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Non-coercive nonlinear parabolic problem with measurable obstacle in modular space

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Abstract In this paper, we shall be concerned with the existence result for an unilateral problem associated to the form, $\frac{\partial u}{\partial t} - \mathrm{div}\Big(a(x,t,u,\nabla u)\Big) - \mathrm{div}(\Phi(x,t,u))) = f$ in $Q_T = \Omega \times (0,T)$, where $Au = -\mathrm{div} a(x,t,u,\nabla u)$ is a Leray-Lions operator defined on $W_0^{1,x}L_M(Q_T)$, The growth and coercivity conditions on the monotone vector field a are prescribed by generalized N-function M, which does not have to satisfy the Δ_2 condition. Therefore we work with a generalized Musielak-Orlicz spaces which are not necessarily reflexive. The right hand side and the initial data belong to $L^1(Q_T)$ and the lower order term Φ is a non-coercive Carathéodory function.

Keywords nonlinear parabolic problems · Musielak-Orlicz-Sobolev spaces · unilateral problems · measurable obstacle · lower order term

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1 Introduction

The study of parabolic equations and inequations arouses much interest with the development of applications in various fields of physics, biology, astronomy, ...

Moreover, the variational inequalities have numerous applications also in the theory of plasticity, hydrodynamics, filtration theory, economics etc. (see, for instance, [4] and [12]). The interest in unilateral problems is due to consider an obstacle function which is only measurable. In the last decade, there has been an increasing interest in the study of various mathematical problems in modular spaces. These problems have many consideration in applications (see [9], [31], [33]) and have resulted in a renewal interest in Lebesgue and Sobolev spaces with variable exponent, or the general Musielak-Orlicz spaces, the origins of which can be traced back to the work of Orlicz in the 1930s. In the 1950s, this study was carried on by Nakano ([26]) who made the first systematic study of spaces with variable exponent.

Later on, Polish and Czechoslovak mathematicians investigated the modular function spaces (see, for instance, Musielak [25], Kováčik and Rákosník [21]). The study of variational problems where the function a satisfies a nonpolynomial growth conditions instead of having the usual p-structure arouses much interest with the development of applications to electro-rheological fluids as an important class of non-Newtonian fluids (sometimes referred to as smart fluids). The electro-rheological fluids are characterized by their ability to drastically change the mechanical properties under the influence of an external electromagnetic field. A mathematical model of electro-rheological fluids was proposed by Rajagopal and Růžička (we refer to [30], [31] for more details). Another important application is related to image processing [27] where this kind of diffusion operator is used to underline the borders of the distorted image and to eliminate the noise.

Our analysis makes extensively use of the notion of modular convergence in Musielak-Orlicz spaces. The fundamental studies in this direction are due to Gossez for the case of elliptic equations ([15, 16]). The appearance of the (x,t)-dependence in an N-function requires the studies on the uniform boundedness of the convolution operator. The considerations of the problem with an x-dependent modular function formulated in Musielak-Orlicz-Sobolev spaces are due to Benkirane et al. ([7]) where the authors formulate an approximation theorem with respect to the modular topology. A particular case of Musielak-Orlicz spaces with an x-dependent modular function are the variable exponent spaces $L^{p(x)}$. The reader is referred to [10] for a comprehensive summary on those spaces.

Following the studies for standard Leray-Lions operators (namely under assumption of polynomial growth conditions for the nonlinear term) on renormalized, entropy solution or unilateral problems [11], [28] and [8], there appeared also results on renormalized solutions for parabolic problems in Orlicz-Sobolev spaces. The existence result of a renormalized solution for a class of nonlinear parabolic equations in Orlicz spaces with no growth assumption made on the nonlinearities and an N-function which does not have to satisfy the Δ_2-condition was achieved in [19] and [5] also for unilateral problems see [20]. The arguments used to prove the convergence of approximate solutions to a renormalized solution are based on an approximation property in Orlicz-Sobolev spaces proved by Gossez in [14]. The author shows that it is possible to approximate the gradient of an $W_0^1 L_M(\Omega)$-function in modular convergence by a sequence of gradient of smooth functions, compactly supported in Ω. As far as we know, there's not much papers concerned with the nonlinear parabolic equations with obstacle in Musielak-Orlicz spaces with L^1 data, in the context of renormalized solution we refer to the work of Gwiazda, Wittbold and al. in [17] where the existence proof related to a nonlinear parabolic problem with L^1-data in Musielak spaces requires a very technical construction of multistage approximation of the solution. In particular it is based on nonlinear semi-group theory of m-accretive operators, but the authors assume that $\overline{M}$

satisfies the $\varDelta_2$-condition and the proof was based on the modular Poincaré inequality which not the same in our approach we refer also to [18] for the elliptic case without $\varDelta_2$-condition on M.

We deal with boundary value problems

$$(P)\begin{cases} u \in K_\psi, \\ \dfrac{\partial u}{\partial t} - \mathrm{div}\Big(a(x,t,u,\nabla u)\Big) - \mathrm{div}(\varPhi(x,t,u)) = f \quad \text{in } Q_T = \varOmega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \varOmega, \\ u = 0 \quad \text{on } \partial\varOmega \times (0,T). \end{cases} \tag{1.1}$$

Where $K_\psi = \{u \in W_0^{1,x}L_M(Q_T);\ u \geq \psi \quad a.e. \text{ in } \quad Q_T\}$, the data f in $L^1(Q_T)$, $Au = -\mathrm{div}\Big(a(x,t,u,\nabla u)\Big)$ is a Leray-Lions operator defined on $W_0^{1,x}L_M(Q_T)$, M is a generalized N-function and the term $\varPhi$ is a Carathéodory function satisfies the following growth condition for a.e. $(x,t) \in Q_T$ and for all $s \in \mathbf{R}$,

$$|\varPhi(x,t,s)| \leq \gamma(x,t) \cdot \overline{M}_x^{-1} M(x,s),$$

with $\gamma \in L^\infty(Q_T)$ and P and Q are three N-functions such that $P \ll M \ll Q$ means that P grows essentially less rapidly than M (see below notations and properties).

As we allow that $\varPhi$ depends on x,t and u, the lower order term will not be handled by the divergence theorem while the N-function M does not have to satisfy the suitable condition $\varDelta_2$ which induces a loss of reflexivity of our framework setting, and more difficulties in the proof of our result. The definition of the unilateral problem solution can be stated as follows.

Definition 1.1 *A measurable function $u : \varOmega \times (0,T) \to \mathbf{R}$ is called a solution of the following unilateral problem,*

$$\begin{aligned} &\int_\varOmega S_k(u-v)(\tau)dx + \langle \frac{\partial v}{\partial t}, T_k(u-v)\rangle_{Q_\tau} \\ &+ \int_{Q_\tau} a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u-v)\ dx\ dt \\ &+ \int_{Q_\tau} \varPhi(x,t,u)\nabla T_k(u-v)\ dx\ dt \\ &\leq \int_{Q_\tau} f T_k(u-v)\ dx\ dt + \int_\varOmega S_k(u_0 - v(0))dx, \end{aligned} \tag{1.2}$$

if the assumptions bellow hold:

- $T_k(u)$ belongs to $W_0^{1,x}L_M(Q_\tau)$ with $u \geq \psi$ a.e in Q_τ and $S_k(u(.,t))$ belongs to $L^1(\varOmega)$, for every $k > 0$ and $t,\tau \in [0,T]$.

- For all $v \in W_0^{1,x}L_M(Q_\tau) \cap L^\infty(Q_\tau)$, $\frac{\partial v}{\partial t}$ belongs to $W^{-1,x}L_{\overline{M}}(Q_\tau) + L^1(Q_\tau)$.

Remark 1.1 Equation (1.2) is formally obtained through pointwise multiplication of the problem (1.1) by $T_k(u-v)$. Note that each term in (1.2) has a meaning since $T_k(u-v) \in W_0^{1,x}L_M(Q_T) \cap L^\infty(Q_T)$. In addition by Lemma 3.8 (to see later), we have $v \in C([0,T]; L^1(\Omega))$ and then the first and last terms of equation (1.2) are well defined.

Our purpose in this paper is to prove an existence results of unilateral problem in the sense of Definition 1.1 in the setting of the inhomogeneous Musielak-Orlicz-Sobolev space $W^{1,x}L_M(Q_T)$ with M is an N-function which does not have to satisfy an upper growth bound described by a Δ_2 condition and in the present of the non-coercive term Φ where the obstacle ψ is only measurable.

2 Musielak-Orlicz spaces : notations and properties

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [25]. We also include the definition of inhomogeneous Musielak-Orlicz-Sobolev spaces and some preliminaries Lemmas to be used later on this paper.

2.1 Musielak-Orlicz spaces

Let Ω be a bounded domain in $\mathbf{R}^d$ and let M be a real-valued function defined in $\Omega \times \mathbf{R}$. M is called generalized Orlicz function or Musielak-Orlicz function if and only if the following conditions are verified :

- $M(x,.)$ is an N-function, i.e., convex, nondecreasing, continuous, $M(x,0)=0$, $M(x,t)>0$ for all $t>0$ and

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{M(x,t)}{t} = 0 \quad \text{for almost all } x \in \Omega, \tag{2.1}$$

$$\lim_{t\to \infty} \inf_{x\in\Omega} \frac{M(x,t)}{t} = \infty \text{ for almost all } x \in \Omega. \tag{2.2}$$

- For all $t \in \mathbf{R}$, $M(\cdot,t)$ is a measurable function.

For a Musielak-Orlicz function $M(x,t)$ we put, for a. a. $x \in \Omega$, $M_x(t) = M(x,t)$ and we associate its inverse function with respect to $t \geq 0$ M_x^{-1}, that is,

$$M_x^{-1}(M(x,t)) = M(x, M_x^{-1}(t)) = t, \text{ for all } t \geq 0.$$

Let M and P be two Musielak-Orlicz functions.

(c) Assume that there exist two positives constants k and t_0 such that for a. a. $x \in \Omega$

$$M(x,t) \leq P(x,kt) \text{ for all } t \geq t_0,$$

then we write $M \prec P$ and we say that P dominates M globally if $t_0 = 0$ and near infinity if $t_0 > 0$.

(d) We say that P grows essentially less rapidly than M at $t = 0$ (respectively, near infinity), and we write $P \ll M$, if for every positive constant k we have

$$\lim_{t \to 0} \sup_{x \in \Omega} \frac{P(x, kt)}{M(x,t)} = 0 \text{ (respectively, } \lim_{t \to \infty} \sup_{x \in \Omega} \frac{P(x, kt)}{M(x,t)} = 0).$$

Remark 2.1 It is easy to check that $P \ll M$ near infinity if and only if

$$\lim_{t \to \infty} \frac{M^{-1}(x, kt)}{P^{-1}(x,t)} = 0 \text{ uniformly for } x \in \Omega \setminus \Omega_0$$

for some null subset $\Omega_0 \subset \Omega$.

We define the functional

$$\varrho_{M,\Omega}(u) = \int_\Omega M(x, u(x))\, \mathrm{d}x,$$

for any Lebesgue measurable function $u\colon \Omega \mapsto \mathbf{R}$ is a Lebesgue measurable function.

The set

$$\mathcal{L}_M(\Omega) = \{u\colon \Omega \mapsto \mathbf{R} \text{ mesurable such that } \varrho_{M,\Omega}(u) < \infty\}$$

is called the Musielak-Orlicz class related to M in Ω or simply the Musielak-Orlicz class.

The Musielak-Orlicz space $L_M(\Omega)$ is the vector space generated by $\mathcal{L}_M(\Omega)$, that is, $L_M(\Omega)$ is the smallest linear space containing the set $\mathcal{L}_M(\Omega)$. Equivalently,

$$L_M(\Omega) = \{u\colon \Omega \mapsto \mathbf{R} \text{ mesurable } / \ \varrho_{M,\Omega}(u/\alpha) < \infty, \text{ for some } \alpha > 0\}.$$

For a Musielak-Orlicz function M, we introduce its complementary function, denoted by $\bar{M}$, as

$$\bar{M}(x,s) = \sup_{t \geq 0} \{st - M(x,t)\},$$

that is $\bar{M}(x,s)$ is the complementary to $M(x,t)$ in the sense of Young with respect to the variable s. It turns out that $\bar{M}$ is another Musielak-Orlicz function and the following Young-Fenchel inequality holds

$$|ts| \leq M(x,t) + \bar{M}(x,s) \text{ for all } t, s \in \mathbf{R} \text{ and a. a. } x \in \Omega. \tag{2.3}$$

In the space $L_M(\Omega)$ we define the following two norms:

$$\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 \,/ \int_\Omega M(x, u(x)/\lambda)\, \mathrm{d}x \leq 1 \right\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm, namely

$$\|u\|_{(M),\Omega} = \sup_{\varrho_{\bar{M},\Omega}(v)\leq 1} \int_\Omega u(x)v(x)\,\mathrm{d}x.$$

where the supremum is taken over all $v \in E_{\bar{M}(\Omega)}$ such that $\varrho_{\bar{M},\Omega}(v) \leq 1$. An important inequality in $L_M(\Omega)$ is the following:

$$\int_\Omega M(x, u(x))\,\mathrm{d}x \leq ||u||_{(M),\Omega} \text{ for all } u \in L_M(\Omega) \text{ such that } ||u||_{(M),\Omega} \leq 1, \tag{2.4}$$

wherefrom we readily deduce

$$\int_\Omega M\Big(x, \frac{u(x)}{||u||_{(M),\Omega}}\Big)\,\mathrm{d}x \leq 1 \text{ for all } u \in L_M(\Omega) \setminus \{0\}. \tag{2.5}$$

It can be shown that the norm $|| \cdot ||_{(M),\Omega}$ is equivalent to the Luxemburg norm $\| \cdot \|_{M,\Omega}$. Indeed,

$$||u||_{M,\Omega} \leq ||u||_{(M),\Omega} \leq 2||u||_{M,\Omega} \text{ for all } u \in L_M(\Omega). \tag{2.6}$$

Also, Hölder's inequality holds

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \leq ||u||_{M,\Omega}||v||_{(\bar{M}),\Omega} \text{ for all } u \in L_M(\Omega) \text{ and } v \in L_{\bar{M}}(\Omega),$$

in particular, if Ω has finite measure, Hölder's inequality yields the continuous inclusion $L_M(\Omega) \subset L^1(\Omega)$.

