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Dual affine translation surfaces in 3-dimensional simply isotropic space $\mathbb{I}_3^1$

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Abstract In this paper, we study dual affine translation surfaces in the 3-dimensional simply isotropic space $\mathbb{I}_3^1$. We obtain the classification results for dual affine translation surfaces with constant dual isotropic Gaussian curvature or constant dual isotropic mean curvature.

Keywords simply isotropic space · dual Gaussian curvature · dual mean curvature · affine translation surface · minimal surface

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1 Introduction

A 3-dimensional simply isotropic space $\mathbb{I}_3^1$ is one of the Cayley-Klein geometries defined from the 3-dimensional projective space $\mathcal{P}(\mathbb{R}^3)$ with an invariant absolute figure [13,15]. From a metric point of view, a simply isotropic space is a real vector space equipped with the degenerated metric

$$\langle x, y\rangle = x_1y_1 + x_2y_2, \tag{1.1}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$. In 3-dimensional projective spaces $\mathcal{P}(\mathbb{R}^3)$, points are duals to planes and conversely planes are duals to points, similarly, straight lines are dual to straight lines and inclusions are reversed. However, duality cannot be applied to metric quantities of Euclidean geometry. In isotropic spaces, the metric duality can be visualized by the polarity of the isotropic unit sphere $\sum : z = \frac{1}{2}(u^2 + v^2)$. A point $q = (q_1, q_2, q_3)$ is mapped to the plane $P : q_1u + q_2v - q_3$. The points q_1 and q_2 with an isotropic distance of d in between are mapped to planes P_1 and P_2 with an isotropic angle d, and conversely. The parallel points and parallel planes correspond to each other in the principle of duality. Let u, v be two independent variables, a surface $r(u, v) = (u, v, z(u, v))$ treated as contact elements of tangent planes and points corresponds to $r^*(u,v)$, parametrized by

$$\begin{cases} u^* = z_u(u,v), \\ v^* = z_v(u,v), \\ z^* = uz_u(u,v) + vz_v(u,v) - z(u,v). \end{cases}$$

Usually, a regular surface $r(u,v)$ parameterized as $r(u,v) = (u,v,z(u,v))$ in a 3-dimensional space is called a translation surface, if

$$z(u,v) = f(u) + g(v), \tag{1.2}$$

where $f(u)$ and $g(v)$ are differentiable functions of independent variables u and v, respectively. Scherk surface [14] is the first non-trivial example of a minimal translation surface in 3-dimensional Euclidean space $\mathbb{E}^3$ and is given by

$$z(u,v) = \frac{1}{b} \log \left| \frac{\cos(bu)}{\cos(bv)} \right|, \tag{1.3}$$

where $b(\neq 0)$ is a constant. Liu [7] studied translation surfaces with constant mean curvature in $\mathbb{E}^3$ and the 3-dimensional Minkowski space $\mathbb{E}^3_1$. López in [11], studied minimal translation surfaces in 3-dimensional Hyperbolic space $\mathbb{H}^3$. For some of the related works, we refer the reader to see [6,12,18,19].

In [9], Liu and Yu generalized the notion of a translation surface into an affine translation surface, which is a surface of the form

$$z(u,v) = f(u) + g(au + v), \tag{1.4}$$

where $a(\neq 0)$ is a constant. Using this generalized notion, they further classified minimal surfaces in $\mathbb{E}^3$. Later in [8], Liu and Jung classified such surfaces with constant non-zero mean curvature in $\mathbb{E}^3$. In [10], dual translation surfaces having constant dual isotropic Gaussian or dual isotropic mean curvatures have been classified in simply isotropic space $\mathbb{I}^1_3$. In [1], Aydin investigated the generalized formulation of translation surfaces in isotropic space, classifying such surfaces with constant curvature. The reader can see some related studies on the generalization of translation surfaces in [3–5,16, 17].

A general notion of the translation surface $r(u,v)$ parameterized as

$$r(u,v) = (u,v,z(u,v)) \tag{1.5}$$

is given by taking $z(u,v)$ to be of the form

$$z(u,v) = f(a_1u + a_2v) + g(a_3u + a_4v), \tag{1.6}$$

where $a_1, \ldots, a_4$ are constants and $a_1a_4 - a_2a_3 \neq 0$.

