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On the summability of random Fourier–Jacobi series

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Abstract This article explores the summability of random Fourier–Jacobi series

$$\sum_{i=0}^{\infty} b_i B_i(\varpi) q_i^{(\gamma,\delta)}(y),\ y \in [-1,1]$$

in various functions spaces, where $q_i^{(\gamma,\delta)}(y), \gamma, \delta > -1$ are the orthonormal Jacobi polynomials and b_i are scalars. The random coefficients $B_i(\varpi)$ are the Fourier–Jacobi coefficients of a symmetric stable process $Z(t,\varpi)$ of index $\chi \in [1,2]$. It is established that the random Fourier series in Jacobi polynomials is summable in Υ means, Cesàro means, Rogosinski means, etc., in the sense of probability if b_i are the Fourier–Jacobi coefficients of a function $f \in C_{[-1,1]}^{(\eta,\tau)}$ space, under different conditions on the Jacobi parameters γ, δ, η and τ. Further, the Nörlund summability, generalized Nörlund summability and lower triangular summability of random Jacobi series associated with the symmetric stable process $Z(t,\varpi)$ of index one is established for functions belonging to the $L_{[-1,1]}^{1,(\gamma,\delta)}$ space. It has been noted that the conditions on the Jacobi parameters for random series differ from that of the Fourier–Jacobi series.

Keywords Cesàro summability · convergence in probability · Jacobi polynomials · lower triangular summability · Nörlund summability · random variables · stochastic integral · symmetric stable process

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1 Introduction

In the early 19th century, the Fourier series was developed to solve the problem of heat diffusion in a continuous medium and has since been widely applied across various branches of science. It is extended to random series of functions in the year 1930 with the pioneering works of Paley and Zygmund [17,18] and opened up a new direction in mathematical science to investigate Fourier series with random coefficients. Kahane, Marcus, and Pisier have

made significant contributions to the study of random Fourier series, yet many problems remain open today. Random Fourier series play a vital role in electrical engineering, signal processing, optics, and other fields because white noise, which is inherently present in all these areas, is a random signal. It is also used for the generation of random noise [5]. Most of the random series that have been studied are formed of independent random coefficients. Rademacher [21], Steinhaus [24, 25], Hunt [7], Paley, and Zygmund [17, 18], Paley et al. [19] discussed random series in the form

$$\sum_{n=-\infty}^{\infty} a_n X_n(\omega) e^{int}. \tag{1.1}$$

The random coefficients $X_n(\omega)$ are independent, identically distributed random variables, such as Rademacher functions, Steinhaus functions, etc. Hide et al. [6] investigated the series (1.1) in the context of the study of "White Noise." The model for white noise was taken to be $\dfrac{dW}{dt}$, where $W(t)$ is the Brownian motion. It is also known as Wiener process, which is a continuous stochastic process with independent increments, such that

$$E\Big(W(t) - W(s)\Big) = 0 \text{ and } E\Big|W(t) - W(s)\Big|^2 = |t-s|^2.$$

Since $W(t)$ is nowhere differentiable, this model seems meaningless and can be assigned in a weak sense. It is observed that the stable process $Z(t, \varpi)$ are better models for white noise than the Wiener process. Stochastic process are useful in modeling extremal events. Recently, it has been found that the stable process $Z(t, \varpi)$ is applied to the analysis of systems under heavy–tailed noise, such as in assessing wind power fluctuations [9].

The random Fourier series converges to the stochastic integral

$$\int_a^b f(t) dZ(t, \varpi), \tag{1.2}$$

if the random coefficients are Fourier coefficients of a symmetric stable process $Z(t, \varpi)$ for f belongs to the function spaces. According to [13], the stochastic integral (1.2) exists in probability if f is a continuous function in $[a, b] \subset \mathbb{R}$ and $Z(t, \varpi), t \in \mathbb{R}$ is a continuous stochastic process with independent increments. The stochastic integral (1.2) is a random variable. Moreover, if $f \in L^p_{[a,b]}$, $p \geq 1$ and $Z(t, \varpi)$ is a symmetric stable process of index $\chi \in [1, 2]$, then the integral (1.2) exists in probability for $p \geq \chi \geq 1$ (c.f. [16]). If $f(t) := e^{2\pi int}$, $n \in \mathbb{N}$, then the integral (1.2) exists and are dependent random variables. These integrals denoted as $\mathbf{A}_n(\varpi)$, are known as Fourier–Stieltjes coefficients of $Z(t, \varpi)$. Further, $Z(t, \varpi)$ can be expressed

as a series expansion

$$\sum_{n=-\infty}^{\infty} \mathbf{A}_n(\varpi) e^{2\pi i n t}.$$

Nayak et al. [16] discussed on random Fourier–Stieltjes series

$$\sum_{n=-\infty}^{\infty} \mathbf{a}_n \mathbf{A}_n(\varpi) e^{2\pi i n t} \tag{1.3}$$

associated with symmetric stable process $Z(t, \varpi)$. The coefficients $\mathbf{a}_n$ and $\mathbf{A}_n(\omega)$ are scalars and random variables respectively. The scalars $\mathbf{a}_n$ are the Fourier–Stieltjes coefficients of a function $f \in L^p_{[0,1]}, p \geq 1$.

As we know, the $L^p_{[a,b]},\ p \geq 1$ space is defined as the set of all measurable functions f on $[a, b]$, such that $\int_a^b |f(t)|^p dt < \infty$.

They established different stochastic convergence of the random series (1.3), such as convergence in probability, convergence in pth mean, quadratic convergence, and almost sure convergence. They also discussed the Cesàro summability in probability of the random series (1.3). It is interesting to look into the random series (1.3) if the orthogonal functions $e^{2\pi i n t}$ are replaced by other orthogonal functions.

Further, the orthogonal polynomials, which originated in the 19th century, play an essential role in mathematical physics. Extensive studies have been conducted on the Fourier series in orthogonal polynomials. Marian and Marian [15] have studied power series involving orthogonal polynomials that occur in problems of quantum optics. The Jacobi polynomials hold great importance in the hierarchy of orthogonal polynomial classes and have found widespread use in all areas of science and engineering. Kvernadze [10], Prasad and Hayashi [20], among others, have discussed the convergence property of Fourier–Jacobi series. However, research on random Fourier series involving orthogonal polynomials has not been extensively explored. The complexity involved in handling such series and the limited availability of literature on their application in physical problems are among the primary reasons for this lack of attention.

In 2007, Liu and Liu [11,12] defined the random Fourier–Hermite transform in the form of random Fourier series in Hermite polynomials as

$$\mathcal{R}[f(y)] := \sum_{n=0}^{\infty} c_n \mathcal{R}(\lambda_n) \varphi_n(y), \tag{1.4}$$

where $\mathcal{R}(\lambda_n) := \exp[i\pi \text{ Random}(n)]$ are randomly chosen values on the unit circle in $\mathbb{C}$. The scalars c_n defined as

$$c_n := \int_{-\infty}^{\infty} f(y) \varphi_n(y) dy$$

are the Fourier–Hermite coefficients of f and $\varphi_n(y)$ are Hermite Gaussian functions. They applied it in the fields of image encryption and decryption, and expected its applications in optics, information technology, and other areas. Further, Imekraz et al. [8] studied the integrability and continuity properties of random series in Hermite functions. This raises the question of what would happen if the random coefficients $\mathcal{R}(\lambda_n)$ selected from the unit circle in $\mathbb{C}$ are replaced by other random variables in the random series (1.4). All these works motivated us and provide a way to study random Fourier series involving orthogonal polynomials.

We study the random Fourier series of the form

$$\sum_{i=0}^{\infty} a_i r_i(\varpi) q_i^{(\gamma,\delta)}(y) \tag{1.5}$$

in orthonormal Jacobi polynomials $q_i^{(\gamma,\delta)}(y)$, where $r_i(\varpi)$ are random variables and a_i are scalars. This series is called as random Fourier–Jacobi series. In our recent work [14], the different modes of convergence of random Fourier–Jacobi series (1.5) in various functions spaces are investigated.