Strong convergence in $L_M(\Omega)$ is rather strict. For most purposes, a mild concept of convergence will be enough, namely, that of modular convergence. The closure in $L_M(\Omega)$ of the bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$. The space $E_M(\Omega)$ is the largest linear space such that $E_M(\Omega) \subset \mathcal{L}_M(\Omega) \subset L_M(\Omega)$, where the inclusion is in general strict.

Definition 2.1 *We say that a sequence $(u_n) \subset L_M(\Omega)$ is modular convergent to $u \in L_M(\Omega)$ if there exists a constant $h > 0$ such that*

$$\lim_{n\to\infty} \varrho_{M,\Omega}((u_n - u)/h) = 0.$$

2.2 Musielak-Orlicz-Sobolev spaces

For any fixed nonnegative integer m we define

$$W^m L_M(\Omega) = \{u \in L_M(\Omega) \,/\, D^\alpha u \in L_M(\Omega) \text{ for all } \alpha, |\alpha| \leq m\}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbf{Z}$, $\alpha_j \geq 0$, $j = 1, \ldots, d$, with $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d$ and $D^\alpha u$ denote the distributional derivative of multiindex α. The space $W^m L_M(\Omega)$ is called the Musielak-Orlicz-Sobolev space (of order m).

Let $u \in W^m L_M(\Omega)$, we define

$$\varrho^{(m)}_{M,\Omega}(u) = \sum_{|\alpha|\leq m} \varrho_{M,\Omega}(D^\alpha u) \text{ and } \|u\|^{(m)}_{M,\Omega} = \inf\{\lambda > 0 \,/\, \varrho^{(m)}_{M,\Omega}(u/\lambda) \leq 1\},$$

and

$$\|u\|_{m,M,\Omega} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{M,\Omega}. \tag{2.7}$$

these functionals are, respectively, convex and equivalent norms on $W^m L_M(\Omega)$. The pair $(W^m L_M(\Omega), \|\cdot\|^{(m)}_{M,\Omega})$ is a Banach space if there exists a constant $c > 0$ such that

$$\operatorname*{ess\,inf}_{x\in\Omega} M(x,1) \geq c \tag{2.8}$$

The space $W^m L_M(\Omega)$ is identified to a subspace of the product $\prod_{|\alpha|\leq m} L_M(\Omega) = \prod L_M$, this subspace is $\sigma(\prod L_M, \prod E_{\bar{M}})$ closed.

Let $W^m_0 L_M(\Omega)$ be the $\sigma(\prod L_M, \prod E_{\bar{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^m L_M(\Omega)$. Let $W^m E_M(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_M(\Omega)$, and $W^m_0 E_M(\Omega)$ is the (norm) closure of $\mathcal{D}(\Omega)$ in $W^m L_M(\Omega)$.

Lemma 2.2 *(cf. [2]) Let Ω be a bounded Lipchitz domain of $\mathbf{R}^N$ and let M be a Musielak-Orlicz function. Then there exists a constant $C > 0$ such that*

$$||u||_{M,\Omega} \leq C||\nabla u||_{M,\Omega}, \tag{2.9}$$

for all $u \in W^1_0 L_M(\Omega)$.

Remark 2.2 Let $u \in W^1_0 L_M(\Omega)$ with M a Musielak-Orlicz function, if

$$\int_\Omega M(x,\nabla u)\,dx \leq C,$$

holds true, then we have

$$||u||_{1,M,\Omega} < C'.$$

With C and C' are two positive constants.
Indeed, we have $||u||_{1,M,\Omega} = ||u||_{M,\Omega} + ||\nabla u||_{M,\Omega}$ by using (2.9) of Lemma 2.2, we get

$$||u||_{1,M,\Omega} \leq C||\nabla u||_{M,\Omega} + ||\nabla u||_{M,\Omega} \leq (C+1)||\nabla u||_{M,\Omega},$$

since $\displaystyle\int_\Omega M(x,\nabla u)\,dx \leq C$,
if $C \geq 1$, we have by convexity of $M(x,.)$,

$$\int_\Omega M(x,\frac{\nabla u}{C})\,dx \leq \frac{1}{C}\int_\Omega M(x,\nabla u)\,dx \leq \frac{C}{C} = 1.$$

then $C \in \{\lambda > 0, \int_\Omega M(x, \frac{\nabla u}{\lambda})\, dx \leq 1\}$, hence

$$||\nabla u||_{M,\Omega} \leq C.$$

If $C < 1$,

$$\int_\Omega M(x, \nabla u)\, dx \leq C < 1,$$

then

$$||\nabla u||_{M,\Omega} \leq 1.$$

Hence,

$$||\nabla u||_{M,\Omega} \leq \max(C, 1).$$

From above, we obtain

$$||u||_{1,M,\Omega} \leq (C+1)\max(C,1) = C'.$$

Since we are going to work with three generalized N-functions, say P, M and Q, such that $P \ll M \ll Q$, we will consider the following assumptions for the three complementary functions $\overline{P}$, $\overline{M}$ and $\overline{Q}$:

$$\lim_{|\xi|\to\infty} \operatorname*{ess\,inf}_{x\in\Omega} \frac{\overline{M}(x,\xi)}{|\xi|} = \infty, \tag{2.10}$$

and

$$\lim_{|\xi|\to\infty} \operatorname*{ess\,inf}_{x\in\Omega} \frac{\overline{P}(x,\xi)}{|\xi|} = \infty. \tag{2.11}$$

and

$$\lim_{|\xi|\to\infty} \operatorname*{ess\,inf}_{x\in\Omega} \frac{\overline{Q}(x,\xi)}{|\xi|} = \infty. \tag{2.12}$$

Remark 2.3 From Remark 2.1 in [17] we have that the assumptions (2.10) and (2.11) provide the following:

$$\sup_{\xi\in B(0,R)} \operatorname*{ess\,sup}_{x\in\Omega} M(x,\xi) < R_1 \text{ for all } 0 < R < +\infty, \tag{2.13}$$

and

$$\sup_{\xi\in B(0,R)} \operatorname*{ess\,sup}_{x\in\Omega} P(x,\xi) < R_2 \text{ for all } 0 < R < +\infty. \tag{2.14}$$

and

$$\sup_{\xi\in B(0,R)} \operatorname*{ess\,sup}_{x\in\Omega} Q(x,\xi) < R_3 \text{ for all } 0 < R < +\infty. \tag{2.15}$$

Definition 2.3 *We say that a sequence of functions $(u_n) \subset W^1L_M(\Omega)$ converges to $u \in W^1L_M(\Omega)$ for the modular convergence in $W^1L_M(\Omega)$ if, for some $h > 0$,*

$$\lim_{n\to\infty} \bar{\varrho}^{(1)}_{M,\Omega}((u_n - u)/h) = 0.$$

The following spaces of distributions will also be used:

$$W^{-1}L_{\bar{M}}(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) \,/\, f = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} \mathcal{D}^{\alpha} f_{\alpha} \text{ for some } f_{\alpha} \in L_{\bar{M}}(\Omega) \right\}$$

and

$$W^{-1}E_{\bar{M}}(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) \,/\, f = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} \mathcal{D}^{\alpha} f_{\alpha} \text{ for some} f_{\alpha} \in E_{\bar{M}}(\Omega) \right\}$$

Lemma 2.4 *If $P \ll M$ and $u_n \to u$ for the modular convergence in $L_M(\Omega)$, then $u_n \to u$ strongly in $E_P(\Omega)$.*

Proof. Let $\epsilon > 0$ be given. Let $\lambda > 0$ be such that

$$\int_{\Omega} M\Big(x, \frac{u_n - u}{\lambda}\Big) \to 0, \text{ as } n \to \infty.$$

Therefore, there exists $h \in L^1(\Omega)$ such that

$$M\Big(x, \frac{u_n - u}{\lambda}\Big) \leq h \text{ a. e. in } \Omega$$

for a subsequence still denoted (u_n).
Since $P \ll M$, then for all $r > 0$ there exists $t_0 > 0$ such that

$$\frac{P(x, rt)}{M(x,t)} \leq 1, \text{ a. e. in } \Omega \text{ and for all } t \geq t_0.$$

For $r = \frac{\lambda}{\epsilon}$ and $t = \frac{t'}{\lambda}$, we get

$$\frac{P(x, \frac{t'}{\epsilon})}{M(x, \frac{t'}{\lambda})} \leq 1, \text{ when } t' \geq t_0 \lambda.$$

Then

$$\begin{aligned} P\Big(x, \frac{u_n - u}{\epsilon}\Big) &\leq M\Big(x, \frac{u_n - u}{\lambda}\Big) + \sup_{t' \in B(0, t_0\lambda)} \operatorname*{ess\,sup}_{x \in \Omega} P(x, t'/\epsilon) \\ &\leq h + \sup_{t' \in B(0, t_0)} \operatorname*{ess\,sup}_{x \in \Omega} P(x, t'/\epsilon) \text{ for a. a. } x \in \Omega. \end{aligned}$$

Since $h + \sup_{t' \in B(0, t_0\lambda)} \operatorname*{ess\,sup}_{x \in \Omega} P(x, \frac{t'}{\epsilon}) \in L^1(\Omega)$ (from Remark 2.3), it yields, by the Lebesgue dominated convergence Theorem,

$$P\Big(x, \frac{u_n - u}{\epsilon}\Big) \to 0 \text{ in } L^1(\Omega),$$

hence, for n big enough, we have $||u_n - u||_{P,\Omega} \leq \epsilon$. That is, $u_n \to u$ in $L_P(\Omega)$.

□

Lemma 2.5 *([24], Lemma 2.2) Let $(w_n) \subset L_M(\Omega)$, $w \in L_M(\Omega)$, $(v_n) \subset L_{\bar{M}}(\Omega)$ and $v \in L_{\bar{M}}(\Omega)$. If $w_n \to w$ in $L_M(\Omega)$ for the modular convergence and $v_n \to v$ in $L_{\bar{M}}(\Omega)$ for the modular convergence, then*

$$\lim_{n\to\infty} \int_\Omega w_n v \,\mathrm{d}x = \int_\Omega wv \,\mathrm{d}x \text{ and } \lim_{n\to\infty} \int_\Omega w_n v_n \,\mathrm{d}x = \int_\Omega wv \,\mathrm{d}x.$$

Lemma 2.6 *(cf. [3], [6]) Let Ω be a bounded Lipschitz domain in $\mathbf{R}^N$ and let M and $\overline{M}$ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

i) There exists a constant $c > 0$ such that

$$\inf_{x\in\Omega} M(x,1) \geq c \tag{2.16}$$

ii) There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ we have

$$\frac{M(x,t)}{M(y,t)} \leq t^{\left(\frac{A}{\log\left(\frac{1}{|x-y|}\right)}\right)} \quad \text{for all } \ t > 1 \tag{2.17}$$

iii)

$$\int_\Omega M(x,1)dx < \infty \tag{2.18}$$

iv) There exists a constant $C > 0$ such that $\overline{M}(x,1) \leq C$ a.e in Ω.

Under these assumptions, $\mathcal{D}(\Omega)$ is dense in $L_M(\Omega)$ with respect to modular convergence, $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1 L_M(\Omega)$ for the modular convergence.

2.3 Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω be a bounded and open subset of $\mathbf{R}^d$ and let $Q_T = \Omega\times]0,T[$ with some given $T > 0$. Let M be a Musielak function. For each $\alpha = (\alpha_1, dots, \alpha_d) \in \mathbf{Z}^d$, $\alpha_j \geq 0$, $j = 1, \dots, d$, we denote by D_x^α the distributional derivative on Q_T of multiindex α with respect to the variable $x \in \mathbf{R}^d$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order one are defined as follows:

$$W^{1,x}L_M(Q_T) = \{u \in L_M(Q_T) \,/\, D_x^\alpha u \in L_M(Q_T) \text{ for all } \alpha,\ |\alpha| \leq 1\}$$

and

$$W^{1,x}E_M(Q_T) = \{u \in E_M(Q_T) \,/\, D_x^\alpha u \in E_M(Q_T) \text{ for all } \alpha,\ |\alpha| \leq 1\}$$

This last space is a subspace of the first one, and both are Banach spaces under the norm

$$||u|| = \sum_{|\alpha|\leq 1} ||D_x^\alpha u||_{M,Q_T}.$$

These spaces are considered as subspaces of the product space $\Pi L_M(Q_T)$ which has $(d+1)$ copies. We shall also consider the weak-$*$ topologies $\sigma(\Pi L_M(Q_T), \Pi E_{\overline{M}}(Q_T))$ and $\sigma(\Pi L_M(Q_T), \Pi L_{\overline{M}}(Q_T))$.
If $u \in W^{1,x}L_M(Q_T)$ then the function $t :\to u(t)$ is defined on $(0,T)$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q_T)$ then this function is a $W^1E_M(\Omega)$-valued and is strongly measurable. The space $W^{1,x}L_M(Q_T)$ is not in general separable. If $u \in W^{1,x}L_M(Q_T)$, we cannot conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \to ||u(t)||_{M,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure in $W^{1,x}E_M(Q_T)$ of $\mathcal{D}(Q)$. We can easily show as in [4] that when a Lipschitz domain then each element u of the closure of $\mathcal{D}(Q_T)$ with respect of the weak-$*$ topology $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ is limit, in $W^{1,x}L_M(Q_T)$, of some subsequence $(u_n) \subset D(Q_T)$ for the modular convergence; i. e., there exists $\lambda > 0$ such that for all α with $|\alpha| \leq 1$

$$\int_{Q_T} M(x, \frac{D_x^\alpha u_n - D_x^\alpha u}{\lambda}) \,\mathrm{d}x \,\mathrm{d}t \to 0 \ as\ n \ \to \infty,$$

and, in particular, this implies that (u_n) converges to u in $W^{1,x}L_M(Q_T)$ for the weak-$*$ topology $\sigma(\Pi L_M, \Pi L_{\bar{M}})$. Consequently

$$\overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi L_{\bar{M}})} = \overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi E_{\bar{M}})}.$$

This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore,

$$W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_{\bar{M}}(Q_T).$$

Poincaré's inequality also holds in $W_0^{1,x}L_M(Q_T)$, i. e. there exists a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q_T)$ one has

$$\sum_{|\alpha| \leq 1} \|D_x^\alpha u\|_{M,Q_T} \leq C \sum_{|\alpha|=1} \|D_x^\alpha u\|_{M,Q_T}. \tag{2.19}$$

The dual space of $W_0^{1,x}E_M(Q_T)$ will be denoted by $W^{-1,x}L_{\bar{M}}(Q_T)$, and it can be shown that

$$W^{-1,x}L_{\bar{M}}(Q_T) = \left\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha \,/\, f_\alpha \in L_{\bar{M}}(Q_T), \text{ for all } \alpha \right\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha| \leq 1} \|D_x^\alpha f_\alpha\|_{\bar{M},Q_T}$$

where the infimum is taken over all possible functions $f_\alpha \in L_{\bar{M}}(Q_T)$ from which the decomposition $f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha$ holds true.