Remark 1.1 Affine translation surfaces will refer to this generalized notion throughout this paper.

By a parameter change, (1.6) can be written as

$$r(x,y)=\left(\frac{a_1x-a_2y}{a_1a_4-a_2a_3},\frac{a_3x+a_1y}{a_1a_4-a_2a_3},f(x)+g(y)\right), \tag{1.7}$$

or

$$r(x,y)=\gamma_1(x)+\gamma_2(y), \tag{1.8}$$

where $\gamma_1(x)$, $\gamma_2(y)$ are 2 plane curves given by

$$\gamma_1(x)=\left(\frac{a_1x}{a_1a_4-a_2a_3},\frac{a_3x}{a_1a_4-a_2a_3},f(x)\right), \tag{1.9}$$

$$\gamma_2(y)=\left(\frac{-a_2y}{a_1a_4-a_2a_3},\frac{a_1y}{a_1a_4-a_2a_3},g(y)\right). \tag{1.10}$$

In this paper, using this generalized notion of translation surfaces in 3-dimensional spaces, we obtain the classification results in a simply isotropic space $\mathbb{I}_3^1$ when either the dual isotropic Gaussian curvature K^* or dual isotropic mean curvature H^* are constants.

2 Preliminaries

A simply isotropic space has an absolute figure which is an ordered triple (ω, f_1, f_2), where ω is an absolute plane in projective space $\mathcal{P}(\mathbb{R}^3)$ and f_1, f_2 are the complex-conjugate straight lines in ω. In $\mathcal{P}(\mathbb{R}^3)$ the homogeneous coordinates are introduced in such a way that ω is given by $x_0 = 0$, and the absolute lines f_1, f_2 by $x_0 = x_1 + ix_2 = 0$, $x_0 = x_1 - ix_2 = 0$. The intersection point of these two lines is called the absolute point. The group of motions of $\mathbb{I}_3^1$ is a group with 6 parameters described in the affine coordinates $u = \frac{x_1}{x_0}, v = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}$ as follows

$$(u,v,z)\mapsto(u',v',z'):\begin{cases} u'=\alpha_1+u\cos\theta-v\sin\theta,\\ v'=\alpha_2+u\sin\theta+v\cos\theta,\\ z'=\alpha_3+\alpha_4u+\alpha_5v+z,\end{cases} \tag{2.1}$$

where $\alpha_1,\dots,\alpha_5,\theta\in\mathbb{R}$. One can define the isotropic distance from the point $A(u_1, v_1, z_1)$ to the point $B(u_2, v_2, z_2)$ as the Euclidean distance of the top view given by

$$d(A,B)=\sqrt{(u_2-u_1)^2+(v_2-v_1)^2}. \tag{2.2}$$

Also, if $X_1 = (u_1, v_1, z_1)$ and $X_2 = (u_2, v_2, z_2)$ are two vectors in $\mathbb{I}_3^1$, then

$$\langle X_1,X_2\rangle=\begin{cases} z_1z_2, & \text{if } u_i=v_i=0,\\ u_1u_2+v_1v_2, & \text{if otherwise.}\end{cases} \tag{2.3}$$

The vectors of the form $(0, 0, z)$ are called isotropic vectors, otherwise non-isotropic. A regular surface $r(u, v) = (u, v, z(u, v))$ without the isotropic tangent plane is called an admissible surface. We shall be discussing surfaces of this kind only. The line element ds^2 in $\mathbb{I}_3^1$ is given by

$$ds^2 = du^2 + dv^2. \tag{2.4}$$

For an admissible surface in $\mathbb{I}_3^1$ the 1^{st} fundamental coefficients $\{E, F, G\}$ are determined with respect to the induced metric. Also, when we calculate the 2^{nd} fundamental coefficients $\{L, M, N\}$, the normal vector field involved is completely isotropic and is given by $(0, 0, 1)$. The isotropic Gaussian curvature K and isotropic mean curvature H are related to their duals K^* and H^*, respectively by

$$K^* = \frac{1}{K} \quad \text{and} \quad H^* = \frac{H}{K},$$

where

$$K = \frac{LN - M^2}{EG - F^2} \quad \text{and} \quad H = \frac{EN - 2FM + GL}{2(EG - F^2)}. \tag{2.5}$$