Generally, a Fourier series does not always exist for all functions. Even if such representation exists, the convergence of Fourier series for such a function may not be possible. However, convergence may be achievable in certain cases by using specific summability methods, such as Cesàro, Riesz, Nörlund, and similar techniques. Summability techniques are useful tools for soft computing problems, such as system stabilization and filtering of audio and image signals represented in the form of series (infinite series, Fourier series, etc.). As a result, summability theory can be utilized to assess the degree and precision of approximation in audio and image signal in digital filters and other applications. Furthermore, some summability methods can enhance the quality of the filters [23].

In this article, various summabilities, like Cesàro, Riesz, Nörlund, etc., of the random Fourier–Jacobi series (1.5) are established. The random coefficients $r_i(\varpi)$ in the random series (1.5) are chosen to be the Fourier coefficients of a symmetric stable process $Z(t,\varpi)$ of index $\chi \in [1,2]$ with respect to the Jacobi weights $\gamma, \delta > -1$ and $\eta, \tau \geq 0$. The random Jacobi series is found to be summable to a stochastic integral. The existence of stochastic integral (1.2) in probability for $f \in L^p_{[a,b]}$, $p \geq 1$ (c.f. [16]), implies the existence of the integral in probability for $\eta, \tau > -1$ if $f \in L^{p,(\eta,\tau)}_{[-1,1]}$, i.e., $f\varrho^{(\eta,\tau)} \in L^p_{[-1,1]}$, where $\varrho^{(\eta,\tau)}$ is the Jacobi weight. This result is stated in Theorem 1.1 below for future reference.

The two weighted functions spaces used in this article are $C^{(\eta,\tau)}_{[-1,1]}$ space and $L^{p,(\eta,\tau)}_{[-1,1]}$ space (see Definition 2.1 and 2.2 in Section 2).

Theorem 1.1 *If $Z(t,\varpi)$ is symmetric stable process of index $\chi \in [1,2]$ and $f \in L^{p,(\eta,\tau)}_{[-1,1]}$, $p \geq 1, \eta, \tau > -1$, then the stochastic integral (1.2) is defined in probability for $p \geq \chi \geq 1$.*

Specifically, when $f(t)$ is the orthonormal Jacobi polynomials $q_i^{(\gamma,\delta)}(t), \gamma, \delta > -1$, then $q_i^{(\gamma,\delta)}(t)\varrho^{(\eta,\tau)}(t)$ belongs to the space $L^{p,(\eta,\tau)}_{[-1,1]}$, since $\varrho^{(\eta,\tau)}(t)$ is continuous in $[-1,1]$ for all $\eta, \tau \geq 0$. Consequently, the integrals

$$B_i(\varpi) := \int_{-1}^{1} q_i^{(\gamma,\delta)}(t)\varrho^{(\eta,\tau)}(t)dZ(t,\varpi) \tag{1.6}$$

exist and are random variables, which are not independent for each $i = 1, 2, \ldots$. We call $B_i(\varpi)$ as the Fourier–Jacobi coefficients of symmetric stable process $Z(t,\varpi)$, and $Z(t,\varpi)$ has the Fourier–Jacobi series expansion $\sum_{i=0}^{\infty} B_i(\varpi)q_i^{(\gamma,\delta)}(t)$. The various type of summability of random Fourier–Jacobi series

$$\sum_{i=0}^{\infty} b_i B_i(\varpi) q_i^{(\gamma,\delta)}(y) \tag{1.7}$$

in the sense of probability are established, if the scalars b_i are Fourier–Jacobi coefficients of a function f in the spaces $C^{(\eta,\tau)}_{[-1,1]}$ and $L^{p,(\eta,\tau)}_{[-1,1]}$ defined as

$$b_i := \int_{-1}^{1} f(t) q_i^{(\gamma,\delta)}(t)\varrho^{(\gamma,\delta)}(t)dt, \ \gamma, \delta > -1. \tag{1.8}$$

Not only from a mathematical point of view, but also from the application of Fourier series and random Fourier series in orthogonal polynomials in various fields of science and technology, gives important to study on these random series in orthogonal functions. Although the methodology is straightforward, it provides insights into the summability of random Fourier–Jacobi series across various function spaces, an area that has not been explored so far.

The summability in the space $C^{(\eta,\tau)}_{[-1,1]}$ is discussed using the summation matrix Υ, which is a lower triangular infinite matrix. Various summability methods are considered based on different entries of the matrix Υ. The summability of the random series (1.7) depends on specific conditions related to this summation matrix Υ. Nörlund summability and its variants are established for the random Fourier–Jacobi series in connection to the $L^{1,(\gamma,\delta)}_{[-1,1]}$ space. Each of these summability depends on particular conditions on the parameters γ, δ, which differ from those conditions for the Jacobi series of functions $f \in L^{1,(\gamma,\delta)}_{[-1,1]}$.

The Fourier–Jacobi series is summable in the Nörlund sense for $-1/2 \leq \gamma < 1/2$ and $\delta > -1/2$ (see [2]). However, the random series is summable

in the Nörlund sense for $0 \leq \gamma < 1/2$ and $\delta \geq 0$. The generalized Nörlund summability of Fourier–Jacobi series hold when $\gamma \geq 0, \delta - \gamma > 1$ and $\delta + \gamma \geq -1$ (c.f. [22]). But, it hold for the random case when $\gamma \geq 0,\ \delta - \gamma > 1$ and $\delta + \gamma > 0$. The Fourier–Jacobi series is lower triangular summable for $-1/2 \leq \gamma \leq 1/2$ and $\delta > -1/2$ (see [4]), where as the random series is summable in the lower triangular sense when $0 \leq \gamma \leq 1/2$ and $\delta \geq 0$.

After presenting the preliminaries in Section 2, different cases of summability, such as Υ, Cesàro, Riesz, Rogosinski, etc., of the series (1.7) in connection to the space $C^{(\eta,\tau)}_{[-1,1]}$ are discussed in Section 3. Section 4, presents the Nörlund summability, generalized Nörlund summability, and lower triangular matrix summability of the series (1.7) in connection to the space $L^{1,(\gamma,\delta)}_{[-1,1]}$ at a specific point only.

These summability methods and some results of the Fourier–Jacobi series are presented in the following section to better understand and develop our results in random cases in the subsequent sections.

2 Preliminaries

Chripkó [3] discussed the Υ–summability of Fourier–Jacobi series of the form

$$\sum_{i=0}^{\infty} b_i q_i^{(\gamma,\delta)}(y) \tag{2.1}$$

in the space $C^{(\eta,\tau)}_{[-1,1]}$ of continuous functions.

Definition 2.1 *The space $C^{(\eta,\tau)}_{[-1,1]}$ is the class of real valued continuous functions f defined on $(-1,1)$ satisfying*

$$\lim_{|y|\to 1} (f\varrho^{(\eta,\tau)})(y) = 0.$$

It is a linear space and is equipped with the norm

$$||f||_{\varrho^{(\eta,\tau)}} := ||f\varrho^{(\eta,\tau)}|| := \max_{y\in[-1,1]} \left|(f\varrho^{(\eta,\tau)})(y)\right|,$$

where

$$\varrho^{(\eta,\tau)}(y) := (1-y)^{\eta}(1+y)^{\tau}$$

is the Jacobi weight with parameters $\eta, \tau \geq 0$.

The matrix Υ is

$$\Upsilon := \begin{bmatrix} \vartheta_{0,1} & & & \\ \vartheta_{0,2} & \vartheta_{1,2} & & \\ \vartheta_{0,3} & \vartheta_{1,3} & \vartheta_{2,3} & \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots\ddots & \end{bmatrix}, \tag{2.2}$$

where $\vartheta_{i,k} \in \mathbb{R}$. The Υ–sum of the Fourier series (2.1) in Jacobi polynomials is defined as

$$\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(y) := \sum_{i=0}^{k-1} \vartheta_{i,k} b_i q_i^{(\gamma,\delta)}(y), \tag{2.3}$$

where b_i are the Fourier–Jacobi coefficients of $f \in C_{[-1,1]}^{(\eta,\tau)}$ and $y \in [-1,1],\ k \in \mathbb{N}$.

