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Average structures associated to a Finsler space

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Abstract Given a Finsler space (M, F) on a manifold M, the averaging method associates to *Finslerian geometric objects affine geometric objects* living on M. In particular, a Riemannian metric is associated with the fundamental tensor g and an affine, torsion free connection is associated with the Chern-Rund connection. As an illustration of the applications of theory, a generalization of the Gauss-Bonnet theorem to Berwald surfaces using the average metric is presented. The parallel transport and curvature endomorphisms of the average connection are obtained. The holonomy group for a Berwald space is discussed. Finally, isometries and symmetric spaces are considered and the heritage of the property of symmetric space from the Finsler space to the average Riemannian metric is proved.

Keywords Finsler structures · averaging method

Mathematics Subject Classification (2010) 53C60 · 53C05

1 Introduction

It is remarkable that many fundamental results can be generalized from the Riemannian to the Finslerian category by properly adapting the corresponding proofs, in some cases in a rather direct way [4]. The present work was partially motivated by the insight that such phenomenon could be understood from the perspective of looking at the Finsler category as determined by *equivalence classes* in such a that each class contains an affine or Riemannian representative. If that conjecture was correct, then a given proposition formulated in the Finsler space that can be casted in terms of notions defined in the quotient space of equivalent classes it can be investigated using affine or Riemannian methods.

The method used to define the equivalence classes relations is build on the *averaging of Finslerian geometric objects*. These operations map geometric objects living in the tangent space and defined by the original Finsler structure to geometric objects living in the base manifold M. The average of the fundamental tensor appeared first in [16], in a pre-print of the author of pre-prints of the author (RGT), of which the present work is the latest version, and in work connection with emergent quantum mechanics [7]. Preliminary versions of the general theory of averaging were also discussed by the author, for instance in [8].

In this paper we consider in detail the average of the Finslerian metric (in the form of the fundamental tensor), the Chern-Rund connection and the average of the curvature operators. Two geometric operators/structures will be equivalent if they have the same average. It turns out that results stated in terms of the defining properties of the coset space and that can be proved using Riemannian or affine methods automatically upgrade to the Finsler category. In particular, the method is very well suited for two specific types of problems. The first is in applications to Berwald spaces. The second in the application to isometry properties of the Finsler metric and related notions.

This manuscript is organized as follows. In *section 2*, the notions of Finsler geometry required for our constructions are introduced. In *section 3* we explain the measure used for the averaging operation. The averaging can be casted in terms of integration along vertical fibers of pull-back bundles $\hat{\pi}^*T^{(p,q)}M$ over the manifold $TM\setminus\{0\}$. The averaging associates a Riemannian structure (M,h) to the initial Finsler space (M,F) by performing an average of the components of the fundamental tensor g on the indicatrix I_x at each point $x\in M$. As an example, we apply the averaging to a Randers metric. In *section 4* we discuss the average connection $\langle\nabla\rangle$ associated to a linear connection ∇ on $\hat{\pi}^*TM$. We show that for Berwald spaces, the Riemann curvature tensor of the average metric h can be obtained by averaging the corresponding tensor of the Chern-Rund connection of g. In *section 5* the parallel transport operators and curvatures of the average connection of $\langle\nabla\rangle$ are discussed in terms of the corresponding parallel transport and curvatures of the linear connection ∇ on $\hat{\pi}^*TM$. The average metric is used to prove several generalizations of Riemannian results to Berwald spaces. In particular, we have considered a generalization of the Gauss-Bonnet theorem for Berwald surfaces as an example of application of the method. We also provide a proof of the metrizability of the holonomy group of a Berwald space is discussed, extending the classical result from Z. Szabó to non-compact holonomies. In *section 6*, the map of isometries induced by the averaging is discussed and a proof of the Mayer-Steenrod theorem sketched from the point of view of the averaging method. Symmetric spaces are very shortly discussed from the point of view of our theory.

2 Basic notions of Finsler geometry

Notation

$(U,\mathbf{x})$ will denote a local coordinate chart of the n-dimensional manifold M, where a point $x\in U$ has local coordinates $(x^1,...,x^n)$ and $U\subset M$ is an open set. TM is the tangent bundle of M. The slit tangent bundle $\hat{\pi} : N \to M$ is the bundle over M with $N = TM \setminus \{0\}$. Fixed a local chart $(U, \mathbf{x})$ on the manifold M, a point $x \in U$ has associated the coordinates $(x^1, ..., x^n)$ and a tangent vector $y = y^i \frac{\partial}{\partial x^i} \in T_xM$ at $x \in M$ is determined by its components $y = (y^1, ..., y^n)$ respect to the basis $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$of T_xM. Note that we will use Einstein's convention for up and down equal indices, if anything else is not directly stated. The set of sections of a bundle $\mathcal{E}$ is denoted by $\Gamma\,\mathcal{E}$. Each local coordinate chart $(U, \mathbf{x})$ on M induces a local coordinate chart on TM denoted by $(TU, \mathbf{x}, \mathbf{y})$ such that a point $u \in TU$ with $\hat{\pi}(u) = x$ and corresponding to the tangent vector $y = y^i \frac{\partial}{\partial x^i} \in T_xM$ has local *natural coordinates* $(x^1, ..., x^n, y^1, ..., y^n)$.

Definition 2.1 *A Finsler space is a pair (M, F) where F is a non-negative, real function $F : M \to [0, \infty[$ such that*

- *It is smooth in the slit tangent bundle N,*
- *Positive homogeneity holds: $F(x, \lambda\, y) = \lambda\, F(x, y)$ for every $\lambda \in [0, +\infty[$,*
- *Strong convexity holds: the Hessian matrix*

$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, \quad i, j = 1, ..., n \tag{2.1}$$

is positive definite on N.

We assume that the function F is $\mathcal{C}^4$-smooth function on N. However, if the second Bianchi identities are required in a particular instance, then it is necessary that F must be at least $\mathcal{C}^5$-smooth on the slit tangent bundle N. The matrix $(g)_{ij} := g_{ij}(x, y)$ is the matrix components of the fundamental tensor g at the point $u = (x, y) \in N$.

Definition 2.2 *Let (M, F) be a Finsler space and $(TU, \mathbf{x}, \mathbf{y})$ a local chart induced on N from the coordinate system $(U, \mathbf{x})$ of M. The components of the Cartan tensor are defined by the expressions*

$$A_{ijk} = \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k}, \quad i, j, k = 1, ..., n. \tag{2.2}$$

The components $\{A_{ijk},\ i, j, k = 1, ..., n\}$ are homogeneous functions of degree zero in the coordinates $(y^1, ..., y^n)$ and totally symmetric in the indices i, j, k. The condition

$$A_{ijk}(x, y) = 0, \quad \forall\, y \in T_xM,\ x \in M$$

characterizes Riemannian geometry among the general class of Finsler geometries.

Definition 2.3 *Let (M, F) be a Finsler space. The indicatrix over the point $x \in M$ is the submanifold $I_x \hookrightarrow T_xM$*

$$I_x := \{y \in T_xM \mid F(x, y) = 1\}.$$

For each $x \in M$ the indicatrix I_x is a compact, strictly convex submanifold of T_xM [4,14]. We denote by $\mathcal{I}$ the fibered manifold $\pi_{\mathcal{I}} : \mathcal{I} \to M$ with $\pi_{\mathcal{I}}^{-1}(x) = I_x$ and where the base manifold is M.

2.1 Definition of a non-linear connection of N

Let us consider the slit bundle $\hat{\pi} : N \to M$. Let us recall that the *vertical distrubution* is $\mathcal{V} = \ker\, d\hat{\pi}$.