We also denote by $W^{-1,x}E_{\bar{M}}(Q_T)$ the subspace of $W^{-1,x}L_{\bar{M}}(Q_T)$ consisting of those linear forms which are $\sigma(\Pi L_M, \Pi E_{\bar{M}})$-continuous. It can be shown that

$$W^{-1,x}E_{\bar{M}}(Q_T) = \left\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha \,/\, f_\alpha \in E_{\bar{M}}(Q_T) \right\}.$$

The following Lemma will be needed later on this paper.

Lemma 2.7 *Let P be a Musielak function such that (2.11) is verified. Assume that $s^2 \leq P(x,s)$, for all a. a. $x \in \Omega$ and all $s \in \mathbf{R}$. Then the following continuous inclusions hold true:*

$$L_P(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\bar{P}}(\Omega).$$

In particular, $W_0^1 L_P(\Omega) \hookrightarrow H_0^1(\Omega)$ and $H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\bar{P}}(\Omega)$. Furthermore, if M is a Musielak function verifying (2.10) and such that $P \ll M$, then the same continuous inclusions hold true for M, that is,

$$L_M(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\bar{M}}(\Omega),$$

and also $W_0^1 L_M(\Omega) \hookrightarrow H_0^1(\Omega)$ and $H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\bar{M}}(\Omega)$.

Proof. From the estimate on P we have

$$\int_\Omega v^2 \,\mathrm{d}x \leq \int_\Omega P(x,v)\,\mathrm{d}x, \text{ for all } v \in \mathcal{L}_P(\Omega). \tag{2.20}$$

Taking $v = u/\|u\|_{(P)}$ with $u \neq 0$ in (2.20) and using (2.5) it yields

$$\|u\|_{L^2(\Omega)} \leq \|u\|_{(P)} \text{ for all } u \in L_P(\Omega),$$

and the first assertions of this Lemma are readily deduced.

Now let $P \ll M$. For $\varepsilon \in (0,1)$ there exists t_0 that

$$P(x,t) \leq M(x,\varepsilon t) \text{ for all } t \geq t_0 \text{ and a. a. } x \in \Omega. \tag{2.21}$$

Then, taking $v \in \mathcal{L}_M(\Omega)$ and using the Remark 2.3, we deduce that for some constant $C_1 = C_1(t_0)$ it is

$$\begin{aligned} \int_\Omega v^2 \,\mathrm{d}x &\leq \int_{\{|v|<t_0\}} P(x,v)\,\mathrm{d}x + \int_{\{|v|\geq t_0\}} P(x,v)\,\mathrm{d}x \\ &\leq C_1 + \int_\Omega M(x,\varepsilon v)\,\mathrm{d}x \\ &\leq C_1 + \varepsilon \int_\Omega M(x,v)\,\mathrm{d}x. \end{aligned}$$

Making $v = u/\|u\|_{(M),Q_T}$, $u \neq 0$, in this last inequality and using (2.5) we finally deduce

$$\|u\|_{L^2(\Omega)} \leq C_3 \|u\|_{(M),Q_T} \text{ for all } u \in L_M(\Omega),$$

where $C_3 = (C_1 + \varepsilon)^{1/2}$.

□

Remark 2.4 Under the assumptions of Lemma 2.7, we have

$$L^2(0,T;H^{-1}(\Omega)) \hookrightarrow W^{-1,x}L_{\bar{P}}(Q_T) \hookrightarrow W^{-1,x}E_{\bar{M}}(Q_T).$$

Indeed, let $f \in L^2(0,T;H^{-1}(\Omega))$. Then, for some $f_\alpha \in L^2(Q_T)$, $f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha$. Since $L^2(Q_T) \subset L_{\bar{P}}(Q_T) \subset E_{\bar{M}}(Q_T)$ we deduce that $f \in W^{-1,x}L_{\bar{P}}(Q_T) \hookrightarrow W^{-1,x}E_{\bar{M}}(Q_T)$.

Finally, we will make use of truncations in the definition of our approximate problems. To do so, for $K > 0$, we introduce the truncation at height K, denoted by $T_K \colon \mathbf{R} \mapsto \mathbf{R}$, as

$$T_K(s) = \min(K, \max(s, -K)) = \begin{cases} s & \text{if } |s| \leq K, \\ Ks/|s| & \text{if } |s| > K, \end{cases} \tag{2.22}$$

3 Compactness results

In the sequel, we will make use of the following results which concern mollification with respect to time and space variables and some trace results. Also, unless stated the contrary, $\Omega \subset \mathbf{R}^d$ is a bounded and open set with a Lipschitz continuous boundary, and M is Musielak function. We put $Q_T = \Omega \times (0,T)$. For a function $u \in L^1(Q_T)$ we introduce the function $\tilde{u} \in L^1(\Omega \times \mathbf{R})$ as $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}$ and define, for all $\mu > 0$, $t \in [0,T]$ and a.e. $x \in \Omega$, the function u_μ given as follows

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \tilde{u}(x,s)\exp(\mu(s-t))\,\mathrm{d}s. \tag{3.1}$$

Lemma 3.1 *([1]).*

1. *Let $u \in L_M(Q_T)$. Then $u_\mu \in C([0,T];L_M(\Omega))$ and $u_\mu \to u$ as $\mu \to +\infty$ in $L_M(Q_T)$ for the modular convergence.*
2. *Let $u \in W^{1,x}L_M(Q_T)$. Then $u_\mu \in C([0,T];W^1L_M(\Omega))$ and $u_\mu \to u$ as $\mu \to +\infty$ in $W^{1,x}L_M(Q_T)$ for the modular convergence.*
3. *Let $u \in E_M(Q_T)$ (respectively, $u \in W^{1,x}E_M(Q_T)$). Then $u_\mu \to u$ as $\mu \to +\infty$ strongly in $E_M(Q_T)$ (respectively, strongly in $W^{1,x}E_M(Q_T)$).*
4. *Let $u \in W^{1,x}L_M(Q_T)$ then $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu) \in W^{1,x}L_M(Q_T)$.*

5. *Let $(u_n) \subset W^{1,x}L_M(Q_T)$ and $u \in W^{1,x}L_M(Q_T)$ such that $u_n \to u$ strongly in $W^{1,x}L_M(Q_T)$ (respectively, for the modular convergence). Then, for all $\mu > 0$, $(u_n)_\mu \to u_\mu$ strongly in $W^{1,x}L_M(Q_T)$ (respectively, for the modular convergence).*

Lemma 3.2 *(cf. [13]). Suppose M be a generalized N-function. Let (u_n) be a sequence of $W^{1,x}L_M(Q_T)$ which verifies $u_n \rightharpoonup u$ weakly in $W^{1,x}L_M(Q_T)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\frac{\partial u_n}{\partial t} = h_n + k_n$ in $D'(Q_T)$ such that h_n is bounded in $W^{-1,x}L_{\overline{M}}(Q_T)$ and k_n is bounded in $L^1(Q_T)$. Then, $u_n \to u$ strongly in $L^1_{Loc}(Q_T)$. Moreover, if $u_n \in W_0^{1,x}L_M(Q_T)$ then u_n converges strongly to u in $L^1(Q_T)$.*

We're going to use the next inequality quite a few times.

Lemma 3.3 *([15]) Suppose Ω be a bounded open subset of $\mathbf{R}^N$ and M a generalized N-function so there exists two positive constants δ and λ*

$$\int_\Omega M(x, \delta|v|)dx \leq \int_\Omega \lambda M(x, |\nabla v|)dx \text{ for all } v \in W_0^1 L_M(\Omega). \tag{3.2}$$

Lemma 3.4 *([1]) Let Y be a Banach space such that the following continuous imbedding holds true $L^1(\Omega) \subset Y$. Then, for all $\epsilon > 0$ and all $\lambda > 0$ there exists a constant C_ϵ such that for all $u \in W^{1,x}L_M(Q_T)$ with $|\nabla u|/\lambda \in L_M(Q_T)$ one has*

$$\|u\|_{L^1(Q_T)} \leq \epsilon\lambda \left(\int_{Q_T} M(x, |\nabla u|/\lambda)\, \mathrm{d}x\mathrm{d}t + T \right) + C_\epsilon \|u\|_{L^1(0,T;Y)} \tag{3.3}$$

Moreover, if F is bounded in $W_0^{1,x}L_M(Q_T)$ and is relatively compact in $L^1(0,T;Y)$ then F is relatively compact in $L^1(Q_T)$.

Lemma 3.5 *The following embedding holds true with continuous injection*

$$E_M(Q_T) \subseteq L^1(0,T; E_M(\Omega)) \tag{3.4}$$

Proof. Since $M(x,t)$ is convex with respect to t, then for every $\alpha \geq 1$, $t \in [0,T]$ and a. a. $x \in \Omega$ we have

$$\alpha M(x,t) \leq M(x, \alpha t) \text{ and } \alpha M(x, t/\alpha) \leq M(x,t). \tag{3.5}$$

Let $u \in E_M(Q_T)\backslash\{0\}$. Owing to the definition of the space $E_M(Q_T)$, we have $\int_{Q_T} M(x, \lambda u(x,t))\, \mathrm{d}x\, \mathrm{d}t < \infty$ for every $\lambda \geq 0$. Hence, $\int_\Omega M(x, \lambda u(x,t))\, \mathrm{d}x < \infty$ for a. a. $t \in [0,T]$ and for all $\lambda \geq 0$. Therefore the function $u(\cdot,t) \in E_M(\Omega)$ for a. a. $t \in [0,T]$. In particular,

$$\int_\Omega M(x, \frac{u(x,t)}{||u(\cdot,t)||_{M,\Omega}})\, \mathrm{d}x = 1 \text{ for a. a. } t \in [0,T].$$

Then, having in mind (3.5),

$$\begin{aligned}
\int_0^T ||u||_{M,\Omega}\,\mathrm{d}t &= \int_{\{||u(\cdot,t)||_{M,\Omega}<1\}} ||u(\cdot,t)||_{M,\Omega}\,\mathrm{d}t + \int_{\{||u(\cdot,t)||_{M}\geq 1\}} ||u(\cdot,t)||_{M,\Omega}\,\mathrm{d}t \\
&\leq T + \int_{\{||u(\cdot,t)||_{M,\Omega}\geq 1\}} ||u(\cdot,t)||_{M,\Omega} \int_\Omega M\left(x, \frac{u(x,t)}{||u(\cdot,t)||_{M,\Omega}}\right)\mathrm{d}x\mathrm{d}t \\
&\leq T + \int_{\{||u(\cdot,t)||_{M,\Omega}\geq 1\}} \int_\Omega ||u(\cdot,t)||_{M,\Omega} M\left(x, \frac{u(x,t)}{||u(\cdot,t)||_{M,\Omega}}\right)\mathrm{d}x\mathrm{d}t \\
&\leq T + \int_{\{||u(\cdot,t)||_{M,\Omega}\geq 1\}} \int_\Omega M(x,u(x,t))\,\mathrm{d}x\mathrm{d}t \\
&\leq T + \int_{Q_T} M(x,u(x,t))\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By taking $u/||u||_{M,Q_T}$ instead of u into the first and last terms of this inequality, using (2.5) and (2.6), it follows that $\int_0^T ||u||_{M,\Omega}\,\mathrm{d}t \leq 2(T+1)||u||_{M,Q_T}$.

□

A straightforward consequence of Lemma 3.5 is given in the next result.

Lemma 3.6 *The following embeddings hold true with continuous injections*

$$W^1E_M(Q_T) \subset L^1\left(0,T;W^1E_M(\Omega)\right), \tag{3.6}$$

$$W^{-1}E_{\bar{M}}(Q_T) \subset L^1\left(0,T;W^{-1}E_{\bar{M}}(\Omega)\right). \tag{3.7}$$

Let B be a Banach space and $0 < h < T$. For a function $f \in L^1(0,T;B)$ we introduce the function $\tau_h f \in L^1(0,T;B)$ given by

$$\tau_h f(t) = \begin{cases} f(t+h) & \text{if } t \in (0,T-h), \\ 0 & \text{if } t \in [T-h,T). \end{cases}$$

The next Lemma is a very well-known result.