Chripkó discussed some possible cases of the summation matrix Υ, which are stated below.

Case-1 If $\vartheta_{i,k} := 1$ $(i = 0, 1, 2 \ldots, k-1, k \in \mathbb{N})$, then the kth Jacobi series (2.3) of f is equal to the kth partial sum of the Jacobi series (2.1).

Case-2 Let $\varphi : [0,1] \to \mathbb{R}$ be a given function with $\varphi(0) = 1$ and $\varphi(1) = 0$. The summation matrix Υ can be represented by $\vartheta_{i,k} := \varphi\left(\dfrac{i}{k}\right)$ $(i = 0, 1, 2, \ldots, k-1, k \in \mathbb{N})$. This case is called φ-summation of the Fourier–Jacobi series (2.1).

Case-3 Let

$$\vartheta_{i,k} := 1 - \frac{i}{k}\ (i = 0, 1, \ldots, k-1, k \in \mathbb{N}).$$

Then $\Upsilon := \vartheta_{i,k}$ is the *Fejér summation matrix* and equal to $\varphi(t) := 1 - t,\ t \in [0,1]$. The kth Fejér means of the Fourier–Jacobi series (2.1) is defined as

$$\xi_k(y) := \sum_{i=0}^{k-1} \left(1 - \frac{i}{k}\right) b_i q_i^{(\gamma,\delta)}(y),\ y \in [-1,1].$$

Case-4 Let

$$\vartheta_{i,k} := \frac{\mathrm{B}_{k-i-1}^{(\phi)}}{\mathrm{B}_{k-1}^{(\phi)}}, (\phi \geq 0,\ i = 0, \ldots, k-1,\ k \in \mathbb{N}),$$

where $\mathrm{B}_0^{(\phi)} := 1,\ \mathrm{B}_m^{(\phi)} := \binom{m+\phi}{m} = \dfrac{(\phi+1)\ldots(\phi+m)}{m!},\ (m \in \mathbb{N})$. Then $\Upsilon := (\vartheta_{i,k})$ is the *Cesàro summation matrix* and the *Cesàro sums* of Fourier–Jacobi series (2.1) is

$$\sigma_k^{(\phi),\gamma,\delta}(y) := \sum_{i=0}^{k-1} \frac{B_{k-i-1}^{(\phi)}}{B_{k-1}^{(\phi)}} b_i q_i^{(\gamma,\delta)}(y),\ y \in [-1,1],\ k \in \mathbb{N} \tag{2.4}$$

$$= \int_{-1}^{1} \mathbf{F}_k^{(\phi),\gamma,\delta}(y,t) \varrho^{(\gamma,\delta)}(t) dt, \tag{2.5}$$

where

$$\mathbf{F}_k^{(\phi),\gamma,\delta}(y,t) := \sum_{i=0}^{k-1} \frac{\mathrm{B}_{k-i-1}^{(\phi)}}{\mathrm{B}_{i-1}^{(\phi)}} q_i^{(\gamma,\delta)}(t) q_i^{(\gamma,\delta)}(y).$$

If $\phi = 0$, then it is same as in the Case-1. For $\phi \geq 1$, the sum (2.4) is known as Cesàro (C, ϕ) sum of Jacobi series (2.1). In particular, if $\phi = 1$, then the Jacobi series (2.1) is *Fejer summable.*

Case-5 The (R, ν, μ) *Riesz summation* of the Fourier–Jacobi series (2.1) is defined by the summation function

$$\varphi_{\nu,\mu}(y) := (1 - y^{\nu})^{\mu},\ y \in [0, 1],$$

where $\nu,\ \mu \geq 0$ are real constant numbers.

Case-6 The *de la Vallée Poussin summation* is defined by the summation function

$$\varphi_u(t) := \begin{cases} 1, & \text{if } 0 \leq t \leq u \\ \dfrac{(t-1)}{(s-1)}, & \text{if } u < t \leq 1, \end{cases}$$

where $u \in (0, 1)$.

Case-7 The *Rogosinski summation* is defined by the summation function

$$\varphi(t) := \cos \frac{\pi t}{2},\ t \in [0, 1].$$

Choudhary [2] discussed the regular Nörlund summability (also known as Nörlund summability) of the Fourier–Jacobi series (2.1) of a function f in the $L^{p,(\gamma,\delta)}_{[-1,1]}$ space for $p = 1$.

Definition 2.2 *The space $L^{p,(\gamma,\delta)}_{[-1,1]}$ is the class of all measurable functions in the segment $[-1, 1]$ with Jacobi weight $\varrho^{(\gamma,\delta)}(t) := (1-t)^{\gamma}(1+t)^{\delta},\ \gamma, \delta > -1$, such that*

$$\int_{-1}^{1} |f(t)\varrho^{(\gamma,\delta)}(t)|^p dt < \infty.$$

The norm endowed in this space is

$$||f||_{L^{p,(\gamma,\delta)}_{[-1,1]}} = \Big\{ \int_{-1}^{1} |f(t)\varrho^{(\gamma,\delta)}(t)|^p dt \Big\}^{\frac{1}{p}} \text{ for } p \geq 1.$$

The regular Nörlund sum of a series is defined below:

Let $\{s_k\}$ be the kth partial sum of the series $\sum\limits_{i=0}^{\infty} a_i$. Consider the sequence $\{u_k\}$ of constants, real or complex and define $\mathrm{U}_k = u_0 + u_1 + \cdots + u_k$. Define the transformation h_k as

$$h_k := \frac{1}{\mathrm{U}_k} \sum_{i=0}^{k} u_{k-i} s_i = \frac{1}{\mathrm{U}_k} \sum_{i=0}^{k} u_i s_{k-i},\ \mathrm{U}_k \neq 0.$$

The h_k is said to be *Nörlund sums* of $\{s_k\}$, generated by $\{u_k\}$. The series $\sum\limits_{i=0}^{\infty} a_i$ is said to be *Nörlund summable* to the sum s if $\lim\limits_{k\to\infty} h_k$ exists and

equal to s.
The Nörlund sums of the Fourier series in Jacobi polynomials at $y = 1$ is

$$\mathbf{h}_k^{(\gamma,\delta)}(y) := \frac{1}{\mathrm{U}_k}\sum_{i=0}^{k} u_{k-i}\mathbf{s}_i^{(\gamma,\delta)}(1),$$

where $\mathbf{s}_i^{(\gamma,\delta)}(1)$ is the ith partial sum of the Fourier–Jacobi series (2.1) at $y = 1$. We use the symbol $(\mathbf{N}, u_n)$ for it. The following theorem is established by Choudhary.

Theorem 2.3 *[2] Let $\{u_k\}$ be the real non–negative monotonic non-increasing sequence, such that U_k tends to ∞ as k tends to ∞. If*

$$\Psi(t) := \int_0^t |F(\psi)|d\psi = O\Big[\frac{u(1/t)}{\mathrm{U}(1/t)}t^{(2\gamma+1)}\Big] \text{ as } t \to 0, \tag{2.6}$$

where

$$F(\psi) := \Big[f(\cos\psi) - A\Big]\Big(\sin\frac{\psi}{2}\Big)^{(2\gamma+1)}\Big(\cos\frac{\psi}{2}\Big)^{(2\delta+1)},$$

A is a constant and

$$\sum_{k=0}^{\infty}\frac{k^{\gamma+1/2}}{\mathrm{U}_k} < \infty, \tag{2.7}$$

then the Jacobi series (2.1) of $f \in L^{1,(\gamma,\delta)}_{[-1,1]}$ is $(\mathbf{N}, u_n)$ summable at $y = 1$ to the sum A with $-1/2 \le \gamma < 1/2$, $\delta > -1/2$ and the condition

$$\int_{-1}^{b}(1+t)^{\delta/2-3/4}|f(t)|dt \tag{2.8}$$

is finite for fixed b.