Definition 2.4 *A non-linear connection of N is a distribution $\mathcal{H} \subset TN$ supplementary to the canonical vertical distribution $\mathcal{V}$.*

Given a Finsler space (M, F), there is defined a non-linear connection in the slit tangent bundle N. Let us consider a local natural coordinate chart $(TU, \mathbf{x}, \mathbf{y})$ of N and then the collection of local sections

$$\left\{\frac{\partial}{\partial y^1}|_u, ..., \frac{\partial}{\partial y^n}|_u,\, u \in \hat{\pi}^{-1}(x),\, x \in U\right\},$$

that determines a local frame for the vertical distribution $\mathcal{V}$. To obtain a supplementary distribution $\mathcal{H}$ we use a standard local construction [4]. First, the non-linear connection coefficients N^i_j are given by the expression

$$\frac{N^i_j}{F} = \gamma^i_{jk}\frac{y^k}{F} - A^i_{jk}\gamma^k_{rs}\frac{y^r}{F}\frac{y^s}{F}, \quad i, j, k, r, s = 1, ..., n,$$

where the *formal second kind Christoffel's symbols* $\gamma^i_{jk}(x, y)$ are defined by

$$\gamma^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^j}\right), \quad i, j, k = 1, ..., n$$

and also $A^i_{jk} := g^{il}A_{ljk}$ and $g^{il}g_{lj} = \delta^i_j$. A tangent basis for T_uN is determined by the vectors

$$\left\{\frac{\delta}{\delta x^1}|_u, ..., \frac{\delta}{\delta x^n}|_u, F\frac{\partial}{\partial y^1}|_u, ..., F\frac{\partial}{\partial y^n}|_u\right\}, \frac{\delta}{\delta x^j}|_u = \frac{\partial}{\partial x^j}|_u - N^i_j\frac{\partial}{\partial y^i}|_u, i,j = 1, ..., n. \tag{2.3}$$

The collection of local sections

$$\left\{\frac{\delta}{\delta x^1}|_u, ..., \frac{\delta}{\delta x^n}|_u,\, u \in \hat{\pi}^{-1}(x),\, x \in U\right\}$$

determines a local frame for the *horizontal distribution* $\mathcal{H}$ [2,4]. Given $\tilde{X} \in \Gamma TN$, the horizontal component is denoted by $H(\tilde{X})$ and the vertical component by $V(\tilde{X})$. The horizontal lift of tangent vectors is defined by the homomorphism

$$\iota_u : T_xM \to T_uN, \quad X = X^i\frac{\partial}{\partial x^i}|_x \mapsto \iota_u(X) = X^i\frac{\delta}{\delta x^i}|_u, \tag{2.4}$$

for fixed $u \in \pi^{-1}(x)$.

The dual basis associated to the local tangent basis (2.3) of $T_u N$ is a local basis of the dual vector space $T_u^* N$,

$$\left\{dx^1|_u, ..., dx^n|_u, \frac{\delta y^1}{F}|_u, ..., \frac{\delta y^n}{F}|_u\right\}, \frac{\delta y^i}{F}|_u = \frac{1}{F}(dy^i + N^i_j dx^j)|_u, i, j=1, ..., n. \tag{2.5}$$

Definition 2.5 *Let (M, F) be a Finsler space. The fundamental and the Cartan tensors are defined in natural local coordinate system $(TU, \mathbf{x}, \mathbf{y})$ by the equations*

– *The fundamental tensor is*

$$g(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}\, dx^i \otimes dx^j, \tag{2.6}$$

– *The Cartan tensor is*

$$A(x,y) := \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k}\,\frac{\delta y^i}{F} \otimes dx^j \otimes dx^k = A_{ijk}\,\frac{\delta y^i}{F} \otimes dx^j \otimes dx^k. \tag{2.7}$$

Note that our definition of the Cartan's tensor (2.7) differs from the usual definition [4], which is the symmetric tensor $A = A_{ijk}\, dx^i \otimes dx^j \otimes dx^k$. Such difference is only formal and does not spoil the properties of the Cartan's tensor in any practical situation.

2.2 The Chern-Rund connection

Let us consider the product $N \times TM$ and the canonical projections

$$\hat{\pi}_1 : N \times TM \to N, \quad (u, \xi) \mapsto u, \qquad \hat{\pi}_2 : N \times TM \to TM, \quad (u, \xi) \mapsto \xi.$$

The pull-back $\hat{\pi}_1 : \hat{\pi}^* TM \to N$ of the tangent bundle $\pi : TM \to M$ by the projection $\hat{\pi} : N \to M$ is the maximal sub-manifold of the product $N \times TM$ such that $\pi \circ \hat{\pi}_2(u, \xi) = \hat{\pi} \circ \hat{\pi}_1(u, \xi)$ holds. It follows that the diagram

$$\begin{tikzcd} \hat{\pi}^* TM \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & TM \arrow[d, "\pi"] \\ N \arrow[r, "\hat{\pi}"] & M \end{tikzcd}$$

commutes. Here $\pi_1 : \hat{\pi}^* TM \to N$ and $\pi_2 : \hat{\pi}^* TM \to TM$ are the restrictions of the natural projections $\hat{\pi}_1 : N \times TM \to N$ and $\hat{\pi}_2 : N \times TM \to TM$ to $\hat{\pi}^* TM$. $\pi_1 : \hat{\pi}^* TM \to N$ is a real vector bundle, with fiber over $u = (x, Z) \in N$ isomorphic to $T_x M$. The fiber dimension of $\hat{\pi}^* TM$ is equal to $n = dim(M)$, while the dimension of the base manifold N is $2n$.

The pull-back of a smooth function $f \in \mathcal{F}(M)$ is the smooth function $\hat{\pi}^* f \in \mathcal{F}(N)$ such that $\hat{\pi}^* f(u) = f(\hat{\pi}(u))$ for every $u \in N$. Also, each tangent field $Z \in T_x M$ can be pulled-back $\hat{\pi}^* Z$ uniquely by the conditions

– $\pi_1(\hat{\pi}^* Z) = (x, Z) \in N_x$,
– $\pi_2(\hat{\pi}^* Z) = Z \in T_x M$.

These conditions extend pointwise to vector fields. Similarly, a pull-back tensor bundle $\hat{\pi}^* T^{(p,q)} M$ can be obtained from each tensor bundle $T^{(p,q)} M$ over M.

Let us consider a linear connection on $\hat{\pi}^* TM$. The associated covariant derivative is the operator

$$\nabla : \Gamma\, \hat{\pi}^* TM \times \Gamma\, TN \to \Gamma\, \hat{\pi}^* TM$$

such that

– For every $\tilde{X} \in \Gamma TN$, $S_1, S_2 \in \Gamma\, \hat{\pi}^* TM$ and $\varphi \in \mathcal{F}(M)$ it holds that

$$\nabla_{\tilde{X}}(\hat{\pi}^* \varphi S_1 + S_2) = (\tilde{X} \cdot \hat{\pi}^* \phi) S_1 + \varphi \nabla_{\tilde{X}} S_1 + \nabla_{\tilde{X}} S_2. \qquad (2.8)$$

– For every $\tilde{X}_1, \tilde{X}_2 \in \Gamma\, TN$, $S \in \Gamma\, \hat{\pi}^* TM$ and $\tilde{\varphi} \in \mathcal{F}(N)$ it holds that

$$\nabla_{\tilde{\varphi} \tilde{X}_1 + \tilde{X}_2} S = \tilde{\varphi} \nabla_{\tilde{X}_1} S + \nabla_{\tilde{X}_2} S. \qquad (2.9)$$

The Chern-Rund connection ${}^{cr}\nabla$ is a linear connection on $\hat{\pi}^* TM$ characterized by the following fundamental result [4,14],

Theorem 2.6 *Let (M, F) be a Finsler space. The vector bundle $\hat{\pi}^* TM$ admits a unique linear connection characterized by the collection of connection 1-forms $\{{}^{cr}\omega^i_j,\ i, j = 1, ..., n\}$ such that the following structure equations hold:*

– *"Torsion free" condition,*

$$d(dx^i) - dx^j \wedge {}^{cr}w^i_j = 0, \quad i, j = 1, ..., n. \qquad (2.10)$$

– *Almost g-compatibility condition,*

$$dg_{ij} - g_{kj}\, {}^{cr}w^k_i - g_{ik}\, {}^{cr}w^k_j = 2A_{ijk} \frac{\delta y^k}{F}, \quad i, j, k = 1, ..., n. \qquad (2.11)$$

The torsion free condition is equivalent to the absence of terms containing δy^i in the connection 1-forms ${}^{cr}\omega^i_j$ and also implies the symmetry of the connection coefficients $\Gamma^i_{jk}(x, y)$,

$${}^{cr}\omega^i_j(x, y) = \Gamma^i_{jk}(x, y)\, dx^k, \quad \Gamma^i_{jk}(x, y) = \Gamma^i_{kj}(x, y), \quad i, j, k = 1, ..., n. \qquad (2.12)$$

We can characterize the Chern-Rund connection by means of the associated covariant derivative operator ${}^{cr}\nabla_{\tilde{X}}$, with $\tilde{X} \in T_u N$. The *generalized torsion tensor* is given by the expression

$$\begin{aligned} T_{{}^{cr}\nabla} :& \Gamma\, TM \times \Gamma\, TM \to \Gamma\, TM \\ & (X, Y) \mapsto \pi_2\big({}^{cr}\nabla_{\tilde{X}} \hat{\pi}^* Y\big) - \pi_2\big({}^{cr}\nabla_{\hat{Y}} \hat{\pi}^* X\big) - [X, Y], \end{aligned} \qquad (2.13)$$

where $\tilde{X}$ and $\tilde{Y}$ are the horizontal lifts of the restrictions $X(x), Y(x)$ at a given point $u \in \hat{\pi}^{-1}(x)$. Then the following corollaries are direct consequences of *Theorem* 2.6.

