An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXXI, 2025, f. 2
https://doi.org/10.47743/anstim.2025.00008

Functions of bounded fractional differential variation: A new Banach algebra

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Abstract Prajapati and Kachhia [10] introduced the concept of functions of bounded fractional differential variation and proposed an open problem whether the class of functions of bounded fractional differential variation on a closed bounded interval is a Banach algebra. We have given a solution to this problem. For a sequence of functions, we have discussed a condition for the convergence of sequence of Caputo fractional derivatives to the Caputo fractional derivative of the limit function. We have proved an additive property of the total fractional differential variation with respect to closed intervals and given an equivalent condition for vanishing of the total fractional differential variation of a function.

Keywords functions of bounded variation · Caputo fractional derivatives · functions of bounded fractional differential variation

Mathematics Subject Classification (2010) 26A45 · 26A33 · 46J10 · 26A46

1 Introduction

Functions of bounded variation have drawn the attention of several researchers. Bhatt *et al.* [2] defined functions of bounded differential variation on a closed interval of $\mathbb{R}$ and studied its various properties. Prajapati and Kachhia [10] introduced a new concept of functions of bounded fractional differential variation on a closed interval of $\mathbb{R}$ and showed that the space of functions of bounded fractional differential variation on a closed, bounded interval is a normed algebra.

In this paper, we have proved that the total fractional differential variation is homogeneous with respect to functions and additive with respect to closed intervals. As an application of the additive property, we have proved that if a function f is absolutely fractional differentiable function on a closed bounded interval, then f is of bounded fractional differential variation. An equivalent condition for vanishing of the total fractional differential variation is also proved. We have established relationships between the total fractional differential variation of a function and the total variation of its

Caputo fractional derivative. For a sequence of functions, we have given a condition for the convergence of sequence of Caputo fractional derivatives to the Caputo fractional derivative of the limit function. As an application of these results, we have proved that $BFDV[a,b]$ is a Banach algebra.

Notations :

$\mathbb{N}$: The set of natural numbers
$\mathbb{Z}$: The set of integers
$\mathbb{R}$: The set of real numbers
$\mathbb{C}$: The set of complex numbers
$\Re(\alpha)$: Real part of a complex number α
$[a,b]$: The closed and bounded interval with end points $a, b \in \mathbb{R}$
P : A partition of $[a,b]$
${}^cD_a^\alpha f$: Caputo fractional derivative of order α on $[a,b]$
$C[a,b]$: The class of real valued continuous functions on $[a,b]$
$V(f,P)$: The variation of f with respect to P
$V(f,[a,b])$: The total variation of f on $[a,b]$
$BV[a,b]$: The class of functions of bounded variation on $[a,b]$
$FDV_l(f,P)$: The left fractional differential variation of f with respect to P
$FDV_r(f,P)$: The right fractional differential variation of f with respect to P
$FDV_l(f,[a,b])$: The total left fractional differential variation of f on $[a,b]$
$FDV_r(f,[a,b])$: The total right fractional differential variation of f on $[a,b]$
$FDV(f,[a,b])$: The total fractional differential variation of f on $[a,b]$
$BFDV[a,b]$: The class of functions of bounded fractional differential variation on $[a,b]$
$AFD[a,b]$: The class of absolutely fractional differentiable functions on $[a,b]$.

Definition 1.1 *([3], [7], [9]) Let $\alpha \in \mathbb{C}$ and $\Re(\alpha) \geq 0$. The* Caputo fractional derivative *of order α on $[a,b]$ is defined as*

$$ {}^cD_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1 < \Re(\alpha) < n). $$

The following mean value theorem for Caputo fractional derivatives ([5], [8]) will be useful for further study.

Theorem 1.2 *Let $0 < \alpha \leq 1$, $a < b$ and $f \in C[a,b]$ be such that ${}^cD_a^\alpha(f) \in C[a,b]$. Then, there exists some $\xi \in (a,b)$ such that*

$$ \frac{f(b)-f(a)}{(b-a)^\alpha} = \frac{1}{\Gamma(\alpha+1)} {}^cD_a^\alpha(f(\xi)). \tag{1.1} $$

Functions of bounded variation is defined in ([1], [11]). $BV[a,b]$ is a Banach algebra relative to the norm (Appell *et al.* [1] and Carothers [4])

$$||f||_{BV} = ||f||_\infty + V(f) \tag{1.2}$$

Bhatt *et al.* [2] defined functions of bounded differential variation as

Definition 1.3 *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function and $P := \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Define the* left differential variation *of f with respect to P by*

$$DV_l(f,P) = \sum_{k=1}^{n} \left| \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} - f'(x_{k-1}) \right|,$$

and the right differential variation *of f with respect to P by*

$$DV_r(f,P) = \sum_{k=1}^{n} \left| \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} - f'(x_k) \right|.$$