Three scenarios concerning the absolute figure can result from the different types of planes in $\mathbb{I}_3^1$. These include both translation curves being in isotropic planes ($v = 0$, resp. $u = 0$); or one curve being in non-isotropic plane ($z = 0$) while the other is in an isotropic plane; or both curves being in non-isotropic perpendicular planes ($v - z = \pi$, resp. $v + z = \pi$). Therefore, in a 3-dimensional simply isotropic space $\mathbb{I}_3^1$, translation surfaces are of the following 3 types (see [10]):

(a) Type 1 translation surface:

$$r(u, v) = (u, v, f(u) + g(v)). \tag{2.6}$$

(b) Type 2 translation surface:

$$r(u, z) = (u, f(u) + g(z), z). \tag{2.7}$$

(c) Type 3 translation surface:

$$r(v, z) = \frac{1}{2}(f(v) + g(z), v - z + \pi, v + z). \tag{2.8}$$

The generalized notion of such surfaces in $\mathbb{I}_3^1$ is obtained by taking f and g to be functions of $a_1u + a_2v$ and $a_3u + a_4v$, respectively instead of being functions of single variables. As argued by the author in [2], it is easy to show that this generalized notion of translation surfaces leads to a single formulation of type 2 and type 3 surfaces. Thus, we have two types of affine translation surfaces in $\mathbb{I}_3^1$.

(i) Affine translation surface of the first kind:

$$r(u,v) = (u, v, z(u,v)), \tag{2.9}$$

such that,

$$z(u,v) = f(a_1 u + a_2 v) + g(a_3 u + a_4 v). \tag{2.10}$$

(ii) Affine translation surface of the second kind:

$$r(u,z) = (u, v(u,z), z), \tag{2.11}$$

such that,

$$v(u,z) = f(a_1 u + a_2 z) + g(a_3 u + a_4 z). \tag{2.12}$$

In both the above cases $a_1, \ldots, a_4$ are constants and $p = a_1 a_4 - a_2 a_3 \neq 0$. We obtain the following partial derivatives for $z(u,v)$ and $v(u,z)$, respectively:

$$\begin{cases} z_u = a_1 f' + a_3 g', \\ z_v = a_2 f' + a_4 g', \\ z_{uu} = a_1^2 f'' + a_3^2 g'', \\ z_{vv} = a_2^2 f'' + a_4^2 g'', \\ z_{uv} = a_1 a_2 f'' + a_3 a_4 g'', \end{cases} \tag{2.13}$$

where $f' = \dfrac{df}{dx}$ and $g' = \dfrac{dg}{dy}$ such that, $x = a_1 u + a_2 v$ and $y = a_3 u + a_4 v$.

$$\begin{cases} v_u = a_1 f' + a_3 g', \\ v_z = a_2 f' + a_4 g', \\ v_{uu} = a_1^2 f'' + a_3^2 g'', \\ v_{zz} = a_2^2 f'' + a_4^2 g'', \\ v_{uz} = a_1 a_2 f'' + a_3 a_4 g'', \end{cases} \tag{2.14}$$

where $f' = \dfrac{df}{dx}$ and $g' = \dfrac{dg}{dy}$ such that, $x = a_1 u + a_2 z$ and $y = a_3 u + a_4 z$.

Some simple calculations lead to the following expressions for K and H of:

(I) Affine translation surface of the first kind:

$$K = p^2 f'' g'', \tag{2.15}$$

$$H = \frac{1}{2}\left[(a_1^2 + a_2^2) f'' + (a_3^2 + a_4^2) g''\right]. \tag{2.16}$$

(II) Affine translation surface of the second kind:

$$K = \frac{p^2 f'' g''}{(a_2 f' + a_4 g')^4} \tag{2.17}$$

$$H = -\frac{\left[a_2^2 + (pg')^2\right] f'' + \left[a_4^2 + (pf')^2\right] g''}{2(a_2 f' + a_4 g')^3}. \tag{2.18}$$

3 Dual affine translation surface of the first kind

Theorem 3.1 *Let $r(u,v) = (u, v, z(u,v))$ be a dual affine translation surface of the first kind in $\mathbb{I}_3^1$ with constant dual isotropic Gaussian curvature K^*. Then*

1. there exists no such dual developable surface in $\mathbb{I}_3^1$.