Corollary 2.7 *Let (M, F) be a Finsler space. The torsion free condition* (2.10) *is equivalent to the following conditions:*

1. *For any $\tilde{X} \in TN$ and $Y \in M$, the following relation holds,*

$$ {}^{cr}\nabla_{V(\tilde{X})}\hat{\pi}^* Y = 0. \tag{2.14}$$

2. *Let us consider $X, Y \in \Gamma TM$. Then the following relation holds,*

$$T_{{}^{cr}\nabla}(X, Y) = 0. \tag{2.15}$$

Proof. Let $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ be a local frame on U and $X = e_i = \frac{\partial}{\partial x^i}$, $Y = e_j = \frac{\partial}{\partial x^j}$. Then we have

$$ {}^{cr}\nabla_{V(\tilde{X})}\hat{\pi}^* Y := \hat{\pi}^* e_k\, w^k_j(V(e_i)) = \hat{\pi}^* e_k \Gamma^k_{aj} dx^a(V(e_i)) = \hat{\pi}^* e_k \Gamma^k_{aj} dx^a(\frac{\partial}{\partial y^i}) = 0.$$

This condition is extended by linearity and by the Leibnitz rule to arbitrary sections X, $Y \in \Gamma TM$.

To prove the relation (2.15), let us consider the torsion condition in local coordinates where the local frame $\{e_j\}$ on ΓTM commutes, $[e_i, e_j] = 0$. Then from the symmetry in the connection coefficients (2.12) one obtains

$$\begin{aligned} \pi_2({}^{cr}\nabla_{\tilde{e}_i}\hat{\pi}^* e_j) - \pi_2({}^{cr}\nabla_{\tilde{e}_j}\hat{\pi}^* e_i) - [e_i, e_j] &= \pi_2({}^{cr}\nabla_{\tilde{e}_i}\hat{\pi}^* e_j) - \pi_2({}^{cr}\nabla_{\tilde{e}_j}\hat{\pi}^* e_i) \\ &= (\Gamma^k_{ij} - \Gamma^k_{ji})\, \hat{\pi}^* e_k = 0. \end{aligned}$$

This relation is extended by linearity to arbitrary vectors $X, Y \in \Gamma TM$. □

Corollary 2.8 *Let (M, F) be a Finsler space and $\tilde{X} \in \Gamma TN$. The almost g-compatibility condition* (2.11) *is equivalent to the conditions*

1. *${}^{cr}\nabla$ is metric compatible in the horizontal directions,*

$$ {}^{cr}\nabla_{H(\tilde{X})} g = 0. \tag{2.16}$$

2. *${}^{cr}\nabla$ is almost-metric compatible in the vertical directions in the sense that*

$$ {}^{cr}\nabla_{V(\tilde{X})} g = 2\, A(\tilde{X}, \cdot, \cdot) \tag{2.17}$$

holds good.

Proof. Using local natural coordinates and reading from the expression (2.11), it follows that

$$^{cr}\nabla g = (dg_{ij} - g_{kj}\, {}^{cr}w^k_i - g_{ik}\, {}^{cr}w^k_j)\, \hat{\pi}^* e^i \otimes \hat{\pi}^* e^j \;=\; 2\, A_{ijk}\frac{\delta y^k}{F} \otimes \hat{\pi}^* e^i \otimes \hat{\pi}^* e^j,$$

By the definition of covariant derivative along an horizontal direction and since $2A_{ijk}\frac{\delta y^k}{F}$ is vertical, the relation (2.16) follows. For the covariant derivative along the vertical component $V(\tilde{X})$,

$$^{cr}\nabla_{V(\tilde{X})}\, g := \; 2\, A_{ijk}\frac{\delta y^k}{F} \cdot V(\tilde{X})\; \left(\hat{\pi}^* e^i \otimes \hat{\pi}^* e^j\right),\; \forall \tilde{X} \in \Gamma TN,$$

from where it follows the condition (2.17).

□

The curvature 2-forms associated with a linear connection ∇ on $\hat{\pi}^* TM$ are

$$\Omega^i_j := dw^i_j - w^k_j \wedge w^i_k, \quad i, j, k = 1, ..., n. \tag{2.18}$$

In local coordinates, the curvature endomorphisms are decomposed as

$$\Omega^i_j = \frac{1}{2} R^i_{jkl}\, dx^k \wedge dx^l + P^i_{jkl}\, dx^k \wedge \frac{\delta y^l}{F} + \frac{1}{2} Q^i_{jkl}\, \frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}. \tag{2.19}$$

The quantities R^i_{jkl}, P^i_{jkl} and Q^i_{jkl} are the hh, hv, and vv-curvature tensor components. For any Finsler space, the *hv*-curvature endomorphisms of the Chern-Rund connection are identically zero [4]. However, for other linear connections in $\hat{\pi}^* TM$ the three curvature types could be different from zero.

3 Riemannian average metrics from a Finsler space

3.1 Definition of the averaging procedure for Finsler metrics

The n-form $d^n y$ is defined in local coordinates by the expression

$$d^n y = \sqrt{\det\, g(x, y)}\; \frac{\delta y^1}{F} \wedge \cdots \wedge \frac{\delta y^n}{F}, \tag{3.1}$$

where $\det\, g(x, y)$ is the determinant of the fundamental tensor $(g)_{ij} = g_{ij}(x, y)$. Since $d^n y$ is invariant by local diffeomorphisms on TM, it defines a section of $\Lambda^n N$. Let us consider for each embedding $i_x : I_x \hookrightarrow N_x$ the volume form on I_x given by

$$dvol_x := i^*_x (d^n y \cdot l)|_x, \tag{3.2}$$

where $l = \frac{y^i}{F}\frac{\partial}{\partial y^i}$ and $d^n y \cdot l$ is the corresponding contraction. Then the *volume function* $vol(I_x)$ is defined by the expression

$$vol_\psi : M \to \mathbb{R}^+, \quad x \mapsto vol_\psi(I_x) := \int_{I_x} |\psi|^2(x,y)\, dvol_x, \tag{3.3}$$

where the weight factor $|\psi|^2 : TM \setminus \{0\} \to \mathbb{R}^+$ is a homogenous of degree zero in y, positive, smooth function on the tangent bundle. The average of a function $f \in \mathcal{F}(\mathcal{I})$ is defined by the expression

$$\langle f\rangle_\psi(x) := \frac{1}{vol_\psi(I_x)} \int_{I_x} |\psi|^2(x,y)\, f(x,y)\, dvol_x.$$

Given the Finsler space (M,F) let us consider the matrix with components

$$h_{ij}(\psi,x) := \langle g_{ij}(x,y)\rangle_\psi,\ i,j = 1,...,n, \tag{3.4}$$

for each point $x \in M$. Then we have,

Proposition 3.1 *Let (M,F) be a Finsler space. Then $\{h_{ij}(\psi,x)\}^n_{i,j=1}$ are the components of a Riemannian metric*

$$h_\psi(x) = h_{ij}(\psi,x)\, dx^i \otimes dx^j, \quad i,j = 1,...,n. \tag{3.5}$$

Proof. The average (3.4) of a positive definite, real and symmetric $n \times n$ matrix is also a positive, real, symmetric matrix. To show this, first we note that

$$g_x(y)(\tilde{y},\tilde{y}) = g_{ij}(x,y)\tilde{y}^i\tilde{y}^j \geq 0,\ y,\tilde{y} \in T_xM.$$

This is because $g_{ij}(x,y)$ for each fixed $y \in T_xM$ is a positive defined scalar product in T_xM. The symmetric property follows in analogous way. The tensorial character of the expression (3.5) is also direct.

