]

\[
= f_{2n+1} + \lambda_3(\lambda_1 f_{2n+5} + \lambda_3(\lambda_2 - \lambda_1))f_{n+1}^2 + \lambda_3(\lambda_1 f_{2n+5} + \lambda_2 f_{n+3})
\]

\[
= f_{2n+1} + \lambda_3(3f_{2n+5} + 2f_{2n+1}) + (\lambda_1\lambda_2 - 1)f_{n+1}^2 + \lambda_3(\lambda_2 - \lambda_1)f_{n+3}^2
\]

\[
= (1 + 2\lambda_3\lambda_1) f_{2n+1} + 3\lambda_3(\lambda_1 f_{2n+2} + \lambda_3(\lambda_2 - \lambda_1))f_{n+1}^2 + \lambda_3(\lambda_2 - \lambda_1)f_{n+3}^2
\]

\[
= h_{2n+2}^1 + 2\lambda_1\lambda_3,3\lambda_1\lambda_3 + (\lambda_1\lambda_2 - 1)f_{n+1}^2 + \lambda_3(\lambda_2 - \lambda_1)f_{n+3}^2
\]

\[
= h_{2n+2}^1 + 2\lambda_1\lambda_3,3\lambda_1\lambda_3 + (\lambda_1\lambda_2 - 1)(f_{2n+2} + 2 f_{n+1} f_n)
\]

\[
+ \lambda_3(\lambda_2 - \lambda_1)(f_{2n+6} - 2 f_{n+3} f_{n+2})
\]

\[
= h_{2n+2}^1 + 2\lambda_1\lambda_3,3\lambda_1\lambda_3 + (\lambda_1\lambda_2 - 1)f_{2n+2} + \lambda_3(\lambda_2 - \lambda_1)f_{2n+6}
\]

\[
- 2[(\lambda_1\lambda_2 - 1)] f_{n+1} f_n + \lambda_3(\lambda_2 - \lambda_1)f_{n+3} f_{n+2}
\]

\[
= h_{2n+2}^1 + 2\lambda_1\lambda_3,3\lambda_1\lambda_3 + (\lambda_1\lambda_2 - 1)f_{2n+2} + \lambda_3(\lambda_2 - \lambda_1)f_{2n+6}
\]

\[
- 2[(\lambda_1\lambda_2 - 1)] f_{n+1} f_n + \lambda_3(\lambda_2 - \lambda_1)f_{n+3} f_{n+2}
\]

\[
+ \lambda_3(\lambda_2 - \lambda_1)f_{n+3} f_{n+2}
\]

\[
= h_{2n+2}^1 + 2\lambda_1\lambda_3,3\lambda_1\lambda_3 + (\lambda_1\lambda_2 - 1)f_{2n+2} + \lambda_3(\lambda_2 - \lambda_1)f_{2n+6}
\]

\[
- 2[(\lambda_1\lambda_2 - 1)] f_{n+1} f_n + \lambda_3(\lambda_2 - \lambda_1)f_{n+3} f_{n+2}
\]

\[
+ \lambda_3(\lambda_2 - \lambda_1)f_{n+3} f_{n+2}
\]

\[
= h_{2n+2}^1 + 2\lambda_1\lambda_3,3\lambda_1\lambda_3 + (\lambda_1\lambda_2 - 1)f_{2n+2} + \lambda_3(\lambda_2 - \lambda_1)f_{2n+6}
\]

\[
- 2[(\lambda_1\lambda_2 - 1)] f_{n+1} f_n + \lambda_3(\lambda_2 - \lambda_1)f_{n+3} f_{n+2}
\]

\[
+ \lambda_3(\lambda_2 - \lambda_1)f_{n+3} f_{n+2}
\]

\[
= h_{2n+2}^1 + 2\lambda_1\lambda_3,3\lambda_1\lambda_3 + (\lambda_1\lambda_2 - 1)f_{2n+2} + \lambda_3(\lambda_2 - \lambda_1)f_{2n+6}
\]