Define the total left differential variation *of f on $[a,b]$ by*

$$DV_l(f) = \sup\{DV_l(f,P) : \ P \text{ a partition of } [a,b]\},$$

and the total right differential variation *of f on $[a,b]$ by*

$$DV_r(f) = \sup\{DV_r(f,P) : \ P \text{ a partition of } [a,b]\}.$$

f is said to be of bounded left differential variation *on $[a,b]$ if $DV_l(f) < \infty$. Similarly, f is said to be of* bounded right differential variation *on $[a,b]$ if $DV_r(f) < \infty$. f is said to be of* bounded differential variation *on $[a,b]$ if $DV_l(f) < \infty$ and $DV_r(f) < \infty$.*

The total differential variation *of f on $[a,b]$ is defined as*

$$DV(f) = DV_l(f) + DV_r(f).$$

Prajapati and Kachhia [10] defined the function of bounded fractional differential variation by using Caputo fractional derivatives instead of first-order derivatives as follows:

Note that the number α satisfying $0 < \alpha \leq 1$ is fixed throughout the paper.

Definition 1.4 *Let $f\colon [a,b] \to \mathbb{R}$ be a Caputo fractional differentiable function and $P := \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Define the* left fractional differential variation *of f with respect to P by*

$$FDV_l(f,P) = \sum_{k=1}^{n} \left| \Gamma(\alpha+1) \left(\frac{f(x_k) - f(x_{k-1})}{(x_k - x_{k-1})^\alpha} \right) - {}^cD_a^\alpha f(x_{k-1}) \right|,$$

and the right fractional differential variation *of f with respect to P by*

$$FDV_r(f,P)=\sum_{k=1}^{n}\left|\Gamma(\alpha+1)\left(\frac{f(x_k)-f(x_{k-1})}{(x_k-x_{k-1})^{\alpha}}\right)-{}^{c}D_a^{\alpha}f(x_k)\right|.$$

Define the total left fractional differential variation *of f on $[a,b]$ by*

$$FDV_l(f)=\sup\{FDV_l(f,P):\ P \text{ a partition of } [a,b]\},$$

and the total right fractional differential variation *of f on $[a,b]$ by*

$$FDV_r(f)=\sup\{FDV_r(f,P):\ P \text{ a partition of } [a,b]\}.$$

f is said to be of bounded left fractional differential variation *on $[a,b]$ if $FDV_l(f)<\infty$. Similarly, f is said to be of* bounded right fractional differential variation *on $[a,b]$ if $FDV_r(f)<\infty$. f is said to be of* bounded fractional differential variation *of on $[a,b]$ if $FDV_l(f)<\infty$ and $FDV_r(f)<\infty$. The* total fractional differential variation *of f on $[a,b]$ is defined as*

$$FDV(f)=FDV_l(f)+FDV_r(f).$$

The class of functions of bounded fractional differential variation on $[a,b]$ is denoted by $BFDV[a,b]$.

Definition 1.5 *([6]) Let $0<\alpha\leq 1$. A Caputo fractional differentiable function f is said to be* absolutely fractional differentiable function *on interval I, if for a given $\epsilon>0$, there exists a $\delta>0$ such that*

$$\sum_{i=1}^{n}\left|\Gamma(\alpha+1)\left(\frac{f(b_i)-f(a_i)}{(b_i-a_i)^{\alpha}}\right)-{}^{c}D_a^{\alpha}(f(a_i))\right|<\epsilon$$

and

$$\sum_{i=1}^{n}\left|\Gamma(\alpha+1)\left(\frac{f(b_i)-f(a_i)}{(b_i-a_i)^{\alpha}}\right)-{}^{c}D_a^{\alpha}(f(b_i))\right|<\epsilon$$

whenever $\{(a_i,b_i)\}_{i=1}^{n}$ is a disjoint collection of open intervals in (a,b) satisfying $\sum_{i=1}^{n}(b_i-a_i)<\delta$.

$AFD[a,b]$ denote the class of absolutely fractional differentiable functions on $[a,b]$.

We will need the following results for further study.

Theorem 1.6 *(Prajapati and Kachhia [10]) A function f is of bounded fractional differential variation on $[a,b]$ if and only if ${}^{c}D_a^{\alpha}f$ is of bounded variation on $[a,b]$.*

Theorem 1.7 *(Rudin [12]) Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and*

$$f'(x)=\lim_{n\to\infty}f_n'(x), \qquad (a\leq x\leq b).$$

2 Certain properties of the total fractional differential variation

Various properties (homogeneous, additive, vanishing point) of the total fractional differential variation have been discussed in this section.