□

Definition 3.2 *Two Finsler spaces (M,F_1), (M,F_2) are said to be h-equivalent if the corresponding average metrics h_1 and h_2 defined by* (3.5) *are the same.*

h-equivalence is an equivalence relation in the set M_F of Finsler metrics defined over M. The equivalence class of g is denoted by [g]. The coset space is defined by $M_F/\sim$. We call this equivalence relation metric-equivalence.

From now we choose the function $|\psi|^2 = 1$, if anything else is not stated. In this case the averaging is called *isotropic*.

3.2 A coordinate-free formula for h

Let us consider the fiber metric

$$\bar{g} = g_{ij}(x,y)\hat{\pi}^* dx^i \otimes \hat{\pi}^* dx^j, \quad \xi \in \pi_2^{-1}(u),\, u \in \hat{\pi}^{-1}(x).$$

Proposition 3.3 *Let (M,F) be a Finsler space. Then the relation*

$$h(x)(X,X) = \frac{1}{vol(I_x)}\Big(\int_{I_x} dvol_x\ \bar{g}\,(\hat{\pi}^*|_u(X), \hat{\pi}^*|_u(X))\Big) \tag{3.6}$$

holds good for each vector field $X \in \Gamma TM$.

Proof. For each point $x \in M$, $h(x) = \langle g_{ij}\rangle(x)\, dx^i|_x \otimes dx^j|_x$, $u \in I_x$ and vector field $X \in \Gamma TM$ one has

$$\begin{aligned} h(x)(X,X) &= \frac{1}{vol(I_x)}\Big(\int_{I_x} dvol_x\ g_{ij}(u)\Big)\, X^i\, X^j \\ &= \frac{1}{vol(I_x)}\Big(\int_{I_x} dvol_x\ \bar{g}_{ij}(u)\Big)\, X^i\, X^j \\ &= \frac{1}{vol(I_x)}\Big(\int_{I_x} dvol_x\ \bar{g}\,(\hat{\pi}^*|_u(X), \hat{\pi}^*|_u(X))\Big). \end{aligned}$$

□

Example 3.1 Let us consider a Randers space [12] of the form $F_R = \alpha + \beta$, with Finsler function in local coordinates given by the expression [4]

$$F_R(x,y) = \sqrt{a_{ij}\, y^i\, y^j} + b_i\, y^i,$$

where $\alpha = \sqrt{a_{ij}\, y^i\, y^j}$ and $\beta = b_i\, y^i$ is the action of the 1-form b on the tangent vector $y \in T_xM$. The associated fundamental tensor is

$$g^r_{ij}(x,y) = \frac{F_R}{\alpha}\left(a_{ij} - \frac{y_i}{\alpha}\frac{y_j}{\alpha}\right) + \frac{y_i}{F_R}\frac{y_j}{F_R}.$$

On the indicatrix I_x the fundamental tensor is

$$g^r_{ij}(x,y)|_{\alpha+\beta=1} = \frac{1}{\alpha}\left(a_{ij} - \frac{y_i}{\alpha}\frac{y_j}{\alpha}\right) + y_i\, y_j. \tag{3.7}$$

Formally, the right hand side of the expression (3.7) does not depend upon the particular details of the 1-form b. Thus we can evaluate the average and it will be equal to the Euclidean case: $b = 0$, $\alpha|_{I_x} = 1$. If we use normal coordinates of the metric a_{ij} centered at x; then we have that $a_{ij}(x) = \delta_{ij}$, that leads to

$$\langle g^r_{ij}(x,y)\rangle = \delta_{ij}.$$

Then in general local coordinates we will have the expression

$$\langle g^r_{ij}(x,y)\rangle = a_{ij}(x). \tag{3.8}$$

4 General form of the average operation and applications

4.1 Average of a family of tensor automorphisms

For each tensor $S_z \in T^{(p,q)}_z M,\ v \in \hat{\pi}^{-1}(z)$ and $z \in U \subset M$, the following isomorphism is defined:

$$\hat{\pi}^*|_v : \hat{\pi}^{-1}(x) \to \pi_1^{-1}(v), \quad S_z \mapsto \hat{\pi}^*_v S_z.$$

Consider a family of fiber preserving automorphisms

$$A := \left\{A_w : \hat{\pi}^*_w T^{(p,q)}M \to \hat{\pi}^*_w T^{(p,q)}M,\, w \in\, I_x,\, x \in M\right\}.$$

Then one can define the vector valued integral operation,

$$\Big(\int_{I_x} \pi_2|_u A_u \hat{\pi}^*_u\Big) \cdot\, S_x := \int_{I_x} \pi_2(\, A_u\, \hat{\pi}^*_u\, S_x)\, dvol_x, \quad S_x, \in T^{(p,q)}_x M, \qquad (4.1)$$

where $u = (x, y)$ is a point on the fiber $I_x \hookrightarrow N_x$. This operation is a fiber integration on a sub-manifold I_x of T_xM and with values of $T^{(p,q)}_x M$, for each $x \in\, M$. The tensor S_x is pulled-back $\{\hat{\pi}^*_u S_x,\, u \in N_x \subset\, N\}$ such that the following diagram commutes,

$$\begin{tikzcd} \hat{\pi}^*_u S_x \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S_x \arrow[d, "\pi"] \\ I_x \arrow[r, "\hat{\pi}"] & x \end{tikzcd}$$

for each $x \in M$, $u \in\, \pi^{-1}(x)$. The chain of compositions defining the integral operation (4.1) is the following: given $x \in M$, $u \in \pi^{-1}(x)$ and $S_x \in\, T^{(p,q)}_X M$,

$$x \mapsto S_x \mapsto \{\hat{\pi}^*_u S_x\} \mapsto \{A_u(\hat{\pi}^*_u S_x)\} \mapsto \{\pi_2(A_u(\hat{\pi}^*_u S_x))\} \mapsto \int_{I_x} \pi_2(A_u(\hat{\pi}^*_u S_x))\, dvol_x. \qquad (4.2)$$

Definition 4.1 *The average operator of the family of automorphism A is the automorphism*

$$\langle A\rangle_x : T^{(p,q)}_x M \to T^{(p,q)}_x M$$
$$S_x \mapsto \frac{1}{vol(I_x)}\Big(\int_{I_x} \pi_2|_u A_u \hat{\pi}^*_u\Big) \cdot\, S_x,\, u \in \pi_1^{-1}(x),\, \forall\, S_x \in T^{(p,q)}_x M.$$

Proposition 4.2 *The average operator $\langle A\rangle_x$ associated with a family of a linear fiber preserving automorphisms $A = \{A_w : \hat{\pi}^*_w T^{(p,q)}M \to \hat{\pi}^*_w T^{(p,q)}M,\, w \in I_x,\, x \in M\}$ is independent of the local frames chosen.*

Proof. Let us consider first sections of the bundle $\hat{\pi}^*TM$ and two local basis $\{e_i,\, i = 1, ..., n\}$ and $\{\tilde{e}_i,\, i = 1, ..., n\}$ of T_xM. Then we have

$$\begin{aligned}\langle A\rangle(S(x)) &= \frac{1}{vol(I_x)}\Big(\int_{I_x} \pi_2|_u A_u \hat{\pi}^*_u\Big)\cdot\, S(x)\\ &= \frac{1}{vol(I_x)}\Big(\int_{I_x} \pi_2|_u A_u \hat{\pi}^*_u\Big)\cdot\, S^i(x)e_i(x)\\ &= \frac{1}{vol(I_x)}\Big(\int_{I_x} \pi_2|_u A_u \hat{\pi}^*_u\, \tilde{S}^k(x)\, \tilde{e}_k(x)\, dvol_x\Big)\\ &= \frac{1}{vol(I_x)}\Big(\int_{I_x} \pi_2|_u A_u \hat{\pi}^*_u\Big)\cdot\, \tilde{S}^k(x)\tilde{e}_k(x).\end{aligned}$$

One can extend this calculation to families of homomorphisms of $\hat{\pi}^*T^{(p,q)}_x M$. □

The averaging operation can be extended to families of linear, local operators acting $\Gamma\hat{\pi}^*T^{(p,q)}M$ by applying the construction (4.2) in the *definition* 4.1 to the evaluation of sections $\hat{\pi}^*_v(S) := (\hat{\pi}^*S)(v)$, where $S \in \Gamma\, T^{(p,q)}M$ and $v \in U_u$ is a point in the open set U_u containing u.

4.2 The average of linear connections on $\hat{\pi}^*TM$

We can average linear connections on $\hat{\pi}^*TM$ following the above general method.