\[
- 2[(\lambda_1\lambda_2 - 1)] f_{n+1} f_n + \lambda_3(\lambda_2 - \lambda_1)f_{n+3} f_{n+2}
\]

\[
+ \lambda_3(\lambda_2 - \lambda_1)f_{n+3} f_{n+2}
\]

\[
= h_{2n+2}^1 + 2\lambda_1\lambda_3,3\lambda_1\lambda_3 + (\lambda_1\lambda_2 - 1)f_{2n+2} + \lambda_3(\lambda_2 - \lambda_1)f_{2n+6}
\]

\[
- 2[(\lambda_1\lambda_2 - 1)] f_{n+1} f_n + \lambda_3(\lambda_2 - \lambda_1)f_{n+3} f_{n+2}
\]

\[
+ \lambda_3(\lambda_2 - \lambda_1)f_{n+3} f_{n+2}
\]
= h_n^{1+2\lambda_1\lambda_3,3\lambda_1\lambda_3} + (\lambda_1\lambda_2 - 1)f_{2n+2} + \lambda_3(\lambda_2 - \lambda_1)2f_{2n+4}
+ \lambda_3(\lambda_2 - \lambda_1), f_{2n+3} - 2[(\lambda_1\lambda_2 - 1) + \lambda_3(\lambda_2 - \lambda_1)]f_{n+1}f_n
- 2\lambda_3(\lambda_2 - \lambda_1)f_{2n+3}
= h_n^{1+2\lambda_1\lambda_3,3\lambda_1\lambda_3} + \lambda_1\lambda_2 - 1)f_{2n+2} + 2\lambda_3(\lambda_2 - \lambda_1)f_{2n+4}
- \lambda_3(\lambda_2 - \lambda_1), f_{2n+3} - 2[(\lambda_1\lambda_2 - 1) + \lambda_3(\lambda_2 - \lambda_1)]f_{n+1}f_n
= h_n^{1+2\lambda_1\lambda_3,3\lambda_1\lambda_3} + [(\lambda_1\lambda_2 - 1) + 2\lambda_3(\lambda_2 - \lambda_1)]f_{2n+2}
+ [2\lambda_3(\lambda_2 - \lambda_1) - \lambda_3(\lambda_2 - \lambda_1)]f_{2n+3} - 2[(\lambda_1\lambda_2 - 1) + \lambda_3(\lambda_2 - \lambda_1)]f_{n+1}f_n
= h_n^{1+2\lambda_1\lambda_3,3\lambda_1\lambda_3} + h_n^{(\lambda_1\lambda_2 - 1) + 2\lambda_3(\lambda_2 - \lambda_1)}, \lambda_3(\lambda_2 - \lambda_1)
- 2[(\lambda_1\lambda_2 - 1) + \lambda_3(\lambda_2 - \lambda_1)]f_{n+1}f_n.

It should be obvious that if we replace \( \lambda_1 = 1, \lambda_2 = \alpha, \) and \( \lambda_3 = \beta \) in our result we get

\[
N(F_n) = h_n^{1+2\beta,3\beta} + h_n^{(\alpha-1) + 2\beta(\alpha-1)}, \beta(\alpha-1) - 2((\alpha - 1) + \beta(\alpha - 1))f_nf_{n+1}
= h_n^{1+2\beta,3\beta} + ((\alpha - 1) + 2\beta(\alpha - 1))f_{2n+2} + \beta(\alpha - 1)f_{2n+3}
- 2(\alpha - 1)(1 + \beta)f_nf_{n+1}
= h_n^{1+2\beta,3\beta} + (\alpha - 1)(1 + 2\beta)f_{2n+2} + \beta f_{2n+3} - 2(\alpha - 1)(1 + \beta)f_nf_{n+1}
= h_n^{1+2\beta,3\beta} + (\alpha - 1)h_n^{1+2\beta,3\beta} - 2(\alpha - 1)(1 + \beta)f_nf_{n+1}.
\]

\[\square\]

Note that the following theorem gives a better generalization of [6, Theorem 2.5].