Lemma 3.7 *([32]) Let $f \in \mathcal{D}'(]0,T[;B)$ such that $\frac{\partial f}{\partial t} \in L^1(0,T;B)$ then $f \in C(]0,T[;B)$ and for all $h > 0$ we have*

$$||\tau_h f - f||_{L^1(0,T;B)} \leq h||\frac{\partial f}{\partial t}||_{L^1(0,T;B)}.$$

Remark 3.1 By the Lemma 3.7, if $F \subset L^1(0,T;B)$ such that $\left\{\frac{\partial f}{\partial t} \,/\, f \in F\right\}$ is bounded in $L^1(0,T;B)$ then $||\tau_h f - f||_{L^1(0,T;B)} \to 0$ as $h \to 0$ uniformly with respect to $f \in F$.

Lemma 3.8 *(see [1]). Let Ω be a bounded open subset of $\mathbf{R}^d$ with the segment property. Consider the Banach space*

$$W = \Big\{ u \in W_0^{1,x} L_M(Q_T) \,/\, \frac{\partial u}{\partial t} \in W^{-1,x} L_{\bar{M}}(Q_T) + L^1(Q_T) \Big\}.$$

Then the embedding $W \subset C([0,T]; L^1(\Omega))$ holds true and is continuous.

Lemma 3.9 *(see [1]). Let Y be a Banach space such that $L^1(\Omega) \subset Y$ with continuous embedding. If $\mathcal{F}$ is bounded in $W_0^{1,x} L_M(Q_T)$ and relatively compact in $L^1(0,T;Y)$ then $\mathcal{F}$ is relatively compact in $E_P(Q_T)$ for all $P \ll M$.*

Lemma 3.10 *(see [1]). If $\mathcal{F}$ is bounded in $W_0^{1,x} L_M(Q_T)$ and $\Big\{ \frac{\partial f}{\partial t} \,/\, f \in \mathcal{F} \Big\}$ is bounded in $W^{-1,x} L_{\bar{M}}(Q_T)$ then $\mathcal{F}$ is relatively compact in $L^1(Q_T)$.*

4 Statement of the problem

Let Ω be a bounded open set on $\mathbf{R}^N$ $(N \geq 2)$, $T > 0$ is given real number and we set $Q_T = \Omega \times (0,T)$. We are interested in existence results for the unilateral problem associated to the following one

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\Big(a(x,t,u,\nabla u)\Big) - \operatorname{div}(\Phi(x,t,u))) = f \;\text{ in } Q_T = \Omega \times (0,T) \\ u(x,0) = u_0(x) \;\text{ in } \Omega \\ u = 0 \;\text{ on } \partial\Omega \times (0,T), \end{cases} \tag{4.1}$$

where f belongs to $L^1(Q_T)$.

Throughout this paper $<,>$ means for either the pairing between $W_0^{1,x} L_M(Q_T) \cap L^\infty(Q_T)$ and $W^{-1,x} L_{\overline{M}}(Q_T) + L^1(Q_T)$ or between $W_0^{1,x} L_M(Q_T)$ and $W^{-1,x} L_{\overline{M}}(Q_T)$ and $Q_\tau = \Omega \times (0,\tau)$ for $\tau \in [0,T]$ and we assume that the following assumptions hold true:

Let M, P and Q be three N-functions such that $P \ll M \ll Q$.
Given an obstacle function $\psi : Q_T \to \overline{\mathbf{R}}$, we consider

$$K_\psi = \{ u \in W_0^{1,x} L_M(Q_T);\; u \geq \psi \;\; a.e. \text{ in } \; Q_T \}, \tag{4.2}$$

this convex set is sequentially $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closed in $W_0^{1,x} L_M(Q_T)$.
Consider a second order partial differential operator $A : D(A) \subset W^{1,x} L_M(Q_T) \to W^{-1,x} L_{\overline{M}}(Q_T)$ in divergence form

$$A(u) = -\operatorname{div}\Big(a(x,t,u,\nabla u)\Big).$$

Throughout this paper, we assume the following assumptions:

$$a : \Omega \times \mathbf{R} \times \mathbf{R}^N \;\to \mathbf{R}^N \text{is a Carathéodory function satisfying} \tag{4.3}$$

for almost every $(x,t)\in Q_T$ and all $s\in \mathbf{R}$, $\xi\neq\xi\in\mathbf{R}^N$

$$|a(x,t,s,\xi)|\leq \beta(c_1(x,t)+\overline{M}_x^{-1}P(x,k_1|s|+\overline{M}_x^{-1}M(x,k_1|\xi|) \tag{4.4}$$

$$[a(x,t,s,\ \xi)-a(x,t,s,\ \xi^{'})][\xi-\xi^{'}]\ >0 \tag{4.5}$$

$$a(x,t,s,\ \xi)\xi\geq\alpha[M(x,|s|)+M(x,|\xi|)] \tag{4.6}$$

with $c_1(x,t)\in E_{\overline{M}}(Q_T)$, $c(x,t)\geq 0$ and $\alpha,\beta,k>0$.

Additionally, we assume that

(B_1) For each $v\in K_\psi\cap L^\infty(Q_T)$ there exists a sequence $v_n\in K_\psi\cap W_0^1E_M(Q_T)\cap L^\infty(Q_T)$ such that $v_n\to v$ for the modular convergence.

(B_2) $K_\psi\cap L^\infty(Q_T)\neq\emptyset$.

(B_3) Φ is a Carathéodory function satisfies the following growth condition for a.e. $(x,t)\in Q_T$ and for all $s\in\mathbf{R}$,

$$|\Phi(x,t,s)|\leq\gamma(x,t)\overline{M}_x^{-1}M(x,|s|),$$

with $\gamma\in L^\infty(Q_T)$.

(B_4) f, u_0 and $\frac{\partial v_0}{\partial t}$ belong to $L^1(Q_T)$.

For $k>0$, we define the truncation function $T_k:\mathbf{R}\to\mathbf{R}$ and it primitive S_k by

$$T_k(s)=\min(k,\max(s,-k))=\begin{cases} s & \text{if}\quad |s|\leq k\\ \frac{ks}{|s|} & \text{if}\quad |s|>k\end{cases}, \tag{4.7}$$

$$S_k(t)=\int_0^t T_k(s)ds=\begin{cases}\frac{t^2}{2} & \text{if}\quad |t|\leq k\\ k|t|-\frac{k^2}{2} & \text{if}\quad |t|\geq k\end{cases}, \tag{4.8}$$

we can deduce from (4.7) and (4.8) that $S_k(t)\geq 0$ and $S_k(t)\leq k|t|$.

Let introduce the following important Lemma.

Lemma 4.1 *Let assumptions* (A_1)*-*(A_4) *be satisfied, and let* (z_n) *be a sequence in* $W_0^{1,x}L_M(Q_T)$ *such that,*

$$z_n\rightharpoonup z \text{ in } W_0^{1,x}L_M(Q_T) \text{ for } \sigma(\Pi L_M(Q_T),\Pi E_{\overline{M}}(Q_T)), \tag{4.9}$$

$$(a(x,t,z_n,\nabla z_n))_n \text{ is bounded in } (L_{\overline{M}}(Q_T))^N, \tag{4.10}$$

$$\int_{Q_T}\Big[a(x,t,z_n,\nabla z_n)-a(x,t,z_n,\nabla z\chi_s)\Big]\Big[\nabla z_n-\nabla z\chi_s\Big]\,dx\,dt\longrightarrow 0, \tag{4.11}$$

as n *and* s *tend to* $+\infty$*, and where* χ_s *is the characteristic function of*

$$Q_s=\Big\{(x,t)\in Q\ ;\ |\nabla z|\leq s\Big\}.$$

Then,

$$\nabla z_n\to\nabla z \quad a.e.\ in\ Q. \tag{4.12}$$

$$\lim_{n\to\infty}\int_{Q_T}a(x,t,z_n,\nabla z_n)\nabla z_n\,dx\,dt=\int_{Q_T}a(x,t,z,\nabla z)\nabla z\,dx\,dt, \tag{4.13}$$

$$M(|\nabla z_n|)\to M(|\nabla z|)\quad in\quad L^1(Q_T), \tag{4.14}$$

Proof. Using the same argument in [5] we get the result.

□

5 Solution of unilateral problem

Our goal now is to show the following existence theorem.

Theorem 5.1 *Let assumptions (A_1)-(A_4) and $(B_1)-(B_4)$ be satisfied, then there exists at least one solution of the problem (4.1) in the sense of Definition 1.1.*

5.1 Uniform estimates. Part I

Let us introduce the following regularization of the data,

$$a_n(x,t,r,\xi) = a(x,t,T_n(r),\xi) \quad \text{a.e} \quad (x,t) \in Q_T, \forall r \in \mathbf{R}, \forall \xi \in \mathbf{R}^N, \tag{5.1}$$

$$\Phi_n(x,t,r) = \Phi(x,t,T_n(r)) \quad \text{a.e } (x,t) \in Q_T, \forall r \in \mathbf{R}, \tag{5.2}$$

$$f_n \in C_0^\infty(Q_T) \ : \ \|f_n\|_{L^1} \leq \|f\|_{L^1} \text{ and } f_n \longrightarrow f \text{ in } L^1(Q_T) \ , \tag{5.3}$$

as n tends to $+\infty$.

$$u_{n0} \in C_0^\infty(\Omega) \ : \ \|u_{n0}\|_{L^1} \leq \|u_0\|_{L^1} \text{ and } \ u_{n0} \longrightarrow u_0 \text{ in } L^1(\Omega), \tag{5.4}$$

as n tends to $+\infty$.

Let us now consider the following regularized problem

$$\begin{cases} u_n \in K_\psi, \\ \dfrac{\partial u_n}{\partial t} - div\Big(a(x,t,u_n,\nabla u_n)\Big) - div\Big(\Phi_n(x,t,u_n)\Big) = f_n \ \text{ in } Q_T, \\ u_n = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ u_n(x,t=0) = u_{n0} \ \text{ in } \Omega. \end{cases} \tag{5.5}$$

The problem (5.5) can be written as follows

$$\begin{cases} \frac{\partial u_n}{\partial t} - \mathrm{div}\Big(F_n(x,t,u_n,\nabla u_n)\Big) = f_n \ \text{ in } Q_T \\ u_n(x,0) = u_{0n}(x) \qquad \text{in } \Omega \\ u_n = 0 \qquad \text{on } \ \partial\Omega \times (0,T), \end{cases} \tag{5.6}$$

with $F_n(x,t,u_n,\nabla u_n) = a(x,t,u_n,\nabla u_n) + (\Phi(x,t,u_n)$.
Note that F_n satisfies the assumptions (A_1), (A_2) and (A_3) as in [23].
Indeed, using (4.4), (4.5) and (B_3) we deduce that F_n satisfies (A_1), (A_2), it remains to prove (A_3). Let $u_n \in W_0^{1,x}L_M(Q_T)$ by (B_3) and Young inequality and the fact that $\overline{M}_x^{-1}M$ is an increasing function, we obtain

$$\begin{aligned}|\Phi(x,t,u_n)\nabla u_n| &\leq |\gamma(x,t)|\overline{M}_x^{-1}M(x,|T_k(u_n)|)|\nabla u_n| \\ &\leq C_\gamma\Big(Q(x,1)+\overline{Q}_x(\overline{M}_x^{-1}M(x,k))\Big)|\nabla u_n|\end{aligned} \tag{5.7}$$

While $M \ll Q \Leftrightarrow \overline{Q} \ll \overline{M}$, we have, for all $\epsilon > 0$ there exists a constant d_ϵ depending on ϵ such that,

$$\overline{Q}(x,t) \leq \overline{M}(x,\epsilon t) + d_\epsilon, \quad \forall t \geq 0. \tag{5.8}$$

Then, for ϵ small enough, we have

$$|\Phi(x,t,u_n)\nabla u_n| \leq C_\gamma\Big(Q(x,1)+\overline{M}_x(\epsilon\overline{M}_x^{-1}M(x,k))+d_\epsilon\Big)|\nabla u_n|. \tag{5.9}$$

We can assume without losing generality that $\epsilon = \frac{1}{1+R_1} \leq 1$, by way of $M(x,k) \leq \sup\limits_{t\in B(0,k)} ess\sup\limits_{x\in\Omega} M(x,t) < R_1$, and since $\overline{M}$ is convex, we have

$$\begin{aligned}|\Phi_n(x,t,u_n)\nabla u_n| \leq C_\gamma\Big(&\sup_{t\in B(0,1)} ess\sup_{x\in\Omega} Q(x,t)+d_\epsilon \\ &+\frac{\sup\limits_{t\in B(0,k)} ess\sup\limits_{x\in\Omega} M(x,t)}{1+R_1}\Big)|\nabla u_n|,\end{aligned} \tag{5.10}$$

while $\xi(x,1) \leq \sup\limits_{t\in B(0,1)} ess\sup\limits_{x\in\Omega}\xi(x,t) < R_3$, then

$$\begin{aligned}|\Phi_n(x,t,u_n)\nabla u_n| &\leq qC_\gamma|\nabla u_n| \\ &\leq C_\gamma\overline{P}\Big(x,q\Big)+C_\gamma P\Big(x,|\nabla u_n|\Big),\end{aligned} \tag{5.11}$$

Recall that by $P \ll M$, we have for all $\epsilon' > 0$ there exists t_0 that

$$P(x,t) \leq M(x,\epsilon' t) + d_{\epsilon'} \text{ for } \text{ all } t \geq 0,\ a.e.x \in \Omega. \tag{5.12}$$

Then

$$|\Phi_n(x,t,u_n)\nabla u_n| \leq C_\gamma\overline{P}\Big(x,q\Big)+d_{\epsilon'}+C_\gamma M\Big(x,\epsilon'|\nabla u_n|\Big), \tag{5.13}$$

We can assume again that $\epsilon' = \frac{\alpha}{2C_\gamma+\alpha}$ which is $\epsilon' \leq 1$, then by convexity of the function $M(x,.)$, one has

$$\begin{aligned}|\Phi_n(x,t,u_n)\nabla u_n| &\leq C_\gamma\overline{P}\Big(x,q\Big)+d_{\epsilon'}+C_\gamma\epsilon' M\Big(x,|\nabla u_n|\Big), \\ &\leq \mathcal{G}+\frac{\alpha}{2}M\Big(x,|\nabla u_n|\Big),\end{aligned} \tag{5.14}$$

then,

$$\Phi_n(x,t,u_n)\nabla u_n \ \geq -\mathcal{G}-\tfrac{\alpha}{2}M(x,|\nabla u_n|)\ . \tag{5.15}$$

Using (5.15) and by (4.6), it is simple to observe that

$$F_n(x,t,u_n,\nabla u_n)\nabla u_n \geq \alpha M(x,|\nabla u_n|) - \text{ fixed } L^1 \text{ function.}$$

Finally, the assumption (A_3) in ([23]) holds.
Then there exists at least one solution u_n of (5.5), (the existence of u_n can be obtained from Galerkin solutions corresponding to the equation (5.5) as in [23], see Theorem 1 of [1] for more details).