2. $r(u,v)$ with non-zero constant K^* is congruent to an open piece of

$$\begin{aligned} z(u,v) = &\frac{c_1(a_1u + a_2v)^2}{2K^*p} + \frac{(a_3u + a_4v)^2}{2c_1} \\ &+ c_2(a_1u + a_2v) + c_3(a_3u + a_4v) + c_4, \end{aligned}$$

where $c_1, \ldots, c_4$ are all constants and $c_1 \neq 0$.

Proof. The dual isotropic Gaussian curvature K^* is given by

$$K^* = \frac{1}{K}. \tag{3.1}$$

From (2.15), it follows that

$$K^* = \frac{1}{p^2 f'' g''}, \tag{3.2}$$

where $f'' \neq 0$, $g'' \neq 0$ and $p = a_1a_4 - a_2a_3 \neq 0$.

By the definition of a dual developable surface, we have $K^* = 0$. Then from (3.2), we have

$$\frac{1}{p^2 f'' g''} = 0, \tag{3.3}$$

which leads to a contradiction. Thus, there exists no dual developable affine translation surface of the first kind in $\mathbb{I}_3^1$.

Now, let K^* be a non-zero constant. It follows from (3.2) that,

$$f'' g'' = \frac{1}{K^* p^2}. \tag{3.4}$$

On solving (3.4), we obtain

$$f(x) = \frac{c_1 x^2}{2K^*p} + c_2 x + c_3,$$

and

$$g(y) = \frac{y^2}{2c_1} + c_4 y + c_5, \tag{3.5}$$

where $c_1, \ldots, c_5$ are all constants and $c_1 \neq 0$. Thus, we have

$$\begin{aligned} z(u,v) &= f(a_1u + a_2v) + g(a_3u + a_4v) \\ &= \frac{c_1(a_1u + a_2v)^2}{2K^*p} + \frac{(a_3u + a_4v)^2}{2c_1} \\ &\quad + c_2(a_1u + a_2v) + c_4(a_3u + a_4v) + c, \end{aligned} \tag{3.6}$$

where $c = c_3 + c_5$. This completes the proof.

□

Now, consider the dual affine translation surface of the first kind parameterized as $r(u,v) = (u, v, f(a_1u + a_2v) + g(a_3u + a_4v))$ in $\mathbb{I}_3^1$ with constant dual isotropic mean curvature H^*. We have

$$H^* = \frac{(a_1^2 + a_2^2)f'' + (a_3^2 + a_4^2)g''}{2p^2 f'' g''}, \tag{3.7}$$

where $f'' \neq 0$, $g'' \neq 0$. H^* being constant leads to the following 2 cases:

Case I: $H^* = 0$. We have from (3.7)

$$(a_1^2 + a_2^2)f'' + (a_3^2 + a_4^2)g'' = 0. \tag{3.8}$$

Case II: $H^* \neq 0$ gives us

$$(a_1^2 + a_2^2)f'' + (a_3^2 + a_4^2)g'' = 2H^*p^2 f'' g''. \tag{3.9}$$

Theorem 3.2 *Let $r(u,v) = (u, v, z(u,v))$ be a dual affine translation surface of the first kind in $\mathbb{I}_3^1$ with zero dual isotropic mean curvature. Then, $z(u,v)$ is congruent to an open piece of*

$$\begin{aligned} z(u,v) =& \frac{(a_3^2 + a_4^2)}{2(a_1^2 + a_2^2)} c_1(a_1u + a_2v)^2 + c_2(a_1u + a_2v) \\ &- \frac{c_1}{2}(a_3u + a_4v)^2 + c_3(a_3u + a_4v) + c_4 \end{aligned}$$

where $c_1, \ldots, c_4$ are all constants and $c_1 \neq 0$.

Proof. We have from (3.8),

$$\frac{(a_1^2 + a_2^2)}{(a_3^2 + a_4^2)} f'' = -g'' = c_1, \tag{3.10}$$

where c_1 is a constant.