**Theorem 5.2** The norm of the \( n \)th generalized Fibonacci quaternion \( H_n^{p,q} \) in a generalized quaternion algebra \( \mathbb{k}(\lambda_1, \lambda_2, \lambda_3) \) is

\[
N(H_n^{p,q}) = p^2h_n^{1+2\lambda_1\lambda_3,3\lambda_1\lambda_3} + p^2h_n^{1+2\lambda_1\lambda_2-1+2\lambda_3(\lambda_2 - \lambda_1)\lambda_3(\lambda_2 - \lambda_1)}
+ q^2h_n^{1+2\lambda_1\lambda_3,3\lambda_1\lambda_3} + q^2h_n^{1+2\lambda_1\lambda_2-1+2\lambda_3(\lambda_2 - \lambda_1)\lambda_3(\lambda_2 - \lambda_1)}
+ [-2p^2((\lambda_1\lambda_2 - 1) + \lambda_3(\lambda_2 - \lambda_1)) + 2pq(1 - \lambda_1\lambda_2)]f_{n-1}f_n
- 2q^2[(\lambda_1\lambda_2 - 1) + \lambda_3(\lambda_2 - \lambda_1)]f_nf_{n+1}
+ h_n^{2pq\lambda_1\lambda_2,3pq\lambda_2\lambda_3} + 2pq\lambda_2\lambda_3(f_{2n} + f_{2n+3}) + 2pq\lambda_3(\lambda_1 - \lambda_2)f_{n+1}f_{n+2}.
\]
Proof. Let $H_{n}^{p,q} = h_n,1 + h_{n+1},e_1 + h_{n+2},e_2 + h_{n+3},e_3$ be the $n$th generalized Fibonacci quaternion. The norm is written as follows:

\[
N(H_{n}^{p,q}) = h_n^2 + \lambda_1 \lambda_2 h_{n+1}^2 + \lambda_1 \lambda_3 h_{n+2}^2 + \lambda_2 \lambda_3 h_{n+3}^2 \\
= (p f_{n-1} + q f_{n})^2 + \lambda_1 \lambda_2 (p f_{n} + q f_{n+1})^2 \\
+ \lambda_1 \lambda_3 (p f_{n+1} + q f_{n+2})^2 + \lambda_2 \lambda_3 (p f_{n+2} + q f_{n+3})^2 \\
= p^2 (f_{n-1}^2 + \lambda_1 \lambda_2 f_n^2 + \lambda_1 \lambda_3 f_{n+1}^2 + \lambda_2 \lambda_3 f_{n+2}^2) \\
+ q^2 (f_n^2 + \lambda_1 \lambda_2 f_{n+1}^2 + \lambda_1 \lambda_3 f_{n+2}^2 + \lambda_2 \lambda_3 f_{n+3}^2) \\
+ 2pq (f_{n-1} f_n + \lambda_1 \lambda_2 f_n f_{n+1} + \lambda_1 \lambda_3 f_{n+1} f_{n+2} + \lambda_2 \lambda_3 f_{n+2} f_{n+3}) \\
= p^2 h_{2n}^2 + h_{2n+1} \lambda_1 \lambda_3 + p^2 h_{2n+1}^2 + q^2 h_{2n+2}^2 + h_{2n+3} \lambda_1 \lambda_3 + q^2 h_{2n+3}^2 \\
- 2[(\lambda_1 \lambda_2 - 1) + \lambda_3 (\lambda_2 - \lambda_1)] f_{n-1} f_n \\
- 2q^2 [(\lambda_1 \lambda_2 - 1) + \lambda_3 (\lambda_2 - \lambda_1)] f_n f_{n+1} + 2pq (1 - \lambda_1 \lambda_2) f_{n-1} f_n \\
+ (\lambda_1 \lambda_3 - \lambda_2 \lambda_3) f_{n+1} f_{n+2} + \lambda_2 \lambda_3 f_{n+2} \\
= p^2 h_{2n}^2 + h_{2n+1} \lambda_1 \lambda_3 + p^2 h_{2n+1}^2 + q^2 h_{2n+2}^2 + h_{2n+3} \lambda_1 \lambda_3 + q^2 h_{2n+3}^2 \\
- 2[(\lambda_1 \lambda_2 - 1) + \lambda_3 (\lambda_2 - \lambda_1)] f_{n-1} f_n \\
- 2q^2 [(\lambda_1 \lambda_2 - 1) + \lambda_3 (\lambda_2 - \lambda_1)] f_n f_{n+1} + 2pq (1 - \lambda_1 \lambda_2) f_{n-1} f_n \\
+ (\lambda_1 \lambda_3 - \lambda_2 \lambda_3) f_{n+1} f_{n+2} + \lambda_2 \lambda_3 f_{n+2}.
\]