Raghuvanshi [22] discussed the generalized Nörlund summability of the Fourier series (2.1) in Jacobi polynomials. The generalized Nörlund summability of a series is defined as follows:
Let $\{\mathrm{r}_k\}$ and $\{\mathrm{q}_k\}$ be positive real constants sequences with R_k and Q_k as their kth partial sums respectively, and let

$$(\mathrm{q}*\mathrm{r})_k := \sum_{i=0}^{k}\mathrm{q}_{k-i}\mathrm{r}_i = \sum_{i=0}^{k}\mathrm{q}_i\mathrm{r}_{k-i} \to \infty$$

as $k \to \infty$. As in [1], define the transformation as

$$t_k^{\mathrm{q},\mathrm{r}} := \frac{1}{(\mathrm{q}*\mathrm{r})_k}\sum_{i=0}^{k}\mathrm{q}_{k-i}\mathrm{r}_i\mathrm{s}_i.$$

If $t_k^{\mathrm{q,r}} \to s$ as $k \to \infty$, then the infinite series $\sum_{k=0}^{\infty} \mathbf{a}_k$ is said to be *generalized Nörlund summable* to s. We use the notation $|\mathbf{N}, \mathrm{q}_k, \mathrm{r}_k|$ for generalized Nörlund summation. The $|\mathbf{N}, \mathrm{q}_k, \mathrm{r}_k|$ means of the Fourier series (2.1) in Jacobi polynomials at $y = 1$ is

$$\mathbf{u}_k^{(\gamma,\delta)}(1) := \frac{1}{(\mathrm{q} * \mathrm{r})_k} \sum_{i=0}^{k} \mathrm{q}_i \mathrm{r}_{k-i} \mathbf{s}_i^{(\gamma,\delta)}(1),$$

where $\mathbf{s}_i^{(\gamma,\delta)}(1)$ is the ith partial sum of the Fourier–Jacobi series (2.1) at the point $y = 1$. Following is the result on generalized Nölund summability of the Fourier–Jacobi series (2.1) by Raghuvanshi.

Theorem 2.4 *[22] Let* $\{\mathrm{q}_k\}$, $\{\mathrm{r}_k\}$ *be the non–negative real constants sequences and let* $\gamma > -1/2$, $\delta - \gamma > 1$, $\delta + \gamma \geq -1$ *such that*

$$\sum_{i=2}^{k} \frac{(\mathrm{q} * \mathrm{r})_i}{i^{\gamma+(1/2)} \log i} = O\Big(\frac{(\mathrm{q} * \mathrm{r})_k}{k^{\gamma+(1/2)}}\Big) \; as \; k \to \infty. \tag{2.9}$$

Also suppose that

$$\int_{1-t}^{t} |f(u) - A| du = O\Big(\frac{t}{\log(1/t)}\Big) \; as \; t \to 0, \tag{2.10}$$

and the condition

$$\int_{-1}^{b} (1+t)^{(\delta-\gamma-1)/2} |f(t)| dt < \infty, \; b \; fixed, \tag{2.11}$$

are satisfied, then the series (2.1) is $|\mathbf{N}, \mathrm{q}_k, \mathrm{r}_k|$ *summable to the sum A at* $y = 1$.

Dhakal [4] studied the lower triangular matrix summability of Jacobi series. This method is defined as follows:
Let M=$(\mathbf{b}_{k,i})$ be an infinite lower triangular matrix satisfying the Silverman–Töeplitz conditions [26] of regularity, i.e.,

$$\sum_{i=0}^{k} \mathbf{b}_{k,i} \to 1 \text{ as } k \to \infty,$$

where $\mathbf{b}_{k,i} = 0$, $i > k$ and $\sum_{i=0}^{k} |\mathbf{b}_{k,i}| \leq \mathbf{C}$, for a finite +ve constant $\mathbf{C}$.
Let $\{\mathrm{t}_k\}$ be the lower triangular matrix sums of $\{s_k\}$, generated by $(\mathbf{b}_{k,i})$, i.e., the transformation is

$$\mathrm{t}_k := \sum_{i=0}^{k} \mathbf{b}_{k,i} s_i.$$

If $\mathrm{t}_k \to s$ as $k \to \infty$, then the series $\sum_{k=0}^{\infty} \mathbf{a}_k$ is said to be *lower triangular matrix summable* to s. Use the notation $(\mathbf{T})$ for lower triangular matrix summation.
The $(\mathbf{T})$ summation of Jacobi series (2.1) at $y = 1$ is

$$\mathbf{t}_k^{(\gamma,\delta)}(1) := \sum_{i=0}^{k} \mathbf{b}_{k,i} \mathbf{s}_i^{(\gamma,\delta)}(1),$$

where $\mathbf{s}_i^{(\gamma,\delta)}(1)$ is the ith sum of Fourier–Jacobi series (2.1) at $y = 1$. The following result is established by Dhakal.

Theorem 2.5 *[4] Let* $\mathrm{M} = (\mathbf{b}_{k,i})$ *be an infinite lower triangular regular matrix such that* $(\mathbf{b}_{k,i})$ *is positive, monotonic increasing in* i, $0 \le i \le k$,

$$D_{k,\tau} = \sum_{i=k-\tau}^{k} \mathbf{b}_{k,i},\ D_{k,k} = 1 \ for\ all\ k, \tag{2.12}$$

and

$$k^{\gamma+1/2} D_{k,[1/\beta]} = o(1), 0 < \beta < \pi\ as\ k \to \infty, \tag{2.13}$$

where $\tau = Integral\ part\ of\ \dfrac{1}{\phi} = \left[\dfrac{1}{\phi}\right]$ *and* $0 \le \phi \le \pi$. *For* $-1/2 \le \gamma \le 1/2$, $\delta > -1/2$, *if*

$$\int_{1-t}^{1} |f(u) - A| du = o\Big(\frac{t}{\xi(1/t)\log(1/t)}\Big)\ as\ t \to 0, \tag{2.14}$$

then the Jacobi series (2.1) is $(\mathbf{T})$ *summable to the sum* A *at* $t = 1$ *provided* $\xi(t)$ *is positive monotonic non–decreasing function of* t *such that*

$$\sum_{i=a}^{k} \frac{D_{k,i}}{i^{(2\gamma+3)/2}\xi(i)\log i} = O\Big(\frac{1}{k^{(2\gamma+1)/2}}\Big), \tag{2.15}$$

and satisfy the condition

$$\int_0^{\frac{1}{k}} t^{\delta-\frac{1}{2}} |f(-\cos t) - A| dt = o(1)\ as\ k \to \infty. \tag{2.16}$$

3 Summability of random Jacobi series in the space $C^{(\eta,\tau)}_{[-1,1]}$

3.1 Υ–Summability

The random Fourier–Jacobi series that we consider here is

$$\sum_{i=0}^{\infty} b_i B_i(\varpi) q_i^{(\gamma,\delta)}(y), \tag{3.1}$$

where the scalars b_i are Fourier–Jacobi coefficients of a function f in $C^{(\eta,\tau)}_{[-1,1]}$, $\eta, \tau \geq 0$ defined as in (1.8), and the random variables $B_i(\varpi)$ defined as in (1.6) are Fourier–Jacobi coefficients of the symmetric stable process $Z(t,\varpi)$ of index $\chi,\ 1 \leq \chi \leq 2$.