The first step in the proof of the Theorem 5.1 is to obtain uniform bounds with respect to n.

Proposition 5.2 *Let assumptions (A_1)-(A_4) and $(B_1)-(B_4)$ be satisfied and let u_n be a solution of the approximate problem (5.5). Then for all $k>0$, we have*

$$\|T_k(u_n)\|_{W_0^{1,x}L_M(Q_\tau)} \leq C(k), \tag{5.16}$$

where C is a constant independently of n, and

$$\lim_{k\to\infty} meas\Big\{(x,t)\in Q_T : |u_n|>k\Big\} = 0 \tag{5.17}$$

uniformly with respect to n.

Proof

First, let see that we have from (5.5) for all $v\in K_\psi(Q_\tau)$ that

$$\begin{aligned}&\langle \tfrac{\partial u_n}{\partial t},(u_n-v)\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla(u_n-v)\,dx\,dt\\ &+\int_{Q_\tau}\Phi_n(x,t,u_n)\nabla(u_n-v)\,dx\,dt \leq \int_{Q_\tau} f(u_n-v)\,dx\,dt.\end{aligned}\tag{5.18}$$

By (B_1) and (B_2), there exists $v_0\in K_\psi\cap L^\infty(\Omega)\cap W_0^1E_M(Q_\tau)$.

Let take $v=u_n-T_k(u_n-v_0)$ as a test function in (5.18) (for more explication concerned this test function see the appendix), we have

$$\begin{aligned}\langle\frac{\partial u_n}{\partial t},T_k(u_n-v_0)\rangle_{Q_\tau} &+ \int_{Q_\tau} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n-v_0)\,dx\,dt\\ &+\int_{Q_\tau}\Phi_n(x,t,u_n)\nabla T_k(u_n-v_0)\,dx\,dt\\ &\leq \int_{Q_\tau} fT_k(u_n-v_0)\,dx\,dt.\end{aligned}\tag{5.19}$$

Then, we get

$$\begin{aligned}\int_{\Omega} S_k(u_n - v_0)(\tau)dx + \langle\frac{\partial v_0}{\partial t}, T_k(u_n - v_0)\rangle_{Q_\tau} \\ + \int_{Q_\tau} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n - v_0)\,dx\,dt \\ + \int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n - v_0)\,dx\,dt \\ \le \int_{Q_\tau} fT_k(u_n - v_0)\,dx\,dt + \int_{\Omega} S_k(u_{n0} - v_0(0))dx.\end{aligned} \tag{5.20}$$

We have

$$S_k(u_n(\tau) - v_0(\tau)) \ge 0,$$

and

$$\int_{\Omega} S_k(u_{n0} - v_0(0))dx \le k\int_{\Omega} |u_{n0} - v_0(0)|\,dx \le k||u_{n0}||_{L^1(\Omega)} + kc = kc'_1.$$

Moreover,

$$\int_{Q_T} f_nT_k(u_n - v_0)\,dx\,dt \le k\|f\|_{L^1(Q_T)} \le kc'_2. \tag{5.21}$$

And by (B_6) we get

$$\langle\frac{\partial v_0}{\partial t}, T_k(u_n - v_0)\rangle_{Q_\tau} \le kc'_3.$$

On the other hand,

$$\begin{aligned}\int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n - v_0)\,dx\,dt \\ \le \int_{Q_\tau} \gamma(x,t)\overline{M}_x^{-1}M(x,T_n(u_n))\nabla T_k(u_n - v_0)\,dx\,dt. \\ \le \int_{\{0\le|u_n-v_0|\le k\}} c_\gamma\overline{M}_x^{-1}M(x,k+||v_0||_\infty)|\nabla(u_n - v_0)|\,dx\,dt. \\ \le \int_{\{0\le|u_n-v_0|\le k\}} c_\gamma\overline{M}_x^{-1}M(x,k+||v_0||_\infty)|\nabla u_n|\,dx\,dt. \\ + \int_{\{0\le|u_n-v_0|\le k\}} c_\gamma\overline{M}_x^{-1}M(x,k+||v_0||_\infty)|\nabla v_0|\,dx\,dt.\end{aligned} \tag{5.22}$$

By Young's inequality and $\nabla v_0 \in (E_M(Q_T))^N$, we get

$$\begin{aligned}\int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n - v_0)\,dx\,dt \\ \le c'_4(k) + \tfrac{\alpha(1-c)}{4}\int_{\{0\le|u_n-v_0|\le k\}} M(x,|\nabla u_n|)\,dx\,dt,\end{aligned} \tag{5.23}$$

with $0 < c < 1$.

Hence

$$\begin{aligned}\int_{\{|u_n-v_0|\le k\}} & a(x,t,u_n,\nabla u_n)\nabla u_n \, dx\, dt \\ & \le c\int_{\{|u_n-v_0|\le k\}} a(x,t,u_n,\nabla u_n)\frac{\nabla v_0}{c} \, dx\, dt \\ & +c_5'(k) + \tfrac{\alpha(1-c)}{4}\int_{\{0\le|u_n-v_0|\le k\}} M(x,|\nabla u_n|) \, dx\, dt.\end{aligned} \tag{5.24}$$

By using (A_3), we have

$$\begin{aligned}c\int_{\{0\le|u_n-v_0|\le k\}} & a(x,t,u_n,\nabla u_n)\frac{\nabla v_0}{c} \, dx\, dt \\ & \le c\Bigg\{\int_{\{0\le|u_n-v_0|\le k\}} a(x,t,u_n,\nabla u_n)\nabla u_n \, dx\, dt \\ & -\int_{\{0\le|u_n-v_0|\le k\}} a(x,t,u_n,\frac{\nabla v_0}{c})(\nabla u_n-\frac{\nabla v_0}{c}) \, dx\, dt\Bigg\},\end{aligned} \tag{5.25}$$

hence, (5.24) becomes

$$\begin{aligned}(1-c)\int_{\{0\le|u_n-v_0|\le k\}} & a(x,t,u_n,\nabla u_n)\nabla u_n \, dx\, dt \\ & \le c_5'(k) + \tfrac{\alpha(1-c)}{4}\int_{\{0\le|u_n-v_0|\le k\}} M(x,|\nabla u_n|) \, dx\, dt \\ & +\int_{\{0\le|u_n-v_0|\le k\}} |a(x,t,u_n,\frac{\nabla v_0}{c})||\frac{\nabla v_0}{c}| \, dx\, dt \\ & +\int_{\{0\le|u_n-v_0|\le k\}} |a(x,t,u_n,\frac{\nabla v_0}{c})||\nabla u_n| \, dx\, dt.\end{aligned} \tag{5.26}$$

While $\frac{\nabla v_0}{c} \in (E_M(Q_T))^N$, (A_2) and Young's inequality, we get

$$\begin{aligned}(1-c)\int_{\{0\le|u_n-v_0|\le k\}} & a(x,t,u_n,\nabla u_n)\nabla u_n \, dx\, dt \\ & \le c_6'(k) + \tfrac{\alpha(1-c)}{4}\int_{\{0\le|u_n-v_0|\le k\}} M(x,|\nabla u_n|) \, dx\, dt \\ & +\tfrac{\alpha(1-c)}{4}\int_{\{0\le|u_n-v_0|\le k\}} M(x,|\nabla u_n|) \, dx\, dt.\end{aligned} \tag{5.27}$$

Hence, by (A_4), one has

$$\alpha(1-c)\int_{\{0\le|u_n-v_0|\le k\}} M(x,|\nabla u_n|)\,dx\,dt \le c_6'(k)+\tfrac{\alpha(1-c)}{2}\int_{\{0\le|u_n-v_0|\le k\}} M(x,|\nabla u_n|)\,dx\,dt. \tag{5.28}$$

Then,

$$\tfrac{\alpha(1-c)}{2}\int_{\{0\le|u_n-v_0|\le k\}} M(x,|\nabla u_n|)\,dx\,dt \le c_6'(k). \tag{5.29}$$

Which implies

$$\int_{\{|u_n-v_0|\le k\}} M(x,|\nabla u_n|)\,dx \le c_7'(k). \tag{5.30}$$

Since $\{x\in\Omega; |u_n(x)|\le k\}\subset\{x\in\Omega; |u_n-v_0|\le k+\|v_0\|_\infty\}$,

$$\begin{aligned}\int_\Omega M(x,|\nabla T_k(u_n)|)\,dx &= \int_{\{|u_n|\le k\}} M(x,|\nabla u_n|)\,dx\\ &\le \int_{\{|u_n-v_0|\le k+\|v_0\|_\infty\}} M(x,|\nabla u_n|)\,dx,\end{aligned}$$

then by (5.30), we conclude (5.16).

Lemma 5.3 *Let assumptions (A_1)-(A_4) and $(B_1)-(B_4)$ be satisfied and let u_n be a solution of the approximate problem (5.5). Then for all $k>0$, there exist two constants C, c (which don't depend on the n and k) such that*

$$\int_{Q_\tau} M(x,|\nabla T_k(u_n-T_h(u_n))|)\,dx \le C+ck. \tag{5.31}$$

Proof. Let take $v=u_n-T_k(u_n-T_h(u_n))$ where $k>h\ge||v_0||_\infty$ as a test function in (5.18), then we have,

$$\begin{aligned}&\langle\frac{\partial u_n}{\partial t}, T_k(u_n-T_h(u_n))\rangle_{Q_\tau}\\ &\qquad+\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n-T_h(u_n))\,dx\,dt\\ &\qquad+\int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n-T_h(u_n))\,dx\,dt\\ &\qquad\le\int_{Q_\tau} fT_k(u_n-T_h(u_n))\,dx\,dt.\end{aligned} \tag{5.32}$$

Hence

$$\begin{aligned}\int_\Omega S_k(u_n - T_h(u_n))(\tau)dx &+ \langle \frac{\partial T_h(u_n)}{\partial t}, T_k(u_n - T_h(u_n))\rangle_{Q_\tau} \\ &+ \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - T_h(u_n))\, dx\, dt \\ &+ \int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n - T_h(u_n))\, dx\, dt \\ &\le \int_{Q_\tau} fT_k(u_n - T_h(u_n))\, dx\, dt + \int_\Omega S_k(u_{n0} - T_h(u_n)(0)dx.\end{aligned} \tag{5.33}$$

We have $S_k(u_n - T_h(u_n))(\tau) \ge 0$ and

$$\int_\Omega S_k(u_{n0} - T_h(u_n)(0)dx \le k(c_7' + h). \tag{5.34}$$

Moreover,

$$\int_{Q_\tau} fT_k(u_n - T_h(u_n))\, dx\, dt \le c_8' k. \tag{5.35}$$

Using (B_3) we get

$$\begin{aligned}\int_{Q_\tau} &\Phi_n(x,t,u_n)\nabla T_k(u_n - T_h(u_n))\, dx\, dt \\ &\le \int_{\{h\le|u_n|\le h+k\}} \gamma(x,t)\cdot \overline{M}_x^{-1} M(x,|u_n|)|\nabla u_n|\, dx\, dt.\end{aligned} \tag{5.36}$$

On the other hand by (5.11) we have

$$\begin{aligned}\int_{Q_\tau} &\Phi_n(x,t,u_n)\nabla T_k(u_n - T_h(u_n))\, dx\, dt \\ &\le \int_{\{h\le|u_n|\le h+k\}} C_\gamma \overline{P}\Big(x,q\Big) + C_\gamma P\Big(x,|\nabla u_n|\Big)\, dx\, dt,\end{aligned} \tag{5.37}$$

Applying (5.12), we obtain

$$\begin{aligned}\int_{\{h\le|u_n|\le h+k\}} &\Phi_n(x,t,u_n)\nabla u_n\, dx\, dt \\ &\le d_\epsilon + C_\gamma \int_{\{h\le|u_n|\le h+k\}} \overline{P}\Big(x,q\Big)\, dx\, dt \\ &+C_\gamma \int_{\{h\le|u_n|\le h+k\}} M(x,\epsilon'|\nabla u_n|)\, dx\, dt,\end{aligned} \tag{5.38}$$

for $\epsilon' = \frac{\alpha}{2C_\gamma+\alpha}$ which is $\epsilon' \leq 1$, and the convexity of $M(x,.)$, we get

$$\begin{aligned}\int_{\{h\leq|u_n|\leq h+k\}} &\Phi_n(x,t,u_n)\nabla u_n \, dx \, dt \\ &\leq d_\epsilon + c'_9 k + C_\gamma \epsilon' \int_{\{h\leq|u_n|\leq h+k\}} M(x,|\nabla u_n|)\, dx\, dt.\end{aligned} \tag{5.39}$$

From above, we get

$$\begin{aligned}\int_{\{h\leq|u_n|\leq h+k\}} &a(x,t,u_n,\nabla u_n)\nabla u_n \, dx \, dt \\ &\leq d_\epsilon + c'_{10} k + \tfrac{\alpha C_\gamma}{2C_\gamma+\alpha} \int_{\{h\leq|u_n|\leq h+k\}} M(x,|\nabla u_n|)\, dx\, dt,\end{aligned} \tag{5.40}$$

with $c'_{10} = (c'_7 + h) + c'_8 + c'_9$.
By using (A_4), we obtain

$$\left(\alpha - \tfrac{\alpha C_\gamma}{2C_\gamma+\alpha}\right) \int_{\{h\leq|u_n|\leq h+k\}} M(x,|\nabla u_n|)\, dx\, dt \leq d_\epsilon + c'_{10} k. \tag{5.41}$$

Since $\left(\alpha - \tfrac{\alpha C_\gamma}{2C_\gamma+\alpha}\right) > 0$, we get

$$\int_{\{h\leq|u_n|\leq h+k\}} M(x,|\nabla u_n|)\, dx\, dt \leq C + ck, \tag{5.42}$$

then, we deduce (5.31).