Proposition 2.1 *If $f\colon [a,b] \to \mathbb{R}$ is a Caputo fractional differentiable function of order α, $0 < \alpha \leq 1$, and $r \in \mathbb{R}$, then*

$$FDV(rf,[a,b]) = |r|FDV(f,[a,b]).$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Then

$$\begin{aligned} FDV_l(rf,P,[a,b]) &= \sum_{i=1}^{n} \left| \Gamma(\alpha+1) \left(\frac{rf(x_i) - rf(x_{i-1})}{(x_i - x_{i-1})^{\alpha}} \right) - {}^{c}D_a^{\alpha} rf(x_{i-1}) \right| \\ &= \sum_{i=1}^{n} \left| r\Gamma(\alpha+1) \left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^{\alpha}} \right) - r\,{}^{c}D_a^{\alpha} f(x_{i-1}) \right| \\ &= |r| \sum_{i=1}^{n} \left| \Gamma(\alpha+1) \left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^{\alpha}} \right) - {}^{c}D_a^{\alpha} f(x_{i-1}) \right| \\ &= |r| FDV_l(f,P,[a,b]). \end{aligned}$$

Similarly,

$$FDV_r(rf,P,[a,b]) = |r|FDV_r(f,P,[a,b]),$$

is proved in precisely the same way. Taking the supremum over all partitions P of $[a,b]$ yields

$$FDV_l(rf,[a,b]) = |r|FDV_l(f,[a,b]),$$

and

$$FDV_r(rf,[a,b]) = |r|FDV_r(f,[a,b]).$$

Consequently,

$$\begin{aligned} FDV(rf,[a,b]) &= FDV_l(rf,[a,b]) + FDV_r(rf,[a,b]) \\ &= |r|FDV_l(f,[a,b]) + |r|FDV_r(f,[a,b]) \\ &= |r|FDV(f,[a,b]). \end{aligned}$$

□

Proposition 2.2 *If $f\colon [a,b] \to \mathbb{R}$ is a Caputo fractional differentiable function of order α, $0 < \alpha \leq 1$, and $a < c < b$, then*

$$FDV(f,[a,b]) = FDV(f,[a,c]) + FDV(f,[c,b]).$$

Proof. Let $P_1 = \{x_0, x_1, \ldots, x_k\}$ and $P_2 = \{x_k, x_{k+1}, \ldots, x_n\}$ be partitions of $[a, c]$ and $[c, b]$ respectively. Consider the partition $P = \{x_0, \ldots, x_k, \ldots x_n\}$ of $[a, b]$. Then,

$$\begin{aligned} FDV_l(f, P, [a, b]) &= \sum_{i=1}^{n} \left| \Gamma(\alpha+1) \left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^\alpha} \right) - {}^cD_a^\alpha f(x_{i-1}) \right| \\ &= \sum_{i=1}^{k} \left| \Gamma(\alpha+1) \left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^\alpha} \right) - {}^cD_a^\alpha f(x_{i-1}) \right| + \\ &\qquad \sum_{i=k+1}^{n} \left| \Gamma(\alpha+1) \left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^\alpha} \right) - {}^cD_a^\alpha f(x_{i-1}) \right| \\ &= FDV_l(f, P_1, [a, c]) + FDV_l(f, P_2, [c, b]), \end{aligned}$$

taking supremum over all partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ gives

$$FDV_l(f, [a, c]) + FDV_l(f, [c, b]) \leq FDV_l(f, [a, b]). \tag{2.1}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Form the partitions $P_1 = \{x_0, x_1, \ldots, x_k, c\}$ of $[a, c]$ and $P_2 = \{c, x_{k+1}, \ldots, x_n\}$ of $[c, b]$.

$$\begin{aligned} FDV_l(f, P, [a, b]) = &\sum_{i=1}^{k} \left| \Gamma(\alpha+1) \left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^\alpha} \right) - {}^cD_a^\alpha f(x_{i-1}) \right| + \\ &\sum_{i=k+1}^{n} \left| \Gamma(\alpha+1) \left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^\alpha} \right) - {}^cD_a^\alpha f(x_{i-1}) \right|, \end{aligned}$$

adding two non-negative terms, this can be written as

$$\begin{aligned} FDV_l(f, P, [a, b]) \leq &\sum_{i=1}^{k} \left| \Gamma(\alpha+1) \left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^\alpha} \right) - {}^cD_a^\alpha f(x_{i-1}) \right| + \\ &\left| \Gamma(\alpha+1) \left(\frac{f(c) - f(x_k)}{(c - x_k)^\alpha} \right) - {}^cD_a^\alpha f(x_k) \right| + \\ &\left| \Gamma(\alpha+1) \left(\frac{f(x_{k+1}) - f(c)}{(x_{k+1} - c)^\alpha} \right) - {}^cD_a^\alpha f(c) \right| + \\ &\sum_{i=k+1}^{n} \left| \Gamma(\alpha+1) \left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^\alpha} \right) - {}^cD_a^\alpha f(x_{i-1}) \right|, \\ = &FDV_l(f, P_1, [a, c]) + FDV_l(f, P_2, [c, b]), \end{aligned}$$