The Υ–sum of the random series (3.1) is

$$\mathbf{S}_k^{\Upsilon,(\gamma,\delta)}(y,\varpi) := \sum_{i=0}^{k-1} \vartheta_{i,k} b_i B_i(\varpi) q_i^{(\gamma,\delta)}(y). \tag{3.2}$$

We will use the following notations on Υ–summation matrix same as in Chripkó [3].

$$\mathrm{T}_1 : \lim_{k\to\infty}(1-\vartheta_{i,k}) = 0 \text{ for all fixed } i = 0,1,2,\ldots.$$

$$\mathrm{T}_2 : \vartheta_{k-1,k} = O\Big(\frac{1}{k}\Big),\ (k \in \mathbb{N}).$$

$$\mathrm{T}_3 : \triangle^2 \vartheta_{i-1,k} = O\Big(\frac{1}{k^2}\Big),\ (i = 1,2,\ldots,k-1, k \in \mathbb{N}).$$

$$\mathrm{T}_4 : \triangle^2 \vartheta_{i-1,k}\ (i = 1,2,\ldots,k-1, k \in \mathbb{N}) \text{ is of constant sign.}$$

$$\mathrm{T}_5 : \operatorname{sgn} \triangle^2 \vartheta_{i-1,k} = \operatorname{sgn} \vartheta_{k-1,k} (i = 1,2,\ldots,k-1, k \in \mathbb{N}),$$

where $\triangle^2 \vartheta_{i,k} := \triangle\vartheta_{i+1,k} - \triangle\vartheta_{i,k}$, $\quad \triangle\vartheta_{i,k} := \vartheta_{i+1,k} - \vartheta_{i,k}$, $(\vartheta_{k,k} := 0)$.
Furthermore, the following notations will be used.
(A) if (T_1), (T_2) and (T_3) hold,
(B) if (T_1), (T_2) and (T_4) hold,
(C) if (T_1), (T_5) hold.
The convergence of Υ–sum (3.2) of the random Jacobi series (3.1) is proved in Theorem 3.2 by imposing some restrictions on $\gamma, \delta, \eta, \tau$ and the entries of summation matrix Υ. This theorem uses the notation $f(y,t) := \sum_{i=0}^{\infty} b_i q_i^{(\gamma,\delta)}(y) q_i^{(\gamma,\delta)}(t)$, which is Υ–summable and Cesàro summable for $f \in C^{(\eta,\tau)}_{[-1,1]}$, since $q_i^{(\gamma,\delta)}(y)$ are bounded in $(-1,1)$. In the subsequent section, this notation $f(y,t)$ will also be used with the space $L^{p,(\gamma,\delta)}_{[-1,1]}$, as it is summable in some means there.

To prove this theorem, we require the following result.

Lemma 3.1 *[16] Let $Z(t,\varpi)$ be a symmetric stable process of index χ, $1 \leq \chi \leq 2$ and $f(t)$ be any function in $L^p_{[a,b]}$, $p \geq 1$, then for all $\epsilon > 0$,*

$$P\left(\left|\int_a^b f(t)dZ(t,\varpi)\right| > \epsilon\right) \leq \frac{C2^{\chi+1}}{(\chi+1)\epsilon'^{\chi}}\int_a^b |f(t)|^{\chi}dt,$$

where C is a positive constant and $\epsilon' < \epsilon$.

Theorem 3.2 *Let $Z(t,\varpi)$ be a symmetric stable process of index χ, $1 \leq \chi \leq 2$. The Υ–sum (3.2) of random series (3.1) converges to the stochastic integral*

$$\int_{-1}^{1} f(y,t)\varrho^{(\eta,\tau)}(t)dZ(t,\varpi) \tag{3.3}$$

in probability if the parameters $\gamma, \delta \geq -1/2, \eta, \tau \geq 0$ satisfy the following conditions

$$\frac{\gamma}{2} - \frac{1}{4} < \eta < \frac{\gamma}{2} + \frac{3}{4} \text{ and } \frac{\delta}{2} - \frac{1}{4} < \tau < \frac{\delta}{2} + \frac{3}{4}, \tag{3.4}$$

with the entries of Υ–matrix satisfying either (A), (B) or (C).

Proof. Consider the kth Υ–sum of the random Fourier series (3.1) involving Jacobi polynomials as

$$\mathbf{S}_k^{\Upsilon,(\gamma,\delta)}(y,\varpi) := \sum_{i=0}^{k-1} \vartheta_{i,k} b_i B_i(\varpi) q_i^{(\gamma,\delta)}(y). \tag{3.5}$$

The integral form of kth sum (3.5) is

$$\begin{aligned}
\mathbf{S}_k^{\Upsilon,(\gamma,\delta)}(y,\varpi) &:= \sum_{i=0}^{k-1} \vartheta_{i,k} b_i B_i(\varpi) q_i^{(\gamma,\delta)}(y) \\
&= \sum_{i=0}^{k-1} \vartheta_{i,k} b_i q_i^{(\gamma,\delta)}(y) \int_{-1}^{1} q_i^{(\gamma,\delta)}(t)\varrho^{(\eta,\tau)}(t)dZ(t,\varpi) \\
&= \int_{-1}^{1} \sum_{i=0}^{k-1} \vartheta_{i,k} b_i q_i^{(\gamma,\delta)}(y) q_i^{(\gamma,\delta)}(t)\varrho^{(\eta,\tau)}(t)dZ(t,\varpi) \\
&= \int_{-1}^{1} \mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(y,t)\varrho^{(\eta,\tau)}(t)dZ(t,\varpi),
\end{aligned}$$

where

$$\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(y,t) := \sum_{i=0}^{k-1} \vartheta_{i,k} b_i q_i^{(\gamma,\delta)}(t) q_i^{(\gamma,\delta)}(y)$$

is the kth partial sum of the Fourier–Jacobi series of f.

Now

$$\begin{aligned}&\mathbf{S}_k^{\Upsilon,(\gamma,\delta)}(y,\varpi)-\mathbf{S}_m^{\Upsilon,(\gamma,\delta)}(y,\varpi)\\&=\int_{-1}^{1}\Big(\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(y,t)-\mathbf{s}_m^{\Upsilon,(\gamma,\delta)}(y,t)\Big)\varrho^{(\eta,\tau)}(t)dZ(t,\varpi).\end{aligned}$$

By inequality in Lemma 3.1,

$$\begin{aligned}&P\Big(\Big|\mathbf{S}_k^{\Upsilon,(\gamma,\delta)}(y,t)-\mathbf{S}_m^{\Upsilon,(\gamma,\delta)}(y,t)\Big|>\epsilon\Big)\\&=P\Big(\Big|\Big(\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(y,t)-\mathbf{s}_m^{\Upsilon,(\gamma,\delta)}(y,t)\Big)\varrho^{(\eta,\tau)}(t)dZ(t,\omega)\Big|>\epsilon\Big)\\&\leq\frac{C2^{\chi+1}}{(\chi+1)\epsilon'^{\chi}}\int_{-1}^{1}\Big|\Big(\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(y,t)-\mathbf{s}_m^{\Upsilon,(\gamma,\delta)}(y,t)\Big)\varrho^{(\eta,\tau)}(t)\Big|^{\chi}dt,\ \text{for}\ \epsilon'<\epsilon.\end{aligned}$$

Invoking Chripkó result (see [3, Theorem 3.2]), we know that $\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}$ converges uniformly in $C_{[-1,1]}^{(\eta,\tau)}$, if b_i are the Fourier–Jacobi coefficients of a function in $C_{[-1,1]}^{(\eta,\tau)}$ and γ,δ,η,τ satisfy the conditions in equation (3.4). Hence $\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(y)$ is a Cauchy sequence. This yields, for $\chi\in[1,2]$ and $f\in C_{[-1,1]}^{(\eta,\tau)}$,

$$\lim_{k,m\to\infty}\int_{-1}^{1}\Big|\Big(\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(y,t)-\mathbf{s}_m^{\Upsilon,(\gamma,\delta)}(y,t)\Big)\varrho^{(\eta,\tau)}(t)\Big|^{\chi}dt=0.$$