□

Now, we turn to prove (5.17), let $k > 0$ large enough. Thanks to Lemma 3.3, there exist two positive constants δ and λ such that,

$$\int_{Q_\tau} M(x,\delta|T_k(u_n - T_h(u_n))|)\, dx\, dt \leq \lambda \int_{Q_\tau} M(x,|\nabla T_k(u_n - T_h(u_n))|)\, dx\, dt. \tag{5.43}$$

Then, we deduce by using Lemma 5.3 that,

$$\begin{aligned}\inf_{x\in\Omega} M(x,\delta k) &\text{meas}\{|u_n - T_h(u_n)| > k\} \\ &\leq \int_{\{|u_n - T_h(u_n)|>k\}} M(x,\delta|T_k(u_n - T_h(u_n))|)\, dx\, dt \\ &\leq \lambda \int_{Q_\tau} M(x,|\nabla T_k(u_n - T_h(u_n))|)\, dx\, dt \\ &\leq \lambda(C + ck).\end{aligned}$$

Hence,

$$\text{meas}(\{|u_n - T_h(u_n)| > k\}) \leq \frac{\lambda(C+ck)}{\inf_{x\in\Omega} M(x,\delta k)} \tag{5.44}$$

for all n and for all k large enough, we have

$$\text{meas}\{|u_n| > k\} \leq \text{meas}\{|u_n - T_h(u_n)| > k-h\} \leq \frac{\lambda c(k-h) + \lambda C}{\inf\limits_{x\in\Omega} M(x,(k-h)\delta)}$$

for all n.

Finally, let tending K to infinity and using (2.2), we deduce

$$\lim_{k\to\infty} \text{meas}\Big\{(x,t) \in Q_\tau : |u_n| > k\Big\} = 0 \tag{5.45}$$

uniformly with respect to n.

Now, we turn to prove that $T_k(u_n) \to T_k(u)$ as n goes to infinity strongly in $L^1(Q_T)$, by applying the Lemma 3.2 and Proposition 5.2.

For that take a $C^2(\mathbf{R})$ nondecreasing function ξ_k such that $\xi_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $\xi_k(s) = k$ for $|s| \geq k$.
Multiplying the approximating equation by $\xi_k'(u_n)$, we get

$$\frac{\partial \xi_k(u_n)}{\partial t} - \text{div}\Big(a(x,t,u_n,\nabla u_n)\xi_k'(u_n)\Big) \tag{5.46}$$
$$+\ a(x,t,u_n,\nabla u_n)\xi_k''(u_n)\nabla u_n - \text{div}\Big(\xi_k'(u_n)\Phi_n(x,t,u_n)\Big)$$
$$+\xi_k''(u_n)\Phi_n(x,t,u_n)\nabla u_n = f_n\xi_k'(u_n).$$

Due to (A_2), (5.1) and the fact that $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$, $-\text{div}\Big(a(x,t,u_n,\nabla u_n)\xi_k'(u_n)\Big) + a(x,t,u_n,\nabla u_n)\xi_k''(u_n)\nabla u_n + f_n\xi_k'(u_n)$ is bounded in $L^1(Q_T) + W_0^{-1,x}L_{\bar{M}}(Q_T)$, so $\xi_n(u_n)$ is bounded in $L^1(Q_T) + W_0^{1,x}L_M(Q_T)$.

Moreover since $supp(\xi_k')$ and $supp(\xi_k'')$ are both included in $[-k,k]$ by (B_3) and (5.2) if follows that,

$$|\int_{Q_T} \xi_k'(u_n)\Phi_n(x,t,u_n)\;dx\;dt| \leq$$
$$||\xi_k'||_{L^\infty}\int_{Q_T}\gamma(x,t)\cdot \overline{M}_x^{-1}M(x,|T_k(u_n)|)\;dx\;dt$$

Furthermore, we have $\gamma \in L^\infty(Q_\tau)$ and $\overline{M}^{-1}M$ is increasing function, hence $|\int_{Q_T}\xi_k'(u_n)\Phi_n(x,t,u_n)\;dx\;dt| \leq C_k^1$, where C_k^1 is a positive constant independent of n.

In the same way, we get
$|\int_{Q_T}\xi_k''(u_n)\Phi_n(x,t,u_n)\nabla u_n\;dx\;dt| \leq C_k^2$, where C_k^2 is a positive constant independent of n.

Then all above implies that

$$\frac{\partial \xi_k(u_n)}{\partial t} \text{ is bounded in } L^1(Q_T) + W^{-1,x}L_{\overline{M}}(Q_T). \tag{5.47}$$

Hence by Lemma 3.2 and using the same technics in [29], we can see that there exists a measurable function u in $L^\infty(0,T;L^1(\Omega))$ such that for every $k > 0$,

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W^{1,x}L_M(Q_T) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}), \tag{5.48}$$

and

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^1(Q_T) \text{ and a.e. in } Q_T. \tag{5.49}$$

Next step, we will use Banach-Steinhaus Theorem to prove the following Lemma but first let remark that for all $s \in \mathbf{R}$, we have

$$a(x,t,T_k(s),\nabla T_k(s)) = \begin{cases} a(x,t,s,\nabla s) & \text{if } |s| \le k \\ 0 & \text{if } |s| > k \end{cases}. \tag{5.50}$$

Lemma 5.4 *Let u_n be a solution of the approximate problem (5.5), then for all $k \geq 0$,*

$$\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))\Big)_n \text{ is bounded in } (L_{\overline{M}}(Q_T))^N. \tag{5.51}$$

Proof. Let $\varphi \in (E_M(Q_T))^N$ be arbitrary. In view of the monotonicity of a, one easily has

$$\Big(a\Big(x,t,u_n,\nabla u_n\Big) - a\Big(x,t,u_n,\varphi\Big)\Big)\Big(\nabla u_n - \varphi\Big) \geq 0. \tag{5.52}$$

Hence

$$\begin{aligned}\int_{\{|u_n|\le k\}} a\Big(x,t,u_n,\nabla u_n\Big)\varphi \, dx\, dt \le &\int_{\{|u_n|\le k\}} a\Big(x,t,u_n,\nabla u_n\Big)\nabla u_n \, dx\, dt \\ &+ \int_{\{|u_n|\le k\}} a\Big(x,t,u_n,\varphi\Big)\Big(\varphi - \nabla u_n\Big)\, dx\, dt.\end{aligned} \tag{5.53}$$

Using (A_2) and since $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$, one easily deduces that

$$\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(u_n)\, dx\, dt \le Ck_1. \tag{5.54}$$

Combining the fact that $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$, (5.53) and (5.54), we get

$$\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\varphi \, dx\, dt \le Ck_2. \tag{5.55}$$

Hence, thanks the Banach-Steinhaus Theorem, the sequence $\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))\Big)_n$ is a bounded in $(L_{\overline{M}}(Q_T))^N$.

□

5.2 Modular convergence of the gradient. Part II

In the proof of the next Proposition, we shall use Lemma 4.1, from above we already verified the conditions (4.9) and (4.10), now our mission is to verify the condition 4.11.

Lemma 5.5 *Let u_n be a solution of the approximate problem (5.5). Then, for any $k \geq 0$:*

$$\lim_{s\to\infty}\lim_{n\to\infty}\int_{Q_T}\Big[a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big] \times\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]\,dx\,dt=0. \tag{5.56}$$

Proof. Let $v_j \in D(Q_T)$ be a sequence such that $v_j \to u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence and let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.

Let $w^i_{\mu,j} = T_k(v_j)_\mu + e^{-\mu t}T_k(\psi_i)$ where $T_k(v_j)_\mu$ is the mollification with respect to time of $T_k(v_j)$, note that $w^i_{\mu,j}$ is a smooth function having the following properties:

$$\frac{\partial w^i_{\mu,j}}{\partial t}=\mu(T_k(v_j)-w^i_{\mu,j}),\ w^i_{\mu,j}(0)=T_k(\psi_i),\ |w^i_{\mu,j}|\leq k, \tag{5.57}$$

$$w^i_{\mu,j}\to T_k(u)_\mu+e^{-\mu t}T_k(\psi_i)\ \text{ in }\ W_0^{1,x}L_M(Q_T), \tag{5.58}$$

for the modular convergence as $j\to\infty$.

$$T_k(u)_\mu+e^{-\mu t}T_k(\psi_i)\to T_k(u)\ \text{ in }\ W_0^{1,x}L_M(Q_T), \tag{5.59}$$

for the modular convergence as $\mu\to\infty$.

Let us consider the function h_m defined on $\mathbf{R}$ by:

$$h_m(s)\begin{cases}1 & \text{if } \ |s|\leq m\\ -|s|+m+1 & \text{if }\ m\leq|s|\leq m+1\\ 0 & \text{if }\ |s|\geq m+1,\end{cases}$$

for any $m \geq k$.

Using the admissible test function $v = u_n - \varphi^{\mu,i}_{n,j,m}$, (for more explication of admissibility of this test function see the appendix of article [20]) where $\varphi^{\mu,i}_{n,j,m} = (T_k(u_n) - w^\mu_{i,j})h_m(u_n)$ as test function in (5.18) leads to

$$\langle\frac{\partial u_n}{\partial t},\varphi^{\mu,i}_{n,j,m}\rangle+\int_{Q_T}a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w^\mu_{i,j})h_m(u_n)\,dx\,dt \tag{5.60}$$

$$+\int_{Q_T}a(x,t,u_n,\nabla u_n)(T_k(u_n)-w^\mu_{i,j})\nabla u_n h'_m(u_n)\,dx\,dt$$

$$+\int_{\{m\leq|u_n|\leq m+1\}} \Phi_n(x,t,u_n)\nabla u_n h'_m(u_n)(T_k(u_n)-w^{\mu}_{i,j})\,dx\,dt$$

$$+\int_{Q_T} \Phi_n(x,t,u_n)h_m(u_n)(\nabla T_k(u_n)-\nabla w^{\mu}_{i,j})\,dx\,dt = \int_{Q_T} f_n\varphi^{\mu,i}_{n,j,m}\,dx\,dt.$$

Denoting by $\epsilon(n,j,\mu,i)$ any quantity such that,

$$\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\epsilon(n,j,\mu,i)=0.$$

By the definition of the sequence $w^{\mu}_{i,j}$, we can establish the following lemma.

Lemma 5.6 *([5]) Let $\varphi^{\mu,i}_{n,j,m} = (T_k(u_n)-w^{\mu}_{i,j})h_m(u_n)$, we have for any $k\geq 0$:*

$$\langle\frac{\partial u_n}{\partial t},\varphi^{\mu,i}_{n,j,m}\rangle \geq \epsilon(n,j,\mu,i), \tag{5.61}$$

where $\langle,\rangle$ denotes the duality pairing between $L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T)$ and $L^\infty(Q_T)\cap W^{1,x}_0L_M(Q_T)$.