Theorem 4.3 *Let ∇ be a linear connection of the vector bundle $\hat{\pi}^*TM \to N$. Then there is defined an affine connection of M determined by the covariant derivative of each section $Y \in \Gamma\, TM$ along each direction $X \in T_xM$,*

$$\langle\nabla\rangle_X Y := \langle \pi_2|_u \nabla_{\iota_u(X)}\hat{\pi}^*_v Y\,\rangle,\; v \in TU_x \setminus 0, \tag{4.3}$$

for each $X \in T_xM$ and $Y \in \Gamma\, TM$, where U_x is an open neighborhood of $x \in M$.

Proof. We check that the properties for a linear covariant derivative hold for $\langle\nabla\rangle$.

1. Using the linearity of the original covariant derivative and the linearity of the averaging operation,

$$\begin{aligned}\langle\nabla\rangle_X(Y_1 + Y_2) &= \langle \pi_2|_u \nabla_{\iota_u(X)}\hat{\pi}^*_v(Y_1 + Y_2)\rangle\\ &= \langle \pi_2|_u \nabla_{\iota_u(X)}\hat{\pi}^*_v Y_1\rangle + \langle \pi_2|_u \nabla_{\iota_u(X)}\hat{\pi}^*_v Y_2\rangle\\ &= \langle\nabla\rangle_X Y_1 + \langle\nabla\rangle_X Y_2,\end{aligned}$$

   $\forall\, Y_1, Y_2\, \Gamma TN$. For the second condition of linearity we have

$$\langle\nabla\rangle_X(\lambda Y) = \langle \pi_2|_u \nabla_{\iota_u(X)}\hat{\pi}^*_v(\lambda Y)\rangle_v = \lambda\langle \pi_2|_u \nabla_{\iota_u(X)}\hat{\pi}^*_v\rangle = \lambda\langle\nabla\rangle_X Y$$

   $\forall\, Y \in \Gamma TM, \lambda \in \mathbb{R},\, X \in T_xM$.

2. $\langle\nabla\rangle_X Y$ is a $\mathcal{F}$-linear respect X, that is,

$$\langle\nabla\rangle_{X_1+X_2} Y = \langle\nabla\rangle_{X_1} Y + \langle\nabla\rangle_{X_2} Y, \qquad \langle\nabla\rangle_{fX}(Y) = f\,\langle\nabla\rangle_X Y, \tag{4.4}$$

   holds good for each $Y \in \Gamma TM, v \in \pi^{-1}(z)$, $X, X_1, X_2 \in T_x M$ and $f \in \mathcal{F}M$. The first condition is proved by the following short calculation,

$$\begin{aligned}\langle\nabla\rangle_{X_1+X_2} Y &= \langle \pi_2|_u(\nabla_{\iota_u(X_1+X_2)})\hat{\pi}_v^* Y\rangle_v \\ &= \langle \pi_2|_u \nabla_{\iota_u(X_1)} \hat{\pi}_v^* Y\rangle_v + \langle \pi_2|_u \nabla_{\iota_u(X_2)} \hat{\pi}_v^* Y\rangle \\ &= \langle\nabla\rangle_{X_1} Y + \langle\nabla\rangle_{X_2} Y.\end{aligned}$$

   For the second condition the proof is similar.
3. The Leibnitz rule holds:

$$\langle\nabla\rangle_X(\varphi Y) = (d\varphi(X))Y + \varphi\langle\nabla\rangle_X Y, \forall\, Y \in \Gamma TM, \varphi \in \mathcal{F}(M), X \in T_x M, \tag{4.5}$$

   where $d\varphi(X)$ is the action of the 1-form $d\varphi \in \Lambda^1 M$ evaluated at $x \in M$ on the tangent vector $X \in T_x M$. In order to prove (4.5) we use the following property,

$$\hat{\pi}_v^*(\varphi Y) = (\hat{\pi}_v^*\varphi)\,(\hat{\pi}_v^* Y),\ \forall\, Y \in TM,\ \varphi \in \mathcal{F}(M).$$

   Then one obtains the following expressions,

$$\begin{aligned}\langle\nabla\rangle_X(\varphi Y) &= \langle \pi_2|_u \nabla_{\iota_u(X)} \hat{\pi}_v^*(\varphi Y)\rangle = \langle \pi_2|_u \nabla_{\iota_u(X)} (\hat{\pi}_v^*\varphi)\hat{\pi}_v^* Y\rangle \\ &= \langle \pi_2|_u(\nabla_{\iota_u(X)}(\hat{\pi}_v^*\varphi))\hat{\pi}_v^*(Y)\rangle + \langle \pi_2|_u(\hat{\pi}_u^* f)\nabla_{\iota_u(X)}\hat{\pi}_v^*(Y)\rangle \\ &= \langle \pi_2|_u(\iota_u(X)\cdot(\hat{\pi}_v^*\varphi))\hat{\pi}_v^*(Y)\rangle + \varphi\,\langle \pi_2|_u \nabla_{\iota_u(X)}\hat{\pi}_v^*(Y)\rangle \\ &= \langle (X\cdot\varphi)\pi_2|_u \hat{\pi}_u^*(Y)\rangle + \varphi\,\langle \pi_2|_v \nabla_{\iota_u(X)}\hat{\pi}_v^*(Y)\rangle.\end{aligned}$$

   For the first term we perform the following simplification,

$$\begin{aligned}\langle (X\cdot\varphi)\pi_2|_u \hat{\pi}_u^*(Y)\rangle &= (X\cdot\varphi)\langle \pi_2|_u \hat{\pi}_u^*(Y)\rangle = (X\cdot\varphi)(\langle \pi_2|_u \hat{\pi}_u^*\rangle)Y \\ &= (X\cdot\varphi)Y = (d\varphi(X))Y.\end{aligned}$$

   Finally we obtain that

$$\langle\nabla\rangle_X(\varphi Y) = (\langle\nabla\rangle_{X_x}\varphi)Y + \varphi\langle\nabla\rangle_X Y = (d\varphi(X))Y + \varphi\langle\nabla\rangle_X Y.$$

□

For each section $Y \in \Gamma TM$, $\langle\nabla\rangle Y \in \Gamma T^{(1,1)}M$ we define

$$\langle\nabla\rangle(X, Y) := \langle\nabla\rangle_X Y, \quad X \in T_x M. \tag{4.6}$$

The average covariant derivative is extended to 1-forms by the requirement that it commutes with contractions. Thus for each $\alpha \in \Lambda^1 M$ and $X \in \Gamma TM$

$$\langle\nabla\rangle_X[\alpha(Z)] = (\langle\nabla\rangle_X\alpha)\cdot Z + \alpha\cdot(\langle\nabla\rangle_X Z)$$

holds by assumption. Then the extension of the covariant derivative $\langle\nabla\rangle_X$ to sections of the tensor bundle $T^{(p,q)}M \to M$ is defined by the rule

$$\begin{aligned}\langle\nabla\rangle_X\, K(X_1,...,X_s,\alpha^1,...,\alpha^r) &= \langle\nabla\rangle_X\, K(X_1,...,X_s,\alpha^1,...,\alpha^r)\\ &- \sum_{i=1}^{s} K(X_1,...,\langle\nabla\rangle_X X_i,...,X_s,\alpha^1,...,\alpha^r)\\ &+ \sum_{j=1}^{r} K(X_1,...,X_s,\alpha^1,...,\langle\nabla\rangle_X \alpha^j,...,\alpha^r).\end{aligned}$$

4.3 General properties of the averaged connection

Proposition 4.4 *Let (M,F) be a Finsler space and ∇ a linear connection on $\hat{\pi}^*TM$. Then the relation*

$$T_{\langle\nabla\rangle} = \langle T_\nabla\rangle. \tag{4.7}$$

holds good.