When $c_1 = 0$, it results in $f'' = 0$ and $g'' = 0$, which can't be the case. Now, consider $c_1 \neq 0$, we have from (3.10)

$$f'' = \frac{(a_3^2 + a_4^2)}{(a_1^2 + a_2^2)} c_1 \quad \text{and} \quad g'' = -c_1, \tag{3.11}$$

which gives us

$$f(x) = \frac{(a_3^2 + a_4^2)}{2(a_1^2 + a_2^2)} c_1 x^2 + c_2 x + c_3, \tag{3.12}$$

$$g(y) = -\frac{c_1}{2} y^2 + c_4 y + c_5, \tag{3.13}$$

where $c_2, \ldots, c_5$ are constants. Thus, we have

$$\begin{aligned} z(u,v) =& f(a_1 u + a_2 v) + g(a_3 u + a_4 v), \\ =& \frac{(a_3^2 + a_4^2)}{2(a_1^2 + a_2^2)} c_1 (a_1 u + a_2 v)^2 + c_2 (a_1 u + a_2 v) \\ & - \frac{c_1}{2} (a_3 u + a_4 v)^2 + c_4 (a_3 u + a_4 v) + c, \end{aligned} \tag{3.14}$$

where $c = c_3 + c_5$. This completes the proof.

□

Theorem 3.3 *Let $r(u,v) = (u, v, z(u,v))$ be a dual affine translation surface of the first kind with a constant non-zero dual mean curvature H^* in $\mathbb{I}_3^1$. Then, $z(u,v)$ is congruent to an open piece of*

$$\begin{aligned} z(u,v) =& \frac{(a_3^2 + a_4^2)(a_1 u + a_2 v)^2}{4H^* p^2 - 2\lambda} + \frac{(a_1^2 + a_2^2)(a_3 u + a_4 v)^2}{2\lambda} \\ & + c_1 (a_1 u + a_2 v) + c_2 (a_3 u + a_4 v) + c_3, \end{aligned}$$

where λ, c_1, c_2, c_3 are constants and $\lambda \neq 0, 2H^* p^2$.

Proof. We have from (3.9),

$$\frac{a_1^2 + a_2^2}{g''} + \frac{a_3^2 + a_4^2}{f''} = 2H^* p^2, \tag{3.15}$$

where $f'' \neq 0$ and $g'' \neq 0$. Differentiating (3.15) w.r.t. u gives us

$$-\frac{(a_1^2 + a_2^2)}{(g'')^2} a_3 g''' - \frac{(a_3^2 + a_4^2)}{(f'')^2} a_1 f''' = 0. \tag{3.16}$$

Again differentiating (3.15) w.r.t. v gives us

$$-\frac{(a_1^2 + a_2^2)}{(g'')^2} a_4 g''' - \frac{(a_3^2 + a_4^2)}{(f'')^2} a_2 f''' = 0. \tag{3.17}$$

Eqn's (3.16) and (3.17) yield

$$f''' = 0, g''' = 0.$$

Thus, $f''(x)$ and $g''(y)$ are constants. Therefore, we have from (3.15)

$$\frac{a_1^2+a_2^2}{g''}=2H^*p^2-\frac{a_3^2+a_4^2}{f''}, \tag{3.18}$$

such that,

$$\frac{a_1^2+a_2^2}{g''}=\lambda,\ 2H^*p^2-\frac{a_3^2+a_4^2}{f''}=\lambda, \tag{3.19}$$

where $\lambda(\neq 0)$ is a constant. On solving the above set of equations, we obtain

$$f(x)=\frac{(a_3^2+a_4^2)x^2}{4H^*p^2-2\lambda}+c_1x+c_2, \tag{3.20}$$

$$g(y)=\frac{(a_1^2+a_2^2)y^2}{2\lambda}+c_3y+c_4, \tag{3.21}$$

where $c_1,\ldots,c_4$ are constants and $\lambda\neq 0, 2H^*p^2$. Hence, we have

$$\begin{aligned} z(u,v)=&f(a_1u+a_2v)+g(a_3u+a_4v)\\ =&\frac{(a_3^2+a_4^2)(a_1u+a_2v)^2}{4H^*p^2-2\lambda}+\frac{(a_1^2+a_2^2)(a_3u+a_4v)^2}{2\lambda}\\ &+c_1(a_1u+a_2v)+c_3(a_3u+a_4v)+c, \end{aligned} \tag{3.22}$$

where $c=c_2+c_4$. This completes the proof.