Now, we turn to prove the following results for any fixed $k\geq 0$.

$$\int_{Q_T} f_n\varphi^{\mu,i}_{n,j,m}\,dx\,dt = \epsilon(n,j,\mu). \tag{5.62}$$

$$\int_{Q_T} \Phi_n(x,t,u_n)h_m(u_n)(\nabla T_k(u_n)-\nabla w^{\mu}_{i,j})\,dx\,dt = \epsilon(n,j,\mu), \tag{5.63}$$

$$\int_{\{m\leq|u_n|\leq m+1\}} \Phi_n(x,t,u_n)\nabla u_n h'_m(u_n)(T_k(u_n)-w^{\mu}_{i,j})\,dx\,dt = \epsilon(n,j,\mu). \tag{5.64}$$

$$\int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n h'_m(u_n)(T_k(u_n)-w^{\mu}_{i,j})\,dx\,dt \leq \epsilon(n,j,\mu,m). \tag{5.65}$$

Proof of (5.62): By the almost everywhere convergence of u_n, we have $(T_k(u_n)-w^{\mu}_{i,j})h_m(u_n)$ converges to $(T_k(u)-w^{\mu}_{i,j})h_m(u)$ in $L^\infty(Q_T)$ weak-* and then,

$$\int_{Q_T} f_n(T_k(u_n)-w^{\mu}_{i,j})h_m(u_n)\,dx\,dt \to \int_{Q_T} f(T_k(u)-w^{\mu}_{i,j})h_m(u)\,dx\,dt.$$

So that,

$$(T_k(u)-w^{\mu}_{i,j})h_m(u) \to (T_k(u)-T_k(u)_\mu - e^{-\mu t}T_k(\psi_i)),$$

in $L^\infty(Q_T)$ weak-*as $j \to \infty$ and also

$$(T_k(u) - T_k(u)_\mu - e^{-\mu t} T_k(\psi_i)) \to 0 \ \text{ in } L^\infty(Q_T) \text{ weak-* as } \mu \to +\infty.$$

Then, we deduce that,

$$\int_{Q_T} f_n(T_k(u_n) - w^\mu_{i,j}) h_m(u_n)\, dx\, dt = \epsilon(n, j, \mu). \tag{5.66}$$

Proof of (5.63) and (5.64): For $n \geq m+1$, we have

$$\Phi_n(x,t,u_n)h_m(u_n) = \Phi(x,t,T_{m+1}(u_n))h_m(T_{m+1}(u_n)) \ \text{ a.e in } \ Q_T. \tag{5.67}$$

On the other hand, since $\overline{P}(x,0) = 0$, $\overline{P}$ is convexe continuous function and Φ is a Carathéodory function and the pointwise convergence of u_n to u as $n \to \infty$, we have $\overline{P}\Big(x, \frac{\Phi(x,t,T_{m+1}(u_n)) - \Phi(x,t,T_{m+1}(u))}{\mu}\Big) \to 0$ a.e in Q_T and we have $\gamma \in L^\infty(Q_T)$, then for μ large enough and $n \geq m+1$, we obtain

$$\begin{aligned}
\Phi(x,t,T_{m+1}(u_n)) - \Phi(x,t,T_{m+1}(u)) &\leq |C_\gamma.\overline{M}_x^{-1} M(x, |T_{m+1}(u_n)|)| \\
&\quad + |C_\gamma.\overline{M}_x^{-1} M(x, |T_{m+1}(u)|)| \\
&\leq 2|C_\gamma.\overline{M}_x^{-1} M(x, (m+1))| \\
&\leq 2C_\gamma.|M(x,1) + M(x,(m+1))| \\
&\leq 2C_\gamma\Big(\sup_{t\in B(0,1)} ess\sup_{x\in\Omega} M(x,t) \\
&\quad + \sup_{t\in B(0,(m+1))} ess\sup_{x\in\Omega} M(x,t)\Big) \\
&\leq 2R_m C_\gamma.
\end{aligned} \tag{5.68}$$

Then,

$$\begin{aligned}
\overline{P}\Big(x, \tfrac{\Phi(x,t,T_{m+1}(u_n)) - \Phi(x,t,T_{m+1}(u))}{\mu}\Big) &\leq \tfrac{1}{\mu}\overline{P}\Big(x, 2R_m C_\gamma\Big) \\
&= g_m,
\end{aligned} \tag{5.69}$$

a.e. in $\mathcal{Q}_\mathsf{T}$, with $g_m \in L^1(Q_T)$.

Then, by Lebesgue's dominated convergence theorem, we obtain

$$\Phi(x,t,T_{m+1}(u_n)) \to \Phi(x,t,T_{m+1}(u)) \tag{5.70}$$

with respect to the modular convergence in $L_{\overline{P}}(Q_T)$ as $n \to +\infty$.

Recall that $P \ll M \Leftrightarrow \overline{M} \ll \overline{P}$ (see [22]). By applying Lemma 2.4, we obtain $\Phi(x,t,T_{m+1}(u_n)) \to \Phi(x,t,T_{m+1}(u))$ in $E_{\overline{M}}(Q_T)$.

Then by virtue of, $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(Q_T))^N$, then

$$\begin{aligned}
&\int_{Q_T} \Phi_n(x,t,u_n)h_m(u_n)(\nabla T_k(u_n) - \nabla w^\mu_{i,j})\, dx\, dt \\
&\qquad \to \int_{Q_T} \Phi(x,t,u)h_m(u)(\nabla T_k(u) - \nabla w^\mu_{i,j})\, dx\, dt
\end{aligned} \tag{5.71}$$

as $n \to +\infty$.

In the other hand, by using the modular convergence of $w_{i,j}^{\mu}$ as $j \to +\infty$ and letting μ tends to infinity, we get (5.63).

Now we turn to prove (5.64).

First, remark for $n \geq m+1$ we have that

$$\nabla u_n h_m'(u_n) = \nabla T_{m+1}(u_n) \quad \text{a.e in} \quad Q_T. \tag{5.72}$$

By the almost everywhere convergence of u_n, we have $(T_k(u_n) - w_{i,j}^{\mu})$ converges to $(T_k(u) - w_{i,j}^{\mu})$ in $L^\infty(Q_T)$ weak-* and since the sequence $(\Phi_n(x,t,T_{m+1}(u_n)))_n$ converges strongly in $E_{\overline{M}}(Q_T)$ then,

$$\Phi_n(x,t,T_{m+1}(u_n))(T_k(u_n) - w_{i,j}^{\mu}) \to \Phi(x,t,T_{m+1}(u))(T_k(u) - w_{i,j}^{\mu})$$

converges strongly in $E_{\overline{M}}(Q_T)$ as n goes to $+\infty$.

Using again the fact that, $\nabla T_{m+1}(u_n) \rightharpoonup \nabla T_{m+1}(u)$ weakly in $(L_M(Q_T))^N$ as n tends to $+\infty$ we obtain

$$\begin{aligned}\int_{\{m \leq |u_n| \leq m+1\}} &\Phi_n(x,t,u_n)\nabla u_n h_m'(u_n)(T_k(u_n) - w_{i,j}^{\mu})\, dx\, dt \\ &\to \int_{\{m \leq |u| \leq m+1\}} \Phi(x,t,u)\nabla u(T_k(u) - w_{i,j}^{\mu})\, dx\, dt,\end{aligned} \tag{5.73}$$

as n tends to $+\infty$.

By using the modular convergence of $w_{i,j}^{\mu}$ as $j \to +\infty$ and letting μ tends to infinity, we get (5.64).

Proof of (5.65): Concerning the third term of the right hand side of (5.60) we obtain that

$$\int_{\{m \leq |u_n| \leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n h_m'(u_n)(T_k(u_n) - w_{i,j}^{\mu})\, dx\, dt \tag{5.74}$$

$$\leq 2k \int_{\{m \leq |u_n| \leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\, dx\, dt.$$

Then by (5.51) we deduce that,

$$\begin{aligned}\int_{\{m \leq |u_n| \leq m+1\}} &a(x,t,u_n,\nabla u_n)\nabla u_n h_m'(u_n)(T_k(u_n) - w_{i,j}^{\mu})\, dx\, dt \\ &\leq \epsilon(n,\mu,m),\end{aligned} \tag{5.75}$$

which is the desired results.

Now, we pass to show the last step of the proof of (5.56). By means of (5.60)-(5.65), we obtain

$$\int_{Q_T} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla w_{i,j}^{\mu})h_m(u_n)\, dx\, dt \leq \epsilon(n,\mu,m). \tag{5.76}$$

Splitting the first integral on the left hand side of (5.76) where $|u_n| \leq k$ and $|u_n| > k$, we can write,

$$\begin{aligned}&\int_{Q_T} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla w^{\mu}_{i,j})h_m(u_n)\,dx\,dt\\&= \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla w^{\mu}_{i,j})h_m(u_n)\,dx\,dt\\&\quad - \int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)\nabla w^{\mu}_{i,j}h_m(u_n)\,dx\,dt.\end{aligned}$$

Since $h_m(u_n) = 0$ if $|u_n| \geq m+1$, one has

$$\int_{Q_T} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla w^{\mu}_{i,j})h_m(u_n)\,dx\,dt \tag{5.77}$$

$$= \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla w^{\mu}_{i,j})h_m(u_n)\,dx\,dt$$

$$- \int_{\{|u_n|>k\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla w^{\mu}_{i,j}h_m(u_n)\,dx\,dt = I_1 - I_2.$$

In the following we pass to the limit in (5.77) as n tends to $+\infty$, then j then μ and then m tends to $+\infty$. Since $a\Big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\Big)$ is bounded in $(L_{\overline{M}}(Q_T))^N$, we have, $a\Big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\Big) \rightharpoonup \varphi_m$ weakly in $L_{\overline{M}}(Q_T)$ in $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ as n tends to infinity and since $\nabla w^{\mu}_{i,j}h_m(u_n)\chi_{\{|u_n|>k\}}$ converges to $\nabla w^{\mu}_{i,j}h_m(u)\chi_{\{|u|>k\}}$ strongly in $E_M(\Omega)$ as n tends to infinity we have,

$$I_2 = \int_{Q_T} \varphi_m \nabla w^{\mu}_{i,j}h_m(u)\chi_{\{|u|>k\}}\,dx\,dt + \epsilon(n).$$

By letting $j \to \infty$, we get,

$$I_2 = \int_{Q_T} \varphi_m(\nabla T_k(u)_\mu - e^{-\mu t}\nabla T_k(\psi_i))h_m(u)\chi_{\{|u|>k\}}\,dx\,dt + \epsilon(n,j).$$

So that, by letting $\mu \to +\infty$: $I_2 = \displaystyle\int_{Q_T} \varphi_m \nabla T_k(u)_\mu h_m(u)\chi_{\{|u|>k\}}\,dx\,dt +$ $\epsilon(n,j,\mu)$.

Using now the term I_1 of (5.77) we conclude that, it is easy to show that,

$$\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)(\nabla T_k(u_n) - \nabla w^{\mu}_{i,j})h_m(u_n)\,dx\,dt \tag{5.78}$$

$$= \int_{Q_T} \Big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j)\chi^s_j)\Big]$$

$$\times\Big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\Big]h_m(u_n)\,dx\,dt$$

$$+\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\Big)\Big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\Big]h_m(u_n)\,dx\,dt$$

$$+\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(v_j)\chi_j^s h_m(u_n)\,dx\,dt$$

$$-\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\nabla w_{i,j}^{\mu} h_m(u_n)\,dx\,dt = J_1+J_2+J_3+J_4,$$

where χ_j^s denotes the characteristic function of the subset

$$\Omega_s^j=\Big\{(x,t)\in Q_T\ :\ |\nabla T_k(v_j)|\le s\Big\}.$$

In the following we pass to the limit in (5.77) as n tends to $+\infty$, then j then μ then m tends and then s tends to $+\infty$ in the last three integrals of the last side. Starting with J_2. Remark first that,

$$J_2=\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\Big)\nabla T_k(u_n)h_m(u_n)\,dx\,dt$$
$$-\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\Big)\nabla T_k(v_j)\chi_j^s h_m(u_n)\,dx\,dt.$$

Since $a\Big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\Big)h_m(u_n)\to a\Big(x,t,T_k(u),\nabla T_k(v_j)\chi_j^s\Big)h_m(u)$ strongly in $(E_{\overline{M}})^N$ and $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ weakly in $(L_M(Q_T))^N$ for $\sigma(\Pi L_M,\Pi E_{\overline{M}})$, moreover, it is easy to show that,

$$\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\Big)\nabla T_k(v_j)\chi_j^s h_m(u_n)\,dx\,dt$$
$$\to\int_{Q_T} a\Big(x,t,T_k(u),\nabla T_k(v_j)\chi_j^s\Big)\nabla T_k(v_j)\chi_j^s h_m(u)\,dx\,dt,$$

as n tends to $+\infty$. We get

$$J_2=\int_{Q_T} a\Big(x,t,T_k(u),\nabla T_k(v_j)\chi_j^s\Big)[\nabla T_k(u)-\nabla T_k(v_j)\chi_j^s]h_m(u)\,dx\,dt+\epsilon(n).$$

Since $\nabla T_k(v_j)\chi_j^s h_m(u)\to\nabla T_k(u)\chi_s h_m(u)$ strongly in $(E_M(Q_T))^N$ as $j\to\infty$ and $a\Big(x,t,T_k(u),\nabla T_k(v_j)\chi_j^s\Big)\to a\Big(x,t,T_k(u),\nabla T_k(u)\chi_s\Big)$ strongly in $(L_{\overline{M}}(Q_T))^N$ as j goes to ∞, we have

$$J_2=\epsilon(n,j).\tag{5.79}$$

About J_3, we have by letting $n \to \infty$ and since $a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varphi_k$ weakly in $(L_{\overline{M}}(Q_T))^N$ and since $h_m(u_n) = 1$ in $\{(x,t)\ ;\ |u_n| \le k\}$,

$$J_3 = \int_{Q_T} \varphi_k \nabla T_k(v_j)\chi_j^s \, dx\, dt + \epsilon(n),$$

which gives by letting $j \to \infty$,

$$J_3 = \int_{Q_T} \varphi_k \nabla T_k(u)\chi_s \, dx\, dt + \epsilon(n,j). \tag{5.80}$$

For what concerns J_4 we can write,

$$J_4 = -\int_{Q_T} \varphi_k \nabla w_{i,j}^{\mu} \mu h_m(u) \, dx\, dt + \epsilon(n), \tag{5.81}$$

which implies that, by letting $j \to \infty$

$$J_4 = -\int_{Q_T} \varphi_k [\nabla T_k(u) - e^{-\mu t}\nabla T_k(\psi_j)] \, dx\, dt + \epsilon(n,j). \tag{5.82}$$

So that, by letting $\mu \to \infty$

$$J_4 = -\int_{Q_T} \varphi_k \nabla T_k(u) \, dx\, dt + \epsilon(n,j,\mu,s). \tag{5.83}$$

We conclude then that,

$$\begin{aligned}
&\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big[\nabla T_k(u_n) - \nabla w_{i,j}^{\mu}\Big] h_m(u_n)\, dx\, dt \qquad (5.84)\\
&= \int_{Q_T} \Big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big]\\
&\qquad \times \Big[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big] h_m(u_n)\, dx\, dt + \ \epsilon(n,j,\mu,s).
\end{aligned}$$

Now remark that,

$$\begin{aligned}
&\int_{Q_T} \Big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\\
&\qquad \times \Big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big] h_m(u_n)\, dx\, dt\\
&= \int_{Q_T} \Big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big]\\
&\qquad \times \Big[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big] h_m(u_n)\, dx\, dt\\
&+ \int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\Big)\Big[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big] h_m(u_n)\, dx\, dt\\
&- \int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big[\nabla T_k(u_n) - \nabla T_k(u)\chi^s\Big] h_m(u_n)\, dx\, dt\\
&+ \int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big[\nabla T_k(v_j)\chi_j^s) - \nabla T_k(u)\chi_s\Big] h_m(u_n)\, dx\, dt.
\end{aligned}$$