Proof. We can evaluate the torsion of the connection $\langle\nabla\rangle$,

$$\begin{aligned}T_{\langle\nabla\rangle}(X,Y) &= \langle\pi_2|_u \nabla_{\iota_u(X)}\hat{\pi}^*_v\rangle Y - \langle\pi_2|_u \nabla_{\iota_u(Y)}\hat{\pi}^*_v\rangle X - [X,Y]\\ &= \langle\pi_2|_u \nabla_{\iota_u(X)}\hat{\pi}^*_v\rangle Y - \langle\pi_2|_u \nabla_{\iota_u(Y)}\hat{\pi}^*_v\rangle X - \langle\pi_2|_u\hat{\pi}^*_u[X,Y]\rangle\\ &= \langle\pi_2|_u\big(\nabla_{\iota_u(X)}\hat{\pi}^* Y - \nabla_{\iota_u(Y)}\hat{\pi}^* X - \hat{\pi}^*|_u[X,Y]\big)\rangle\\ &= \langle T_\nabla(X,Y)\rangle.\end{aligned}$$

□

Corollary 4.5 *Let (M,F) be a Finsler space with average Chern-Rund connection $\langle{}^{cr}\nabla\rangle$. Then the torsion $T_{\langle{}^{cr}\nabla\rangle}$ is zero.*

Definition 4.6 *Two Finsler spaces (M,F_1) and (M,F_2) are Γ-related iff the corresponding average connections are the same.*

Γ-equivalence is an equivalence relation. Γ-equivalence classes are denoted as $[g]_\Gamma$. Let us remark that the averaging procedure depends on the particular Finsler spacer (M,F) that one starts and that this equivalence relation is between different types of averaging procedures. Also note that in general the Γ-equivalence relation is different from the *metric*-equivalence relation.

4.4 Average connection of a Berwald space

In order to avoid cluttering in the notation, the Chern-Rund connection will be denoted simply by ∇ instead of the most specific notation ${}^{cr}\nabla$ used until now.

Definition 4.7 *A Berwald space (M, F_B) is a Finsler space such that its Chern-Rund connection also defines an affine connection on M.*

For a Berwald space the connection coefficients ${}^{cr}\Gamma^i_{jk}(x, y)$ depend on $x \in M$ only. Thus, we have the following,

Theorem 4.8 *For a Berwald space (M, F)*

– *The average of the Chern-Rund connection coincides with the Chern-Rund connection in the sense that*

$$(\hat{\pi}^* \langle \nabla \rangle_X S) = \nabla_{\iota_u(X)} \hat{\pi}^* S. \tag{4.8}$$

– *The average of the Chern-Rund connection coincides with the Levi-Civita connection of the average metric,*

$$\langle \nabla \rangle \; = \; {}^h\nabla. \tag{4.9}$$

Proof. The relation (4.8) is direct from the definition of average connection. A detailed proof can be found in [9]. For the proof of relation (4.9), let us first evaluate the covariant derivative of the metric h for $\langle \nabla \rangle$,

$$\begin{aligned}
\langle \nabla \rangle_{X = \frac{\partial}{\partial x^i}} h &= \Big(\frac{\partial h_{jk}}{\partial x^i} - \langle \nabla \rangle^l_{ik} h_{jl} - \langle \nabla \rangle^l_{ij} h_{lk} \Big) dx^j \otimes dx^k \\
&= \Big(\frac{\partial h_{jk}}{\partial x^i} - {}^{cr}\Gamma^l_{ik} h_{jl} - {}^{cr}\Gamma^l_{ij} h_{lk} \Big) dx^j \otimes dx^k \\
&= \frac{1}{vol(I_x)} \Big(\int_{I_x} \big(\frac{\partial g_{jk}}{\partial x^i} - {}^{cr}\Gamma^l_{ik} g_{jl} - {}^{cr}\Gamma^l_{ij} g_{lk} \big) \, dvol_x \Big) dx^j \otimes dx^k \\
&= \frac{1}{vol(I_x)} \Big(\int_{I_x} \big(\nabla_{\frac{\partial}{\partial \tilde{x}^i}} g \big) \big)_{jk} \, dvol_x \Big) dx^j \otimes dx^k.
\end{aligned}$$

Since the horizontal metric compatibility of the Chern-Rund connection (*equation* (2.17)), the integrand is zero. Therefore,

$$\langle \nabla \rangle_{X = \frac{\partial}{\partial x^i}} h = 0, \quad i = 1, ..., n.$$

Moreover, by *Corollary* 4.5 $\langle \nabla \rangle$ is torsion free. Therefore, $\langle \nabla \rangle$ must be the Levi-Civita connection ${}^h\nabla$ of h.

$\square$

5 The parallel transport of the average connection

The parallel transport of a linear connection ∇ along a path $\gamma : [a,b] \to M$ with $\gamma(a) = x$ and $\gamma(b) = z$ is defined as the linear homomorphism

$$p_{xz}(\gamma) : T^{(p,q)}_x M \to T^{(p,q)}_z M, \quad S_x \mapsto S_z$$

such that the section $S(t)$ along γ is a solution of the linear differential equation

$$\nabla_{\dot\gamma(0)} p(\gamma)(S)(t) = 0, \quad p(\gamma)(S)(0) = S(0). \tag{5.1}$$

A *polygonal approximation* $\bar\gamma$ of γ is determined by a set of points

$$\{\gamma(0) = x, ..., \gamma(t_i), ..., \gamma(t_{A-1}), \gamma(t_A) = z, \gamma(t_i) \in \gamma([a,b])\}$$

joined by geodesic segments $\bar\gamma_{k,k+1}$ of F, with initial and ending points $\gamma(t_k)$ and $\gamma(t_{k+1})$ respectively. One can also consider the case when $t_k - t_{k-1} = \epsilon$.

Let us define $\epsilon = \frac{b-a}{A}$. The double limit $A \to +\infty$ and $\epsilon = t_k - t_{k-1} \to 0$ is taken in this parallel transport operation, under the constraint

$$\lim_{A \to +\infty,\, \epsilon \to 0} A\epsilon = b - a.$$

We can write the solution of the parallel transport equation, at least formally, by the following result:

Lemma 5.1 *The parallel transport operator along $\bar\gamma$ can be re-recast as a parallel transports*

$$p(\bar\gamma) := \prod_{k=1}^{A} \circ\, p_{t_k, t_{k-1}},$$

where the composition of elementary parallel transport $p_{t_k,t_{k-1}}$ is taken along the geodesic segment $\bar\gamma_{k,k+1}$ and is

$$p_{t_k,t_{k-1}} : T_{\gamma(t_{k-1})}M \to T_{\gamma(t_k)}M, X^i e_i \mapsto \big(\delta^{j_k}_{i_{k-1}} X^{i_{k-1}} - \epsilon\, \Gamma^{j_k}_{i_{k-1} l_{k-1}} X^{i_{k-1}} \dot\gamma^{l_{k-1}}\big) e_{j_k}.$$

$\dot\gamma^{l_{k-1}}$ is the tangent vector at the point $\gamma(k-1)$, where $\{e_i(t),\, i = 1, ..., n\}$ is a frame along $\gamma : I \to M$.

Proof. Then parallel transport of $X \in T_{\gamma(0)}M$ along γ between the point $x = \gamma(a)$ and $z = \gamma(b)$ is given by

$$(p_{xz}X)^j = \lim_{A \to +\infty, \epsilon \to 0} \Big((\delta^{j_A}_{i_{A-1}} - \epsilon\, \Gamma^{j_A}_{i_{A-1} l_{A-1}} \dot\gamma^{l_{A-1}}) \Big) (p_{x\gamma(t_{A-1})} X)^{i_{A-1}}, \; j = 1, ..., n, \tag{5.2}$$

with $(p_{xx}X)^{j_0} = X^{j_0}$, $\lim_{A \to +\infty} \gamma(t_{A-1}) = \gamma(b)$ and $j_A = j$.

Infinitesimally one has the finite difference expression

$$\begin{aligned}(p_{x\gamma(t+\epsilon)}X)^j - (p_{x\gamma(t)}X)^j &= \Big((\delta^{j_A}_{i_{A-1}} - \epsilon\,\Gamma^{j_A}{}_{i_{A-1}l_{A-1}}\,\dot\gamma^{l_{A-1}})\Big)(p_{x\gamma(t)}X)^{i_{A-1}}\\ &\quad - (p_{x\gamma(t)}X)^j\\ &= -\,\epsilon\,(p_{x\gamma(t)}\Gamma^j_{ik}(\gamma(t))\,\dot\gamma^i(t)(p_{x\gamma(t)}X)^k.\end{aligned}$$

For smooth vector fields and connections, one can take the limit

$$\lim_{\epsilon\to 0}\frac{(p_{x\gamma(t+\epsilon)}X)^j - (p_{x\gamma(t)}X)^j}{\epsilon} = -(p_{x\gamma(t)}\Gamma^j_{ik}(\gamma(t))\,\dot\gamma^i(t)(p_{x\gamma(t)}X)^k,$$

showing that the expression (5.2) is the solution of the parallel transport equation (5.1).