This result holds uniformly for all y,t in the range $[-1,1]$, implying that $\mathbf{S}_k^{\Upsilon,(\gamma,\delta)}(y,\varpi)$ is a Cauchy sequence that converges in the sense of probability. Hence, it will converge to a random variable. Now we will show that the sequence $\mathbf{S}_k^{\Upsilon,(\gamma,\delta)}(y,\varpi)$ converges to the integral $\int_{-1}^{1}f(y,t)\varrho^{(\eta,\tau)}(t)dZ(t,\varpi)$ in probability. We know that the integral (3.3) exists in probability according to [13]. By Lemma 3.1,

$$\begin{aligned}&P\left(\left|\mathbf{S}_k^{\Upsilon,(\gamma,\delta)}(y,\varpi)-\int_{-1}^{1}f(y,t)\varrho^{(\eta,\tau)}(t)dZ(t,\varpi)\right|>\epsilon\right)\\&=P\left(\left|\int_{-1}^{1}\Big(\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(y,t)-f(y,t)\Big)\varrho^{(\eta,\tau)}(t)dZ(t,\varpi)\right|>\epsilon\right)\\&\leq\frac{C2^{\chi+1}}{(\chi+1)\epsilon'^{\chi}}\int_{-1}^{1}\Big|\Big(\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(y,t)-f(y,t)\Big)\varrho^{(\eta,\tau)}(t)\Big|^{\chi}dt\ \text{for}\ \epsilon'<\epsilon.\end{aligned}$$

If γ,δ,η,τ satisfy the conditions in (3.4), then by Chripkó [3, Theorem 3.2],

$$\lim_{k\to\infty}\int_{-1}^{1}\Big|\Big(\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(t)-f(t)\Big)\varrho^{(\eta,\tau)}(t)\Big|dt=0.$$

Hence, for $\chi \in [1,2]$,

$$\lim_{k\to\infty}\int_{-1}^{1}\left|\left(\mathbf{s}_k^{\Upsilon,(\gamma,\delta)}(y,t)-f(y,t)\right)\varrho^{(\eta,\tau)}(t)\right|^{\chi}dt=0.$$

This proves that the Υ–sum (3.2) of random Fourier–Jacobi series (3.1) converges in probability to the integral (3.3).

□

3.2 $(C,\phi),\ \phi\geq 1$ summability

The following theorem establishes the $(C,\phi),\phi\geq 1$ summability of the random Jacobi series (3.1) if b_i are the Fourier–Jacobi coefficients of a function f in the space $C^{(\eta,\tau)}_{[-1,1]}$.

Theorem 3.3 *The random Fourier series (3.1) in Jacobi polynomials is Cesàro $(C,\phi),\ \phi\geq 1$ summable in probability to the integral (3.3), if $\eta,\ \tau\geq 0,\gamma,\delta\geq -1/2$ and entries of the Υ–matrix satisfy the conditions (3.4) in Theorem 3.2.*

Proof. The kth $(C,\phi),\phi\geq 1$ means of random Fourier–Jacobi series (3.1) is

$$\mathbf{Q}_k^{(\phi),\gamma,\delta}(y,\varpi):=\sum_{i=0}^{k-1}\frac{\mathrm{B}_{k-i-1}^{(\phi)}}{\mathrm{B}_{k-1}^{(\phi)}}\mathbf{S}_i^{(\gamma,\delta)}(y,\varpi),\tag{3.6}$$

where $\mathrm{B}_m^n=\dfrac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)}$ and $\mathbf{S}_i^{(\gamma,\delta)}(y,\varpi)$ is the ith partial sum of random series (3.1). The (C,ϕ) means of (3.6) can be expressed in the integral form as

$$\begin{aligned}\mathbf{Q}_k^{(\phi),\gamma,\delta}(y,\varpi)&:=\sum_{i=0}^{k-1}\frac{\mathrm{B}_{k-i-1}^{(\phi)}}{\mathrm{B}_{k-1}^{(\phi)}}\mathbf{S}_i^{(\gamma,\delta)}(y,\varpi)\\&=\int_{-1}^{1}\sigma_k^{(\phi),\gamma,\delta}(y,t)\varrho^{(\eta,\tau)}(t)dZ(t,\varpi),\end{aligned}$$

where

$$\sigma_k^{(\phi),\gamma,\delta}(y,t):=\sum_{i=0}^{k-1}\frac{\mathrm{B}_{k-i-1}^{(\phi)}}{\mathrm{B}_{k-1}^{(\phi)}}\mathbf{s}_i^{(\gamma,\delta)}(y,t),$$

and $\mathbf{s}_i^{(\gamma,\delta)}(y)$ is the ith partial sum of Fourier–Jacobi series (2.1).
Now

$$\begin{aligned}&\mathbf{Q}_k^{(\phi),\gamma,\delta}(y,\varpi)-\mathbf{Q}_m^{(\phi),\gamma,\delta}(y,\varpi)\\&=\int_{-1}^{1}\left(\sigma_k^{(\phi),\gamma,\delta}(y,t)-\sigma_m^{(\phi),\gamma,\delta}(y,t)\right)\varrho^{(\eta,\tau)}(t)dZ(t,\omega).\end{aligned}$$

By the inequality in Lemma 3.1, we obtain $\mathbf{Q}_k^{(\phi),\gamma,\delta}(y,\varpi)$ is a Cauchy sequence. With the help of Chripkó result (see [3, Corollary 3.4]) on $(C,\phi), \phi \geq 1$ summation and the Lemma 3.1, the random series (3.1) is Cesàro summable in probability to the stochastic integral (3.3).

□

4 Example

Cesàro $(C,1)$ summability:

Example 4.1 Let $f(y) = 1,\ y \in [-1,1]$ and f is in the space $C_{[-1,1]}^{(\eta,\tau)}, \eta, \tau \geq 0$. The Fourier–Jacobi coefficients of f are

$$b_0 = 1,\ b_k = 0 \text{ for } k \geq 1.$$

The Fourier–Jacobi series of f is

$$\mathbf{s}^{(\gamma,\delta)}(y) := b_0 q_0^{(\gamma,\delta)}(y), \text{ where } \gamma, \delta \geq 0.$$

Since, $q_0^{(\gamma,\delta)}(y) = 1$ for $\gamma, \delta \geq 0,\ \mathbf{s}^{(\gamma,\delta)}(y) = b_0 = 1$ and hence the kth partial sum

$$\mathbf{s}_k^{(\gamma,\delta)}(y) = 1,\ k \in \mathbb{N} \cup \{0\}.$$

The kth $(C,1)$ sums of the Fourier–Jacobi series $\mathbf{s}^{(\gamma,\delta)}(y)$ is

$$\sigma_k^{(\phi),\gamma,\delta}(y) := 1 \text{ for } \phi = 1,\ k \in \mathbb{N} \cup \{0\}.$$

Consider the kth partial sum of the random Fourier–Jacobi series

$$\begin{aligned}\mathbf{S}_k^{(\gamma,\delta)}(y,\varpi) &:= \sum_{i=0}^{k} b_i B_i(\varpi) q_i^{(\gamma,\delta)}(y) \\ &= B_0(\varpi) \\ &= \int_{-1}^{1} dZ(t,\varpi).\end{aligned}$$

The kth $(C,1)$ sums of the random Fourier–Jacobi series $\mathbf{S}^{(\gamma,\delta)}(y,\varpi)$ is

$$\mathbf{Q}_k^{(\phi),\gamma,\delta}(y,\varpi) = \int_{-1}^{1} dZ(t,\varpi) \text{ for } \phi = 1 \text{ and } k \in \mathbb{N} \cup \{0\}.$$

Now,

$$\begin{aligned}&P\Big(\varpi : \Big|\mathbf{Q}_n^{(1),\gamma,\delta}(y,\varpi) - \int_{-1}^{1} dZ(t,\varpi)\Big| > \epsilon\Big) \\ &= P\Big(\varpi : \Big|B_0(\varpi) - \int_{-1}^{1} dZ(t,\varpi)\Big| > \epsilon\Big) \\ &= 0,\end{aligned}$$

since $\{\varpi : \left|B_0(\varpi) - \int_{-1}^{1} dZ(t,\varpi)\right| > \epsilon\}$ is a null set.