□

4 Dual affine translation surface of the second kind

Theorem 4.1 *Let $r(u,z)=(u,v(u,z),z)$ be a dual affine translation surface of the second kind in $\mathbb{I}_3^1$ with a constant dual isotropic Gaussian curvature K^*. Then,*

1. there exists no such dual developable surface in $\mathbb{I}_3^1$.

2. $r(u,z)$ with non-zero constant K^* is congruent to an open piece of

$$\begin{aligned} v(u,z)=&-\frac{(-3c(a_1u+a_2z)+c_1)^{\frac{2}{3}}}{2c}+\frac{a_2^2u^2}{2K^*c}\\ &+c_2a_3u+c_3, \end{aligned}$$

where c,c_1,c_2,c_3 are constants and $c\neq 0$.

Proof. The dual isotropic Gaussian curvature is given by $K^* = \dfrac{1}{K}$. Thus, (2.18) yields

$$K^* = \frac{(a_2 f' + a_4 g')^4}{p^2 f'' g''}, \tag{4.1}$$

where $p = a_1 a_4 - a_2 a_3 \neq 0$ and $f'' \neq 0, g'' \neq 0$.

We first assume $K^* = 0$, it follows from (4.1)

$$a_2 f' + a_4 g' = 0. \tag{4.2}$$

We arrive at a solution in (4.2) when f' and g' are constants. Therefore, $f'' = 0$ and $g'' = 0$, which can't be the case. Thus, there exists no dual developable affine translation surface of the second kind in $\mathbb{I}_3^1$.

Now assuming $K^* \neq 0$. It follows from (4.1)

$$(a_2 f' + a_4 g')^4 = K^* p^2 f'' g''. \tag{4.3}$$

Differentiating (4.3) w.r.t. x gives us

$$4a_2 (a_2 f' + a_4 g')^3 = K^* p^2 f''' g''. \tag{4.4}$$

Using (4.3) in (4.4), gives the following expression

$$\frac{4a_2 f''}{f'''} - a_2 f' = a_4 g'. \tag{4.5}$$

We arrive at a conclusion, either $a_4 = 0$ or g' is a constant. But g' can't be a constant as $g'' \neq 0$. Substituting $a_4 = 0$ in (4.1), results in

$$(a_2 f')^4 = K^* p^2 f'' g''. \tag{4.6}$$

Thus, we have

$$\frac{a_2^2}{K^* a_3^2 g''} = \frac{f''}{(f')^4} = c, \tag{4.7}$$

where $c(\neq 0)$ is a constant. On solving (4.7), we obtain

$$f(x) = -\frac{(-3cx + c_1)^{\frac{2}{3}}}{2c} + c_2, \tag{4.8}$$

$$g(y) = \frac{a_2^2}{2K^* a_3^2 c} y^2 + c_3 y + c_4, \tag{4.9}$$

where $c_1, \ldots, c_4$ are constants. Hence,

$$\begin{aligned} v(u,z) =& f(a_1u + a_2z) + g(a_3u) \\ =& -\frac{(-3c(a_1u + a_2z) + c_1)^{\frac{2}{3}}}{2c} + \frac{a_2^2u^2}{2K^*c} \\ &+ c_3a_3u + c_2 + c_4. \end{aligned} \tag{4.10}$$

This completes the proof.

□

Remark 4.1 When we take $a_4 = 0$ in (4.5), the affine translation surface of the second kind results in a weaker case of such surface, given by

$$v(u,z) = f(a_1u + a_2z) + g(a_3u). \tag{4.11}$$

Notably, $p = -a_2a_3 \neq 0$ in this case.