□

The expression (5.2) applies to the parallel transport of any linear connection. In particular, it can be applied to the average connection,

Proposition 5.2 *Let $\langle\Gamma^i_{jk}\rangle$ be the connection coefficients of the average connection $\langle\nabla\rangle$. Then the parallel transport operation is*

$$(p_{xz}X)^j = \lim_{A\to\infty,\,\epsilon\to 0}\prod_{k=1}^{A}\Big((\delta^{j_k}{}_{i_{k-1}} - \epsilon\,\langle\Gamma^{j_k}_{i_{k-1}l_k}\rangle(\gamma(t_k))\,\dot\gamma^{l_k})\Big)X^{i_0},\qquad j=1,...,n. \tag{5.3}$$

If we explicitly insert the weight factor $|\psi|^2 : N\to\mathbb{R}^+$ in each integration, one obtains the expression for the parallel transport

$$\begin{aligned}(p_{xz}X)^j &= \lim_{A\to+\infty,\epsilon\to 0}\Big(\prod_{k=1}^{A}\frac{1}{vol(I_{\gamma(t_{k-1})})}\Big(\int_{I_{\gamma(t_{k-1})}} dvol_x(\gamma(t_{k-1})) \qquad (5.4)\\ &|\psi|^2(\gamma(t_k),y_{\gamma(t_k)})\,(\delta^{j_k}{}_{i_{k-1}} - \epsilon\,\Gamma^{j_k}{}_{i_{k-1}l_{k-1}}(\gamma(t_k),y_{\gamma(t_k)})\,\dot\gamma^{l_{k-1}}(\gamma(t_{k-1}))\Big)\Big)X^{i_0}.\end{aligned}$$

Remark 5.1 In the general case the average of the parallel transport operation does not coincide with the parallel transport of the average connection: the average of the parallel transport of ∇ involves only one fiber integration, while the parallel transport of the average connection involves a formal product of an infinite number of integral operations along each fiber $\hat\pi^{-1}(\gamma(t))$, $t\in[a,b]$.

5.1 Curvature of the average connection

Let us consider the curvature endomorphisms for the average connection $\langle\nabla\rangle$,

$$R^{\langle\nabla\rangle}(X_1,X_2)Z=\big(\langle\nabla\rangle_{X_1}\langle\nabla\rangle_{X_2}-\langle\nabla\rangle_{X_2}\langle\nabla\rangle_{X_1}-\langle\nabla\rangle_{[X_1,X_2]}\big)Z.$$

Developing this expression in terms of the original connection ∇ one obtains

$$\begin{aligned}R^{\langle\nabla\rangle}_x(X_1,X_2)(Z)=\frac{1}{vol^2(\Sigma_x)}&\int_{I_x}\int_{I_x}dvol_x(v)dvol_x(u)\,\pi_2(v)\\ &\Big(\nabla_{\iota_v(X_1)}\hat{\pi}^*_v\,\pi_2(u)\,\nabla_{\iota_u(X_2)}\hat{\pi}^*_u\,Z-\nabla_{\iota_v(X_2)}\hat{\pi}^*_v\,\pi_2(u)\,\nabla_{\iota_u(X_1)}\hat{\pi}^*_u\,Z\\ &-\nabla_{\iota_u([X_1,X_2])}\,\hat{\pi}^*_u\,Z\Big).\end{aligned}$$

Analogously to the *remark* 5.1, the curvature $R^{\langle\nabla\rangle}$ is not equal to the *average curvature* of the linear connection ∇. For instance, the averaged hh-curvature is

$$\begin{aligned}\langle R^{\nabla}(\iota_u(X_1),\iota_u(X_2))\rangle Z:=&\frac{1}{vol(I_x)}\int_{I_x}dvol_x\,\pi_2(u)\,\nabla_{\iota_u(X_1)}\nabla_{\iota_u(X_2)}\hat{\pi}^*_u\,Z\\ &-\frac{1}{vol(I_x)}\int_{I_x}dvol_x\,\pi_2(u)\nabla_{\iota_u(X_2)}\,\nabla_{\iota_u(X_1)}\hat{\pi}^*_u\,Z\\ &-\frac{1}{vol(I_x)}\int_{I_x}dvol_x\,du\,\pi_2(u)\,\nabla_{\iota_u([X_1,X_2])}\,\hat{\pi}^*_u\,Z\\ &=\frac{1}{vol(I_x)}\int_{I_x}dvol_x\,\pi_2(u)\,\Big(\nabla_{\iota_u(X_1)}\nabla_{\iota_u(X_2)}\\ &-\nabla_{\iota_u(X_2)}\,\nabla_{\iota_u(X_1)}-\nabla_{\iota_u([X_1,X_2])}\Big)\hat{\pi}^*_u\,Z.\end{aligned}$$

Therefore, given a linear connection on π^*TM ∇, there are two notions of *average curvature endomorphisms*, $R^{\langle\nabla\rangle}_x(X_1,X_2)$ and $\langle R^{\nabla}(\iota_u(X_1),\iota_u(X_2))\rangle$. In the general case the tensors $R^{\langle\nabla\rangle}_x(X_1,X_2)$ and $\langle R^{\nabla}(\iota_u(X_1),\iota_u(X_2))\rangle$ do not coincide because the covariant derivative $\nabla_{\iota_u(Y)}$ depends on $u\in N$ for a general Finsler space.

Also, note that the average of the curvature operation can be generalized to other curvature endomorphisms, specifically, to the hv and vv-curvature endomorphisms of ∇.

5.2 Average curvature of a Berwald space

The geodesic deviation equation for a Finsler space is formally the same than for a Riemannian space. In particular, the Jacobi equation for the Chern-Rund connection [4] is the linear differential equation along the geodesic

$X : I \to M$

$$\nabla_X \nabla_X J + R(X, J)X = 0. \tag{5.5}$$

When (M, F_B) is a Berwald space, then the equation (5.5) is formally the same than the equation for the geodesics of the average connection,

$$\langle \nabla \rangle_{\tilde{X}} \langle \nabla \rangle_{\tilde{X}} \tilde{J} + R^{\langle \nabla \rangle}(\tilde{X}, \tilde{J})\tilde{X} = 0,$$

or

$$\nabla_{\tilde{X}} \nabla_{\tilde{X}} \tilde{J} + R(\tilde{X}, \tilde{J})\tilde{X} = 0,$$

where $\tilde{X} : \tilde{I} \to M$ is a geodesic of the averaged connection $\langle \nabla \rangle$ and $R^{\langle \nabla \rangle}$ is the curvature endomorphism. The Jacobi fields $J : I \to X(I)$ and $\tilde{J} : \tilde{I} \to \tilde{X}(\tilde{I})$ are the same (at least for small values of the time values of the time parameter, the geodesics X and $\tilde{X}$ as un-parameterized geodesics), except that they are parameterized by different parameters. Then the condition

$$R^{\langle \nabla \rangle}(\tilde{X}, \tilde{J}) = R(X, J)$$

must hold for Berwald spaces. Since for a Berwald space $\langle \nabla \rangle = {}^h\nabla$, the following result holds:

Theorem 5.3 *If (M, F_B) is a Berwald space, then*

$$^hR(X, Y) = R(X, Y), \quad X, Y \in \Gamma TM. \tag{5.6}$$

Corollary 5.4 *If (M, F_B) is a Berwald space, then ${}^hR^i{}_{jkl} = \langle {}^gR^i{}_{jkl} \rangle$.*

Proof. This is a direct consequence of the above calculation and *Theorem* 5.3.

□

A straightforward calculation illustrates that for a general Finsler space, the average of the curvature endomorphism does not coincide with the curvature endomorphism of the average metric,

$$\begin{aligned} {}^hR^i_{jkl} &= h^{im}\, {}^hR_{mjkl} = h^{im} \langle {}^gR_{mjkl} \rangle = \langle h^{im}\, {}^gR_{mjkl} \rangle \\ &= \langle h^{im} g_{ms} g^{sa}\, {}^gR_{ajkl} \rangle = \langle \vartheta^i{}_s\, {}^gR^s_{jkl} \rangle, \end{aligned}$$

where the tensor $\vartheta^i{}_s := h^{im} g_{ms} \neq \delta^i_s$ measures the departure of the fundamental tensor g of being Riemannian. Similarly, for a general Finsler spacer, here is no direct relation between the average of the flag curvature of F and the sectional curvatures of h, even in the case of constant sign flag curvature spaces.