Example 4.2 Let $f(y) = y,\ y \in [-1,1]$. The function f is in the space $C^{(\eta,\tau)}_{[-1,1]}, \eta, \tau \geq 0$. For $\gamma = 0$ and $\delta = 0$, we obtain the Jacobi polynomials

$$\begin{aligned} q_0(y) &= 1, \\ q_1(y) &= y, \\ q_2(y) &= \frac{1}{2}(3y^2 - 1), \\ q_3(y) &= \frac{1}{2}(5y^3 - 3y), \\ q_4(y) &= \frac{1}{8}(35y^4 - 30y^2 + 3). \end{aligned}$$

The Fourier–Jacobi coefficients of f are

$$b_0 = 0,\ b_1 = 1,\ b_2 = 0,\ b_3 = 0\ b_4 = 0.$$

The Fourier–Jacobi series of f is

$$\mathbf{s}^{(\gamma,\delta)}(y) := b_1 q_1^{(\gamma,\delta)}(y) \text{ for } \gamma = 0, \delta = 0.$$

Since $q_1^{(\gamma,\delta)}(y) = y$ for $\gamma = 0, \delta = 0,\ \mathbf{s}^{(\gamma,\delta)}(y) = b_1 = y$ and hence all kth partial sum $\mathbf{s}_k^{(0,0)}(y) = y,\ k \in \mathbb{N} \cup \{0\}$.
The kth $(C,1)$ sums of the Fourier–Jacobi series $\mathbf{s}^{(\gamma,\delta)}(y)$ is

$$\sigma_k^{(\phi),(\gamma,\delta)}(y) := y \text{ for } \phi = 1,\ k \in \mathbb{N} \cup \{0\} \text{ and } \gamma = 0, \delta = 0.$$

Consider the kth partial sum of random Fourier–Jacobi series

$$\mathbf{S}_k^{(\gamma,\delta)}(y,\varpi) := \sum_{i=0}^{k} b_i B_i(\varpi) q_i^{(\gamma,\delta)}(y).$$

For $\gamma = 0, \delta = 0$ and $k = 0$, the value of $\mathbf{S}_0^{(0,0)}(y,\varpi) = 0$. For all kth partial sum of the random Fourier–Jacobi series for $k \geq 1$ is

$$\begin{aligned} \mathbf{S}_k^{(0,0)}(y,\varpi) &:= \sum_{i=1}^{k} b_i B_i(\varpi) q_i^{(0,0)}(y) \\ &= B_1(\varpi) \\ &= \int_{-1}^{1} y dZ(t,\varpi). \end{aligned}$$

The kth $(C,1)$ sums of random Fourier–Jacobi series $\mathbf{S}^{(\gamma,\delta)}(y,\varpi)$ is

$$\mathbf{Q}_k^{(\phi),(\gamma,\delta)}(y,\varpi)=\int_{-1}^{1} dZ(t,\varpi),\ \text{for}\ \phi=1,\ \gamma=0,\delta=0\ \text{and}\ k\in\mathbb{N}\cup\{0\}.$$

Now,

$$\begin{aligned}&P\Big(\varpi:\Big|\mathbf{Q}_k^{(1),\gamma,\delta}(y,\varpi)-\int_{-1}^{1} dZ(t,\varpi)\Big|>\epsilon\Big)\\&=P\Big(\varpi:\Big|B_1(\varpi)-\int_{-1}^{1} dZ(t,\varpi)\Big|>\epsilon\Big)\\&=0.\end{aligned}$$

Since $\{\varpi:\Big|B_1(\varpi)-\int\limits_{-1}^{1} ydZ(t,\varpi)\Big|>\epsilon\}$ is a null set.

Note:
The summability of random Fourier–Jacobi series (3.1) in the sense of probability to the stochastic integral (3.3) can be established in some other sums like:
(a) *Riesz* (R,μ,ν) for $\mu,\nu\geq 1$,
(b) *de la Vallée Poussin* for every $s\in(0,1)$,
(c)*Rogosinski*
under the same conditions (3.4) in Theorem 3.2, and using the Lemma 3.1 and Corollary 3.4 in [3] for f in $C_{[-1,1]}^{(\eta,\tau)}$, $\eta,\tau\geq 0$ and $\gamma,\delta\geq -1/2$.

5 Summability of random Jacobi series in $L_{[-1,1]}^{1,(\gamma,\delta)}$ space

Consider the random Fourier series in Jacobi polynomials (3.1), where b_i are the Fourier–Jacobi coefficients of a function $f\in L_{[-1,1]}^{1,(\gamma,\delta)}$, $\gamma,\delta\geq 0$ and $B_i(\varpi)$ are Fourier–Jacobi coefficients of symmetric stable process $Z(t,\varpi)$ of index 1 defined as (1.6). In the next theorem it is shown that the $(\mathbf{N},u_k)$ sums of random series (3.1) converges to the stochastic integral

$$\int_{-1}^{1} A\rho^{(\gamma,\delta)}(t)dZ(t,\varpi) \tag{5.1}$$

in probability, where A is a positive fixed constant.

Theorem 5.1 *If* U_k *is the sum of the first* k *terms of the non–negative, non–increasing sequence* $\{u_k\}$, *which tends to* ∞ *as* $k\to\infty$ *satisfying the conditions (2.6), (2.7) along with the condition (2.8) stated in Theorem 2.3 for* $0\leq\gamma<1/2$, $\delta\geq 0$, *then the random series (3.1) is* $(\mathbf{N},u_k)$ *summable in probability to the stochastic integral (5.1) at the point one.*

Proof. Let

$$\mathbf{S}_k^{(\gamma,\delta)}(y,\varpi) := \sum_{i=0}^{k} b_i B_i(\varpi) q_i^{(\gamma,\delta)}(y) \tag{5.2}$$

be the kth partial sum of the random Fourier–Jacobi series (3.1). For $y = 1$, the integral form of $\mathbf{S}_k^{(\gamma,\delta)}(1,\varpi)$ is

$$\begin{aligned}\mathbf{S}_k^{(\gamma,\delta)}(1,\varpi) &:= \sum_{i=0}^{k} b_i B_i(\varpi) q_i^{(\gamma,\delta)}(1) \\ &= \int_{-1}^{1} \mathbf{s}_k^{(\gamma,\delta)}(1,t)\varrho^{(\gamma,\delta)}(t)dZ(t,\varpi),\end{aligned} \tag{5.3}$$

where

$$\mathbf{s}_k^{(\gamma,\delta)}(1,t) := \sum_{i=0}^{k} b_i q_i^{(\gamma,\delta)}(t) q_i^{(\gamma,\delta)}(1).$$

The $(\mathbf{N}, u_k)$ sums of the random series (3.1) at the point $y = 1$ is

$$\mathbf{H}_k^{(\gamma,\delta)}(1,\varpi) := \frac{1}{\mathrm{U}_k} \sum_{i=0}^{k} u_{k-i} \mathbf{S}_i^{(\gamma,\delta)}(1,\varpi). \tag{5.4}$$

The integral form of $(\mathbf{N}, u_k)$ sums (5.4) is

$$\begin{aligned}\mathbf{H}_k^{(\gamma,\delta)}(1,\varpi) &:= \frac{1}{\mathrm{U}_k} \sum_{i=0}^{k} u_{k-i} \mathbf{S}_i^{(\gamma,\delta)}(1,\varpi) \\ &= \int_{-1}^{1} \mathbf{h}_k^{(\gamma,\delta)}(1,t)\varrho^{(\gamma,\delta)}(t)dZ(t,\varpi),\end{aligned}$$

where

$$\mathbf{h}_k^{(\gamma,\delta)}(1,t) := \frac{1}{\mathrm{U}_k} \sum_{i=0}^{k} u_{k-i} \mathbf{s}_i^{(\gamma,\delta)}(1,t).$$

By Lemma 3.1, for $\chi = 1$,

$$\begin{aligned}&P\Big(\Big|\mathbf{H}_k^{(\gamma,\delta)}(1,\varpi) - \mathbf{H}_m^{(\gamma,\delta)}(1,\varpi)\Big| > \epsilon\Big) \\ &\leq \frac{2C}{\epsilon'} \int_{-1}^{1} \Big|\Big(\mathbf{h}_k^{(\gamma,\delta)}(1,t) - \mathbf{h}_m^{(\gamma,\delta)}(1,t)\Big)\varrho^{(\gamma,\delta)}(t)\Big| dt.\end{aligned}$$

If the equations (2.6) and (2.7) are satisfied by U_k with the condition (2.8), then the Nörlund $(\mathbf{N}, u_k)$ sums of the Fourier–Jacobi series (2.1) at the point $y = 1$ converges to A. So, $\{\mathbf{h}_k^{(\gamma,\delta)}(1,t)\}$ forms a Cauchy sequence.