Now, the dual mean curvature is given by, $H^* = \dfrac{H}{K}$. For affine translation surface of the second kind, it follows from (2.17) and (2.18) that,

$$H^* = -\frac{\left[\left\{a_2^2 + (pg')^2\right\}f'' + \left\{a_4^2 + (pf')^2\right\}g''\right](a_2f' + a_4g')}{2p^2f''g''}. \tag{4.12}$$

Theorem 4.2 *Let $r(u,z) = (u, v(u,z), z)$ be a dual affine translation surface of the second kind in $\mathbb{I}_3^1$ with a vanishing dual isotropic mean curvature. Then, $v(u,z)$ is congruent to an open piece of*

$$v(u,z) = \frac{1}{cp^2}\log\left|\frac{\cos\left[cpa_4(a_1u + a_2z) + c_1\right]}{\cos\left[cpa_2(a_3u + a_4z) + c_2\right]}\right| + c_3,$$

where c, c_1, c_2, c_3 are constants and $c \neq 0$.

Proof. For $H^* = 0$, from (4.12) we get

$$\left[\left\{a_2^2 + (pg')^2\right\}f'' + \left\{a_4^2 + (pf')^2\right\}g''\right](a_2f' + a_4g') = 0. \tag{4.13}$$

Either

$$\left\{a_2^2 + (pg')^2\right\}f'' + \left\{a_4^2 + (pf')^2\right\}g'' = 0$$

or

$$a_2f' + a_4g' = 0.$$

$a_2f' + a_4g' = 0$ results in $f'' = 0$ and $g'' = 0$, which is a contradiction. Hence, we have

$$-\frac{f''}{a_4^2 + (pf')^2} = \frac{g''}{a_2^2 + (pg')^2} = \lambda, \tag{4.14}$$

where $\lambda(\neq 0)$ is constant. Thus, the above expression gives us

$$f(x) = \frac{1}{\lambda p^2} \log |\cos(\lambda p a_4 x + c_1)| + c_2, \tag{4.15}$$

$$g(y) = -\frac{1}{\lambda p^2} \log |\cos(\lambda p a_2 y + c_3)| + c_4, \tag{4.16}$$

where $c_1, \ldots, c_4$ are constants. Thus, we have

$$\begin{aligned} v(u,z) =& f(a_1 u + a_2 z) + g(a_3 u + a_4 z) \\ =& \frac{1}{\lambda p^2} \log \left| \frac{\cos\left[\lambda p a_4 (a_1 u + a_2 z) + c_1\right]}{\cos\left[\lambda p a_2 (a_3 u + a_4 z) + c_3\right]} \right| + c, \end{aligned} \tag{4.17}$$

where $c = c_2 + c_4$. This completes the proof.

□

Theorem 4.3 *Let $r(u,z) = (u, v(u,z), z)$ be a dual affine translation surface of the second kind in $\mathbb{I}_3^1$ with constant non-zero dual isotropic mean curvature. Then, there exists no such surface in $\mathbb{I}_3^1$.*

Proof. The dual isotropic mean curvature of the affine translation surface of the second kind in $\mathbb{I}_3^1$ is given by

$$H^* = -\frac{\left[\left\{a_2^2 + (pg')^2\right\} f'' + \left\{a_4^2 + (pf')^2\right\} g''\right](a_2 f' + a_4 g')}{2p^2 f'' g''}, \tag{4.18}$$

which writes as

$$-\frac{2p^2 H^*}{a_2 f' + a_4 g'} = \frac{a_2^2 + (pg')^2}{g''} + \frac{a_4^2 + (pf')^2}{f''}, \tag{4.19}$$

where $f'' \neq 0$, $g'' \neq 0$.

Differentiating (4.19) w.r.t. x, we get

$$\frac{2p^2 H^*}{(a_2 f' + a_4 g')^2} a_2 f'' = \frac{d}{dx} \left\{ \frac{a_4^2 + (pf')^2}{f''} \right\}. \tag{4.20}$$

Again differentiating (4.19) w.r.t. y, we obtain

$$\frac{2p^2 H^*}{(a_2 f' + a_4 g')^2} a_4 g'' = \frac{d}{dy} \left\{ \frac{a_2^2 + (pg')^2}{g''} \right\}. \tag{4.21}$$

Thus, we have

$$\frac{2p^2 H^*}{(a_2 f' + a_4 g')^2} = \frac{1}{a_2 f''} \frac{d}{dx} \left\{ \frac{a_4^2 + (pf')^2}{f''} \right\} = \frac{1}{a_4 g''} \frac{d}{dy} \left\{ \frac{a_2^2 + (pg')^2}{g''} \right\}, \tag{4.22}$$

which results in the following

$$\frac{2p^2 H^*}{(a_2 f' + a_4 g')^2} = \lambda, \tag{4.23}$$

where λ is a constant.