\[\square\]

If $\lambda_1 = 1$, $\lambda_2 = \alpha$, $\lambda_3 = \beta$, we can easily arrive at

\[
N(H_{n}^{p,q}) = p^2 h_{2n}^2 + q^2 (\alpha - 1) h_{2n+1}^2, \beta + p^2 h_{2n+2}^2 + q^2 (\alpha - 1) h_{2n+3}^2 \beta \\
- 2p(\alpha - 1)(p + q) f_{n-1} f_n - 2q^2 (\alpha - 1)(1 + \beta) f_n f_{n+1} + \\
+ h_{2n+1}^{2pq \beta, 2pq \alpha \beta} + 2pq \alpha (f_{2n} + f_{2n+3}) + 2pq \beta (1 - \alpha) f_{n+1} f_{n+2}.
\]
As a result, [6, Theorem 2.5] is obtained. According to [7] the nth term of a Fibonacci element is written as follows:

\[ f_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n) = \frac{\alpha^n}{\sqrt{5}} \left( 1 - \frac{\beta^n}{\alpha^n} \right), \]

where \( \alpha = \frac{1 + \sqrt{5}}{2} \) and \( \beta = \frac{1 - \sqrt{5}}{2} \). Thus we can easily compute the following by using the previous expressions:

\[
\lim_{n \to +\infty} N(F_n) = \lim_{n \to +\infty} \left( f_n^2 + \lambda_1 \lambda_2 f_{n+1}^2 + \lambda_1 \lambda_3 f_{n+2}^2 + \lambda_2 \lambda_3 f_{n+3}^2 \right) \\
= \lim_{n \to +\infty} \left( \frac{\alpha^{2n}}{5} + \lambda_1 \lambda_2 \frac{\alpha^{2n+2}}{5} + \lambda_1 \lambda_3 \frac{\alpha^{2n+4}}{5} + \lambda_2 \lambda_3 \frac{\alpha^{2n+6}}{5} \right) \\
= \text{sgn} \ E(\lambda_1, \lambda_2, \lambda_3) \cdot \infty
\]

\[
E(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{5} (1 + \lambda_1 \lambda_2 \alpha^2 + \lambda_1 \lambda_3 \alpha^4 + \lambda_2 \lambda_3 \alpha^6) \\
= \frac{1}{5} (1 + \lambda_1 \lambda_2 (\alpha + 1) + \lambda_1 \lambda_3 (3\alpha + 2) + \lambda_2 \lambda_3 (8\alpha + 5)) \\
= \frac{1}{5} (1 + \lambda_1 \lambda_2 + 2\lambda_1 \lambda_3 + 5\lambda_2 \lambda_3 + \alpha(\lambda_1 \lambda_2 + 3\lambda_1 \lambda_3 + 8\lambda_2 \lambda_3)).
\]

- If \( E(\lambda_1, \lambda_2, \lambda_3) > 0 \), then \( \lim_{n \to +\infty} N(F_n) = +\infty \) thus there exists a number \( n_1 \in \mathbb{N} \) such that for all \( n \geq n_1 \), we have \( N(F_n) > 0 \).
- If \( E(\lambda_1, \lambda_2, \lambda_3) < 0 \), then \( \lim_{n \to +\infty} N(F_n) = -\infty \) thus there exists a number \( n_2 \in \mathbb{N} \) such that for all \( n \geq n_2 \), we have \( N(F_n) < 0 \).