This implies $\mathbf{H}_k^{(\gamma,\delta)}(1,\varpi)$ is a Cauchy sequence and hence will converge in probability to a random variable. With the help of Lemma 3.1, we obtain

$$P\left(\left|\mathbf{H}_k^{(\gamma,\delta)}(1,\varpi) - \int_{-1}^{1} A\varrho^{(\gamma,\delta)}(t)dZ(t,\varpi)\right| > \epsilon\right)$$
$$\leq \frac{2C}{\epsilon'}\int_{-1}^{1}\left|\left(\mathbf{h}_k^{(\gamma,\delta)}(1,t) - A\right)\varrho^{(\gamma,\delta)}(t)\right|dt.$$

By Theorem 2.3,

$$\lim_{k\to\infty}\int_{-1}^{1}\left|\left(\mathbf{h}_k^{(\gamma,\delta)}(1,t) - A\right)\varrho^{(\gamma,\delta)}(t)\right|dt = 0.$$

This completes the proof of theorem.

□

The following two theorems state the generalized Nörlund summability and lower triangular summability of the random Fourier–Jacobi series (3.1). These can be proved by following the same steps as in Theorem 5.1 and using Lemma 3.1.

Theorem 5.2 *If* $\gamma \geq 0,\ \delta - \gamma > 1,\ \delta + \gamma > 0$ *and* $\{\mathrm{q}_k\},\ \{\mathrm{r}_k\}$ *are the non–negative real constants sequences satisfying the conditions (2.9), (2.10) and (2.11) stated in Theorem 2.4, then the random series (3.1) is* $|\mathbf{N}, \mathrm{q}_k, \mathrm{r}_k|$ *summable in probability to the stochastic integral (5.1) at* $y = 1$.

Theorem 5.3 *If* $0 \leq \gamma \leq 1/2,\ \delta \geq 0$ *and the infinite triangular regular matrix* $\mathrm{M} = (\mathbf{b}_{k,i})$ *be such that its entries* $(\mathbf{b}_{k,i})$ *are positive, monotonic increasing in* $i,\ 0 \leq i \leq k$, *and satisfy the conditions (2.12), (2.13), (2.14) and (2.15) with (2.16) stated in Theorem 2.5, then the* $(\mathbf{T})$ *sums of random Fourier–Jacobi series (3.1) converges in probability to the stochastic integral (5.1) at* $y = 1$.

Remark

We discussed the summability in the sense of probability of random Fourier–Jacobi series in the functions spaces $C_{[-1,1]}^{(\eta,\tau)}$ and $L_{[-1,1]}^{p,(\gamma,\delta)}$ for $p = 1$ and the parameters $\eta, \tau \geq 0,\ \gamma, \delta > -1$. We expect the summability in the sense of mean of random Fourier–Jacobi series (3.1) for the function $f \in C_{[-1,1]}^{(\eta,\tau)}$.

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References

1. Borwein, D. – *On product of sequences*, J. Lond. Math. Soc., **33** (1958), 352-357.
2. Choudhary, R.S. – *On Nörlund summability of Jacobi series*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **(8) 52** (1972), 644-652.
3. Chripkó, Á. – *Weighted approximation via Θ–summations of Fourier–Jacobi series*, Studia Sci. Math. Hungar., **47** (2010), no. 2, 139-154.
4. Dhakal, B.P. – *Summability of a Jacobi series by lower triangular matrix method*, Amer. J. Math. Anal., **1** (2013), no. 3, 42-47.
5. Grauer, J. – *Random noise generation using Fourier series*, J. Aircraft, **55** (2018), no. 3, 1-7.
6. Hida, T.; Kuo, H.H.; Potthoff, J.; Streit, L. – *White Noise. An infinite-dimensional calculus.*, Mathematics and its Applications, 253, Kluwer Academic Publishers Group, Dordrecht (1993).
7. Hunt, G.A. – *Random Fourier transforms*, Trans. Amer. Math. Soc., **71** (1951), 38-69.
8. Imekraz, R.; Robert, D.; Thomann, L. – *On random Hermite series*, Trans. Amer. Math. Soc., **368** (2016), no. 4, 2763-2792.
9. Kashima, K.; Aoyama, H.; Ohta, Y. – *Stable process approach to analysis of systems under heavy–tailed noise: modeling and stochastic linearization*, IEEE Trans. Automat. Control, **64** (2019), no. 4, 1344-1357.
10. Kvernadze, G. – *Uniform convergence of Fourier-Jacobi series*, J. Approx. Theory, **117** (2002), no. 2, 207-228.
11. Liu, Z.; Liu, S. – *Randomization of the Fourier transform*, Optics Letters, **32** (2007), no. 5, 478-480.
12. Liu, Z.; Liu, S. – *Random fractional Fourier transform*, Optics Letters, **32** (2007), no. 15, 2088-2090.
13. Lukacs, E. – *Stochastic convergence*, Second Ed., Probability and Mathematical Statistics, **30**, Academic Press, London (1975).
14. Maharana, P.; Sahoo, S. – *On the convergence of random Fourier–Jacobi series of continuous functions*, Commun. Math., **32** (2024), no. 1, 49-61.
15. Marian, P; Marian, T.A. – *On a power series involving classical orthogonal polynomials*, Romanian J. Phys., **55** (2010), no. 5-6, 631-644.
16. Nayak, C.; Pattanayak, S.; Mishra, M.N. – *Random Fourier–Stieltjes series associated with stable process*, Tohoku Math. J., **(2) 39** (1987), no. 1, 1-15.
17. Paley, R.E.A.C.; Zygmund, A. – *On some series of functions, I*, Proc. Camb. Philos. Soc., **26** (1930), 337-357.
18. Paley, R.E.A.C.; Zygmund, A. – *On some series of functions, III*, Proc. Camb. Philos. Soc., **28** (1932), 190-205.
19. Paley, R.E.A.C.; Wiener, N; Zygmund, A. – *Notes on random functions*, Math. Z., **37** (1933), 647-668.
20. Prasad, J.; Hayashi, H. – *On the uniform convergence of Fourier–Jacobi series*, SIAM J. Numer. Anal., **10** (1973), 23-27.
21. Rademacher, H. – *Einige Sätze über reihen von allgemeinen orthogonal functionen* (German), Math. Ann., **87** (1922), no. 1-2, 112-138.
22. Raghuvanshi, A.K. – *On $|N, q_n, r_n|$–summability of Jacobi series*, Int. J. Sci. Inno. Math. Res., **2** (2014), no. 2, 139-148.
23. Ram., M.; Davim, J.P. – *Soft computing techniques and applications in mechanical engineering*, IGI Global (2018).
24. Steinhaus, H. – *Les probabilités dénombrables et leur rapport à la théorie de la mesure* (French), Fund. Math., **4** (1923), 286-310.
25. Steinhaus, H. – *Über die wahrscheinlichkeit dafür, dass der konvergenzkries einer potenzreihe, ihre natürlich grenze ist* (German), Mathematische Zeitschrift, **31** (1930), 408-416.
26. Töeplitz, O. – *Über allgemeine lineare Mittelbildungen* (German), Prace Matematyczno-Fizyczne, **22** (1911), 113-119.

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