Case 1: When $\lambda = 0$. It follows from (4.23) that, $H^* = 0$, which is contradiction to the fact that, $H^* \neq 0$

Case 2: When $\lambda \neq 0$. It follows from (4.23) that, $a_2 f' + a_4 g'$ is a constant, which results in $f'' = 0$, $g'' = 0$, which is again a contradiction.

Therefore, there exists no dual affine translation surface of the second kind in the isotropic space $\mathbb{I}_3^1$ with a non-zero constant dual isotropic mean curvature. This completes the proof.

□

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References

1. Aydin, M.E. – *A generalization of translation surfaces with constant curvature in the isotropic space*, J. Geom., **107** (2016), no. 3, 603–615.
2. Aydin, M.E. – *Classifications of translation surfaces in isotropic geometry with constant curvature*, Ukr. Mat. Zh.., **72** (2020), no. 3, 291-306.
3. Bozok, H.G.; Ergüt, M. – *Polynomial affine translation surfaces in Euclidean 3-space*, Bol. Soc. Parana. Mat., **(3) 37** (2019), no. 3, 195-202.
4. Dillen, F.; Verstraelen, L.; Zafindratafa, G. – *A generalization of the translation surfaces of Scherk*, Differential Geometry in honor of Radu Rosca, K.U.L. (1991), 107-109.
5. Fu, Y.; Hou, Z.-H. – *Affine translation surfaces with constant Gaussian curvature*, Kyungpook Math. J., **50** (2010), no. 2, 337–343.
6. Inoguchi, J.-I.; López, R.; Munteanu, M.I. – *Minimal translation surfaces in the Heisenberg group Nil_3*, Geom. Dedicata, **161** (2012), 221-231.
7. Liu, H. – *Translation surfaces with constant mean curvature in 3-dimensional spaces*, J. Geom., **64** (1999), no. 1-2, 141-149.
8. Liu, H.; Jung, S.D. – *Affine translation surfaces with constant mean curvature in Euclidean 3-space*, J. Geom., **108** (2017), no. 2, 423-428.
9. Liu, H.; Yu, Y. – *Affine translation surfaces in Euclidean 3-space*, Proc. Japan Acad. Ser. A Math. Sci., **89** (2013), no. 9, 111-113.
10. Lone, M.S.; Karacan, M.K. – *Dual translation surfaces in the three dimensional simply isotropic space $\mathbb{I}_3^1$*, Tamkang J. Math, **49** (2018), no. 1, 67-77.
11. López, R. – *Minimal translation surfaces in hyperbolic space*, Beitr. Algebra Geom., **52** (2011), no. 1, 105-112.
12. López, R.; Munteanu, M.I. – *Minimal translation surfaces in Sol_3*, J. Math. Soc. Japan, **64** (2012), no. 3, 985-1003.
13. Onishchik, A.L.; Sulanke, R. – *Projective and Cayley-Klein geometries*, Springer Monographs in Mathematics, Springer-Verlag, Berlin (2006).
14. Scherk, H.F. – *Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen* (German), J. Reine Angew. Math., **13** (1835), 185-208.
15. Struve, H.; Struve, R. – *Non-euclidean geometries: the Cayley-Klein approach*, J. Geom., **98** (2010), no. 1-2, 151-170.

16. Verstraelen, L.; Walrave, J.; Yaprak, S. – *The minimal translation surfaces in Euclidean space*, Soochow J. Math., **20** (1994), no. 1, 77-82.
17. Yang, D.; Fu, Y. – *On affine translation surfaces in affine space*, J. Math. Anal. Appl., **440** (2016), no. 2, 437-450.
18. Yoon, D.W. – *Minimal translation surfaces in* $\mathbb{H}^2\times\mathbb{R}$, Taiwanese J. Math., **17** (2013), no. 5, 1545-1556.
19. Yoon, D.W.; Lee, C.W.; Karacan, M.K. – *Some translation surfaces in the 3-dimensional Heisenberg group*, Bull. Korean Math. Soc., **50** (2013), no. 4, 1329-1343.

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