Therefore for all \( \lambda_1, \lambda_3, \lambda_3 \in \mathbb{R} \) with \( E(\lambda_1, \lambda_2, \lambda_3) \neq 0 \), in the algebra \( \mathbb{K}(\lambda_1, \lambda_2, \lambda_3) \) there is a natural number \( n_0 = \max(n_1, n_2) \) such that \( N(F_n) \neq 0 \), hence \( F_n \) is an invertible element for all \( n \geq n_0 \). Using the same arguments, we can compute

\[
\lim_{n \to +\infty} N(H_n^{pq}) = \lim_{n \to +\infty} \left( h_n^2 + \lambda_1 \lambda_2 h_{n+1}^2 + \lambda_1 \lambda_3 h_{n+2}^2 + \lambda_2 \lambda_3 h_{n+3}^2 \right) \\
= \lim_{n \to +\infty} \left( (pf_{n-1} + qf_n)^2 + \lambda_1 \lambda_2 (pf_n + qf_{n+1})^2 \right) \\
+ \lambda_1 \lambda_3 (pf_{n+1} + qf_{n+2})^2 + \lambda_2 \lambda_3 (pf_{n+2} + qf_{n+3})^2 \\
= \text{sgn} \ E'(\lambda_1, \lambda_2, \lambda_3) \cdot \infty,
\]
where
\[
E'(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{5}((p + \alpha q)^2 + \lambda_1 \lambda_2 (p\alpha + \alpha^2 q)^2 + \lambda_1 \lambda_3 (p\alpha^2 + \alpha^3 q)^2 \\
+ \lambda_2 \lambda_3 (p\alpha^3 + \alpha^4 q)^2)
= \frac{1}{5}(p + \alpha q)^2(1 + \lambda_1 \lambda_2 \alpha^2 + \lambda_1 \lambda_3 \alpha^4 + \lambda_2 \lambda_3 \alpha^6)
= \frac{1}{5}(p + \alpha q)^2 E(\lambda_1, \lambda_2, \lambda_3).
\]

From the above, there exists a natural integer \(n'_0\) such that \(N(H_{p,q}^n) \neq 0\) for all \(n \geq n'_0\), therefore \(H_{p,q}^n\) is an invertible element for all \(n \geq n'_0\).

**Theorem 5.3** For all \(\lambda_1, \lambda_3, \lambda_3 \in \mathbb{R}\) with \(E'(\lambda_1, \lambda_2, \lambda_3) \neq 0\), there exists a natural number \(n'_0\) such that for all \(n \geq n'_0\) Fibonacci elements \(F_n\) and generalized Fibonacci elements \(H_{p,q}^n\) are invertible elements in the algebra \(E(\lambda_1, \lambda_2, \lambda_3)\).

**Remark 5.1** Algebra \(k(\lambda_1, \lambda_2, \lambda_3)\) is not always a division algebra, and obtaining an example of an invertible element can be difficult at times. In this algebra, the above theorem gives us infinite sets of invertible elements, namely Fibonacci and generalized Fibonacci elements.

**References**


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AUTHORS

RACHID CHAKER,
Department of Mathematics,
Polydisciplinary Faculty,
Sidi Mohamed Ben Abdellah University,
Taza, Morocco,
E-mail: rachid.chaker@usmba.ac.ma

ABDELKARIM BOUA (Corresponding author),
Department of Mathematics,
Polydisciplinary Faculty,
Sidi Mohamed Ben Abdellah University,
Taza, Morocco,
E-mail: abdelkarimboua@yahoo.fr