We shall pass to the limit in n and j in the last three terms of the right hand side of the last inequality, we get

$$\begin{aligned}
&\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\Big)\Big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\Big]h_m(u_n)\,dx\,dt\\
&-\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]h_m(u_n)\,dx\,dt\\
&=\epsilon(n,j),
\end{aligned}$$

and

$$\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big[\nabla T_k(v_j)\chi_j^s-\nabla T_k(u)\chi_s\Big]h_m(u_n)\,dx\,dt = \epsilon(n,j). \tag{5.85}$$

This implies,

$$\begin{aligned}
&\int_{Q_T}\Big[a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big] \qquad (5.86)\\
&\qquad\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]h_m(u_n)\,dx\,dt\\
&=\int_{Q_T}\Big[a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\Big)\Big]\\
&\qquad\times\Big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\Big]h_m(u_n)\,dx\,dt+\epsilon(n,j).
\end{aligned}$$

On the other side, we have

$$\begin{aligned}
&\int_{Q_T}\Big[a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big] \qquad (5.87)\\
&\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]=\int_{Q_T}\Big[a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\\
&-a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big]\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]h_m(u_n)\,dx\,dt\\
&+\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi^s\Big](1-h_m(u_n))\,dx\,dt\\
&-\int_{Q_T} a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big](1-h_m(u_n))\,dx\,dt.
\end{aligned}$$

Since $h_m(u_n)=1$ in $\{|u_n|\leq m\}$ and since $\{|u_n|\leq k\}\subset\{|u_n|\leq m\}$ for m large enough, we deduce from (5.87)

$$\begin{aligned}&\int_{Q_T}\Big[a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big]\\&\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]=\int_{Q_T}\Big[a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\\&-a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]\\&+\int_{\{|u_n|>k\}}a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\nabla T_k(u)\chi_s(1-h_m(u_n))\,dx\,dt.\end{aligned}$$

It's easy to see that, the last terms of the last inequality tends to zero as $n\to+\infty$, which implies that,

$$\begin{aligned}&\int_{Q_T}\Big[a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big]\\&\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]=\int_{Q_T}\Big[a\Big(x,t,T_k(u),\nabla T_k(u)\Big)\\&-a\Big(x,t,T_k(u),\nabla T_k(u)\chi_s\Big)\Big]\Big[\nabla T_k(u)-\nabla T_k(u)\chi_s\Big]h_m(u_n)\,dx\,dt+\epsilon(n,j).\end{aligned}$$

Combining (5.61), (5.78), (5.79), (5.80), (5.83) and (5.87) we deduce,

$$\begin{aligned}&\int_{Q_T}\Big[a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big]\\&\qquad\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]\,dx\,dt\leq\epsilon(n,j,\mu,m,s).\end{aligned}\tag{5.88}$$

When we pass to the limit in (5.88) as $n,\ \mu,\ j,\ m,\ s$ tend to infinity, we obtain

$$\begin{aligned}&\lim_{s\to\infty}\lim_{n\to\infty}\int_{Q_T}\Big[a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big]\\&\qquad\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]\,dx\,dt=0.\end{aligned}\tag{5.89}$$

□

Proposition 5.7 *Let u_n be a solution of the approximate problem (5.5). Then, for any $k\geq 0$:*

$$\nabla u_n\to\nabla u\quad a.e.\ in\ \ Q_T,\tag{5.90}$$

$$a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\rightharpoonup a\Big(x,t,T_k(u),\nabla T_k(u)\Big)\tag{5.91}$$

weakly in $(L_{\overline{M}}(Q_T))^N$.

$$M(x,|\nabla T_k(u_n)|)\to M(x,|\nabla T_k(u)|)\quad strongly\ in\ \ L^1(Q_T),\tag{5.92}$$

as n tends to $+\infty$.

Proof. By using (5.16), Lemma 5.4 and the Lemma 5.5 all the desired statement are fulfilled of the Lemma 4.1, hence the proof of Proposition 5.7 is achieved.

5.3 Proof of Theorem 5.1

In this step, we pass to the limit.

Let $v \in W^{1,x}L_M(Q_T) \cap L^\infty(Q_T)$ such that $\frac{\partial v}{\partial t}$ belongs to $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ and $v \geq \psi$ a.e in Q_T.

By Lemma 5 and Theorem 3 in [13], there exists a prolongation $\overline{v} = v$ on Q_T, $\overline{v} \in W^{1,x}L_M(\Omega \times \mathbf{R}) \cap L^1(\Omega \times \mathbf{R}) \cap L^\infty(\Omega \times \mathbf{R})$, and

$$\frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times \mathbf{R}) + L^1(\Omega \times \mathbf{R}).$$

There exists also a sequence $(\omega_j) \subset \mathcal{D}(\Omega \times \mathbf{R})$ ([13]) such that

$$\begin{array}{l} \omega_j \to \overline{v} \;\text{ in }\; W_0^{1,x}L_M(\Omega \times \mathbf{R}), \;\text{ and} \\ \frac{\partial \omega_j}{\partial t} \to \frac{\partial \overline{v}}{\partial t} \;\text{ in }\; W^{-1,x}L_{\overline{M}}(\Omega \times \mathbf{R}) + L^1(\Omega \times \mathbf{R}). \end{array} \tag{5.93}$$

For the modular convergence and $||\omega_j||_{\infty,Q_T} \leq (N+2)|v||_{\infty,Q_T}$.

Now, let us take $v = u_n - T_k(u_n - \omega_j) \cdot \chi_{(0,\tau)}$ as a test function in (5.18), thus for every $\tau \in [0,T]$, we get

$$\begin{aligned} \langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_j) \rangle_{Q_\tau} & \\ &+ \int_{Q_\tau} a(x,t,T_{\mathcal{K}}(u_n), \nabla T_{\mathcal{K}}(u_n)) \nabla T_k(u_n - \omega_j)\, dx\, dt \\ &+ \int_{Q_\tau} \Phi(x,t,T_{\mathcal{K}}(u_n)) \nabla T_k(u_n - \omega_j)\, dx\, dt \\ &= \int_{Q_\tau} f_n T_k(u_n - \omega_j)\, dx\, dt, \end{aligned} \tag{5.94}$$

where $\mathcal{K} = k + C||v||_{\infty,Q_T}$, which implies

$$\begin{aligned} \langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_j) \rangle_{Q_\tau} & \\ &+ \int_{Q_\tau \cap \{|u_n - \omega_j| \leq k\}} a(x,t,T_{\mathcal{K}}(u_n), \nabla T_{\mathcal{K}}(u_n)) \nabla u_n\, dx \\ &- \int_{Q_\tau \cap \{|u_n - \omega_j| \leq k\}} a(x,t,T_{\mathcal{K}}(u_n), \nabla T_{\mathcal{K}}(u_n)) \nabla \omega_j\, dx \\ &+ \int_{Q_\tau} \Phi(x,t,T_{\mathcal{K}}(u_n)) \nabla T_k(u_n - \omega_j)\, dx\, dt \\ &= \int_{Q_\tau} f_n T_k(u_n - \omega_j)\, dx\, dt. \end{aligned} \tag{5.95}$$

By Fatou's lemma and the fact that

$$a(x,t,T_{\mathcal{K}}(u_n),\nabla T_{\mathcal{K}}(u_n)) \rightharpoonup a(x,t,T_{\mathcal{K}}(u),\nabla T_{\mathcal{K}}(u))$$

weakly in $(L_{\overline{M}}(Q_T))^N$ for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$, one easily sees that

$$\begin{aligned}\int_{Q_\tau\cap\{|u_n-\omega_j|\le k\}} & a(x,t,T_{\mathcal{K}}(u_n),\nabla T_{\mathcal{K}}(u_n))\nabla u_n\,dx \\ & -\int_{Q_\tau\cap\{|u_n-\omega_j|\le k\}} a(x,t,T_{\mathcal{K}}(u_n),\nabla T_{\mathcal{K}}(u_n))\nabla\omega_j\,dx \\ & \ge \int_{Q_\tau\cap\{|u-\omega_j|\le k\}} a(x,t,T_{\mathcal{K}}(u),\nabla T_{\mathcal{K}}(u))\nabla u\,dx \\ & -\int_{Q_\tau\cap\{|u-\omega_j|\le k\}} a(x,t,T_{\mathcal{K}}(u),\nabla T_{\mathcal{K}}(u))\nabla\omega_j\,dx.\end{aligned} \tag{5.96}$$

As in (5.68), we obtain $\Phi(x,t,T_{\mathcal{K}}(u_n)) \to \Phi(x,t,T_{\mathcal{K}}(u))$ in $E_{\overline{M}}(Q_T)$ as $n\to+\infty$ and using the fact that $\nabla T_k(u_n-\omega_j) \rightharpoonup \nabla T_k(u-\omega_j)$ in $L_M(Q_T)$, as $n\to+\infty$, we can easy see that

$$\begin{aligned}\int_{Q_\tau} & \Phi(x,t,T_{\mathcal{K}}(u_n))\nabla T_k(u_n-\omega_j)\,dx\,dt \\ & \to \int_{Q_\tau} \Phi(x,t,T_{\mathcal{K}}(u))\nabla T_k(u-\omega_j)\,dx\,dt.\end{aligned} \tag{5.97}$$

Since $T_k(u_n-\omega_j) \to T_k(u-\omega_j)$ weakly* in L^∞ as $n\to+\infty$, we have

$$\int_{Q_\tau} f_n T_k(u_n-\omega_j)\,dx\,dt \to \int_{Q_\tau} f T_k(u-\omega_j)\,dx\,dt.$$

Turn now to see the first term of (5.94),

$$\begin{aligned}\langle\frac{\partial u_n}{\partial t},T_k(u_n-\omega_j)\rangle_{Q_\tau} = & \int_\Omega S_k(u_n-\omega_j)dx + \langle\frac{\partial\omega_j}{\partial t},T_k(u_n-\omega_j)\rangle_{Q_\tau} \\ & -\int_\Omega S_k(u_{n0}-\omega_j(0))dx.\end{aligned} \tag{5.98}$$

First, let see that $u_n\to u$ in $C([0,T];L^1(\Omega))$ (see [13]). Moreover, since $S_k(u_n-\omega_j)(\tau)\le k|u_n(\tau)|+k|\omega_j(\tau)|$, we have by Lebesgue Theorem

$$\int_\Omega S_k(u_n-\omega_j)(\tau)dx \to \int_\Omega S_k(u-\omega_j)(\tau)dx,$$

as $n\to+\infty$. Then, we can pass to the limit in (5.98) as $n\to+\infty$ we obtain

$$\begin{aligned}\lim_{n\to+\infty}\langle\frac{\partial u_n}{\partial t},T_k(u_n-\omega_j)\rangle_{Q_\tau} = & \int_\Omega S_k(u-\omega_j)dx + \langle\frac{\partial\omega_j}{\partial t},T_k(u-\omega_j)\rangle_{Q_\tau} \\ & -\int_\Omega S_k(u_0-\omega_j(0))dx.\end{aligned} \tag{5.99}$$

Now, let n goes to infinity in (5.94), we get

$$\begin{aligned}
&\int_\Omega S_k(u-\omega_j)dx + \langle \frac{\partial \omega_j}{\partial t}, T_k(u-\omega_j)\rangle_{Q_\tau} \\
&\qquad + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-\omega_j)\ dx\ dt \\
&\qquad + \int_{Q_\tau} \Phi(x,t,u)\nabla T_k(u-\omega_j)\ dx\ dt \\
&\qquad \leq \int_{Q_\tau} fT_k(u-\omega_j)\ dx\ dt + \int_\Omega S_k(u_0-\omega_j(0))dx.
\end{aligned} \tag{5.100}$$

By (5.93), as j tends to $+\infty$ we have

$$\langle \frac{\partial \omega_j}{\partial t}, T_k(u-\omega_j)\rangle_{Q_\tau} \to \langle \frac{\partial v}{\partial t}, T_k(u-v)\rangle_{Q_\tau}.$$

Moreover, for every $\tau \in [0,T]$, we have $||\omega_j - v(\tau)||_{L^1(\Omega)} \to 0$ as $j \to +\infty$. Therefore, we pass now to the limit as $j \to +\infty$ in (5.100), we get

$$\begin{aligned}
&\int_\Omega S_k(u-v)dx + \langle \frac{\partial v}{\partial t}, T_k(u-v)\rangle_{Q_\tau} \\
&\qquad + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-v)\ dx\ dt \\
&\qquad + \int_{Q_\tau} \Phi(x,t,u)\nabla T_k(u-v)\ dx\ dt \\
&\qquad \leq \int_{Q_\tau} fT_k(u-v)\ dx\ dt + \int_\Omega S_k(u_0-v(0))dx.
\end{aligned} \tag{5.101}$$

The proof of theorem 5.1 is complete.

□

5.4 Appendix

The function $v = u_n - T_k(u_n - v_0)$ is an admissible function i.e that $v \in K_\psi$, first, it is clear that $v \in W^{1,x}L_M(Q_\tau)$ it remains to verify that $v \geq \psi$ indeed, we have

$$v = u_n - T_k(u_n - v_0) = \begin{cases} u_n + k & \text{if} \quad u_n - v_0 \leq -k \\ v_0 & \text{if} \quad |u_n - v_0| < k \\ u_n - k & \text{if} \quad u_n - v_0 \geq k \end{cases}, \tag{5.102}$$

and the fact that $u_n, v_0 \in K_\psi$ and $k \geq 0$, we can easy deduce that $v \geq \psi$ a.e in Q_τ, then $v \in K_\psi$.

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