Let (M, F_B) be a Berwald space. Then the Riemann tensor of g is a $(0,4)$-tensor along the map $\hat{\pi} : N \to M$ whose components are given in normal coordinates of g by the expression[1]

$$^{g}R_{ijkl} := g_{il,jk} - g_{ik,jl} + g_{jk,il} - g_{jl,ik}, \tag{5.7}$$

where $g_{ij,kl}$ stands for $\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}$, etc...

Proposition 5.5 *Let (M, F_B) be a Berwald space and consider the isometric average metric h_{ij}. Then the following relation holds,*

$$^{h}R_{ijkl}(x) = \langle\, ^{g}R_{ijkl}(x,y)\rangle. \tag{5.8}$$

Proof. In normal coordinates for h, the Riemann tensor $^{h}R_{ijkl}$ is linear on the second derivatives of the components of the curvature tensor of h. Then the tensor can be expressed as

$$^{h}R_{ijkl} = h_{il,jk} - h_{ik,jl} + h_{jk,il} - h_{jl,ik},$$

where for instance $h_{il,jk} = \frac{\partial^2 h_{il}(x)}{\partial x^j \partial x^k}$, etc... From the definition of h it follows that

$$^{h}R_{ijkl} = \langle\, g_{il}\rangle_{,jk} - \langle g_{ik}\rangle_{,jl} + \langle g_{jk}\rangle_{,il} - \langle h_{jl}\rangle_{,ik}$$

holds in the normal coordinate chard of h. Since the weight factor is $|\psi|^2 = 1$ and the volume function $vol(I_x)$ is constant for a Berwald space [3], the partial derivatives can be introduced in the integrals,

$$\begin{aligned} ^{\langle g\rangle}R_{ijkl}(x) &= {}^{h}R_{ijkl} = \langle\, g_{il}\rangle_{,jk} - \langle g_{ik}\rangle_{,jl} + \langle g_{jk}\rangle_{,il} - \langle h_{jl}\rangle_{,ik} \\ &= \langle\, g_{il,jk} - g_{ik,jl} + g_{jk,il} - g_{jl,ik}\rangle = \langle\, ^{g}R_{ijkl}(x,y)\rangle, \end{aligned}$$

where from the first to the second line in the above expression we have use the equality $^{cr}\Gamma^{i}_{jk}(x) = {}^{h}\Gamma^{i}_{jk}(x) = 0$ in normal coordinates of the average metric h. In this step is essential the Berwald condition. This formulae has been proved in normal coordinates of h, but since it is an identity between tensor components, it holds in any coordinate system.

□

As an example of direct application of *Proposition* 5.5 is a generalization of Riemann's characterization of Euclidean space in terms of curvature. Let h_o be the Euclidean metric in $\mathbb{R}^n$. Then we have

Corollary 5.6 *Let (M, F) be a Berwald space such that $^{g}R_{ijkl} = 0$. Then $M \cong \mathbb{R}^n$ and the fundamental tensor $g = h_0 + \delta g$, with $\langle \delta g\rangle = 0$.*

Proof. If $^{g}R_{ijkl} = 0$, then the relation (5.8) implies $^{h}R_{ijkl}(x) = 0$ and the result follows from the classical result of Riemann [13].

□

[1] For a Berwald space, normal coordinate system exists and are $\mathcal{C}^2$, see for instance [4].

5.3 Gauss-Bonnet theorem for Berwald surfaces

The construction of the metric h as an average of the fundamental tensor over the indicatrix opens the possibility to generalize results from Riemannian to Berwald geometry, using directly the Riemannian results. We consider here an example of this technique: a weak version of the Gauss-Bonnet theorem for arbitrary Berwald surfaces. Let $(\mathcal{I}, \pi_{\mathcal{I}}, M)$ be the fibered manifold whose fiber over $x \in M$ is the indicatrix I_x and note that for a Berwald space the function $x \mapsto vol(I_x)$ is constant. Then we have the following result,

Theorem 5.7 *Let (M, F_B) be a compact Berwald surface with average metric h and Gaussian curvature ${}^hK = -\,{}^hR_{1212}$ in some orthonormal basis of h. Then the following formula holds [5],*

$$\frac{1}{vol(I_x)} \int_{\mathcal{I}} {}^gR_{1212}(x,y)\, dvol_x \wedge d\mu(x) = \, -2\pi\,\chi(M), \tag{5.9}$$

where $\chi(M)$ is the Euler's characteristic of M and $d\mu$ is the Riemannian volume form of h on M.

Proof. For the Riemannian metric h one can apply the classical Gauss-Bonnet theorem for compact surfaces M. Thus the relation

$$\int_M {}^hR_{1212}(x)\, d\mu = \, -2\pi\,\chi(M) \tag{5.10}$$

holds. Fixed the integration measure as in *Proposition* 5.5 an using an orthonormal frame associated to h, one obtains the relations

$${}^hK(x) = -\,{}^hR_{1212}(x) = -\langle\, {}^gR_{1212}\rangle,$$

from which the relation (5.9) follows.

□

5.4 Holonomy of a Berwald space

For a Berwald space, the Chern-Rund connection is defined on the manifold M and the curvature endomorphisms of the average connection coincide with the average of the curvature endomorphisms of the original linear connection on $\hat{\pi}^*TM$. Therefore, as an application of the Ambrose-Singer theorem on holonomy [1,11] and *Theorem* 5.3 it holds the following result. Let $Hol(\nabla)$ be the holonomy group of the Chern-Rund connection ∇. Then we have the following

Theorem 5.8 *Let (M, F_B) be a Berwald space. Then the holonomy group $Hol(\nabla)$ is a Riemannian holonomy.*

Proof. If the space (M, F_B) is a Berwald space, then the hv-curvature endomorphisms are identically zero, $P = 0$. Then the result follows from a direct application of the Ambrose-Singer theorem on holonomy and the relation (5.6).

□

An interesting consequence of *Corollary* 5.6 is that, if the space (M, F_B) is a Berwald space, then the average holonomy group is not only an affine holonomy group, but it is indeed metrizable. This is an extension of a theorem from Z. Szabó on metrizability of compact holonomies of Berwald spaces [15],

Theorem 5.9 *Let (M, F_B) be a Berwald space. Then the holonomy of the Chern-Rund connection is metrizable.*

The proof of *Theorem* 5.9 does not make use of any assumption on the compactness of $Hol(\nabla)$. Therefore, we have also proved that

Corollary 5.10 *For a Berwald space (M, F_B), the holonomy group of the Chern-Rund connection is compact.*

6 Isometries of the average metric

Definition 6.1 *Given two Finsler spaces (M_1, F_1) and (M_2, F_2), a base manifold Finsler isometry (or simply a Finsler isometry) is a diffeomorphism $\Phi : M_1 \to M_2$ such that preserves the Finsler function,*

$$F_2(\Phi(x), d\Phi(y)) = F_1(x, y). \tag{6.1}$$

As a direct consequence of this definition and in the case when $M_1 = M_2$, the components of the fundamental tensor transform locally under an isometry as

$$(g_1)_{ij}(\tilde{x}(x), \tilde{y}(x, y)) = \frac{\partial x^l}{\partial \tilde{x}^i}\frac{\partial x^k}{\partial \tilde{x}^j}(g_2)_{lk}(x, y). \tag{6.2}$$

In such context, it was proved using a construction that involves the Sasaki type metric [10] that

Proposition 6.2 *The group of isometries of (M, F) is contained as a closed subgroup of the isometries of (M, h) (in the compact-open topology).*

Then it follows that the group of isometries of (M, F) is a Lie group [6] and it is a subgroup of the group of isometries of the average metric (M, h).

Definition 6.3 *A Finsler space (M, F) is symmetric if for each point $x \in M$ there is a Finsler isometry $\varphi_x : M \to M$ such that*

– $\varphi_x(x) = x$,
– $(d\varphi_x)|_x = -Id|_{T_xM}$.

The two conditions of the isometry φ are formally the same for h than for F. Then by direct application of *Proposition* 6.2 we have

Theorem 6.4 *If (M, F) is a Finsler symmetric space, then (M, h) is a Riemannian symmetric space.*

This result holds for locally symmetric spaces and globally symmetric spaces. In particular, we have

Corollary 6.5 *If (M, F) is a global symmetric space, then it is an homogeneous space.*

Proof. Let us consider the average metric h, which is necessarily globally symmetric. Then by application of the analogous Riemannian result, $M \cong G/H$, where G is the isometry group and H the isotropy group of the average metric h.

□

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