taking supremum over all partitions P of $[a, b]$ gives

$$FDV_l(f, [a, b]) \leq FDV_l(f, [a, c]) + FDV_l(f, [c, b]). \tag{2.2}$$

Combining inequalities (2.1) and (2.2) together results in

$$FDV_l(f,[a,b]) = FDV_l(f,[a,c]) + FDV_l(f,[c,b]).$$

Next,

$$FDV_r(f,[a,b]) = FDV_r(f,[a,c]) + FDV_r(f,[c,b]).$$

is proved in precisely the same way. This leads to

$$\begin{aligned} FDV(f,[a,b]) &= FDV_l(f,[a,b]) + FDV_r(f,[a,b]) \\ &= FDV_l(f,[a,c]) + FDV_l(f,[c,b]) + \\ &\qquad FDV_r(f,[a,c]) + FDV_r(f,[c,b]) \\ &= FDV_l(f,[a,c]) + FDV_r(f,[a,c]) + \\ &\qquad FDV_l(f,[c,b]) + FDV_r(f,[c,b]) \\ &= FDV(f,[a,c]) + FDV(f,[c,b]). \end{aligned}$$

□

Using Proposition 2.2, we prove the next proposition.

Proposition 2.3 *If $f \in AFD[a,b]$, then $f \in BFDV[a,b]$.*

Proof. Let $\epsilon = 1$. There exists a $\delta > 0$ such that for any pairwise disjoint intervals $\{(a_i,b_i)\}$ in $[a,b]$ satisfying $\sum_{i=1}^{n} |b_i - a_i| < \delta$, we have

$$\sum_{i=1}^{n} \left| \Gamma(\alpha+1) \left(\frac{f(b_i) - f(a_i)}{(b_i - a_i)^\alpha} \right) - {}^cD_a^\alpha(f(a_i)) \right| < 1$$

and

$$\sum_{i=1}^{n} \left| \Gamma(\alpha+1) \left(\frac{f(b_i) - f(a_i)}{(b_i - a_i)^\alpha} \right) - {}^cD_a^\alpha(f(b_i)) \right| < 1$$

Choose $n \in \mathbb{N}$ such that $\frac{b-a}{n} < \delta$. Let $b_i = a + i\left(\frac{b-a}{n}\right)$, $i = 0,1,\ldots,n$. For a given partition $P = \{y_0, y_1, \ldots, y_k\}$ of $[b_{i-1}, b_i]$, we always have

$$\sum_{j=1}^{k} |y_j - y_{j-1}| < \frac{b-a}{n} < \delta.$$

For $i = 1,2,\ldots,n$, we have

$$FDV_l(f,[b_{i-1},b_i]) \leq 1$$

and

$$FDV_r(f,[b_{i-1},b_i])) \leq 1$$

Therefore, for $i = 1, 2, \ldots, n$

$$\begin{aligned} FDV(f, [b_{i-1}, b_i])) &= FDV_l(f, [b_{i-1}, b_i]) + FDV_r(f, [b_{i-1}, b_i])) \\ &\leq 2. \end{aligned}$$

By Proposition 2.2, we conclude that

$$\begin{aligned} FDV(f, [a, b]) &= \sum_{i=1}^{n} FDV(f, [b_{i-1}, b_i]) \\ &\leq \sum_{i=1}^{n} 2 \\ &= 2n < \infty, \end{aligned} \tag{2.3}$$

hence, $f \in BFDV[a, b]$.

□

Proposition 2.4 *Let $f\colon [a, b] \to \mathbb{R}$ be a Caputo fractional differentiable function of order α, $0 < \alpha \leq 1$. Then $FDV(f, [a, b]) = 0$ if and only if ${}^cD_a^\alpha f$ is a constant function.*

Proof. Assume first that $FDV(f, [a, b]) = 0$. By definition of $FDV(f, [a, b])$,

$$FDV_l(f, [a, b]) = 0 \quad \text{and} \quad FDV_r(f, [a, b]) = 0$$

Let $x, y \in [a, b]$ be given. For $x = y$, ${}^cD_a^\alpha f(x) = {}^cD_a^\alpha f(y)$. Without loss of generality assume that $x < y$. Consider the partition $P = \{a, x, y, b\}$ of $[a, b]$. By our assumption,

$$FDV_l(f, P) = 0 \quad \text{and} \quad FDV_r(f, P) = 0,$$

consequently,

$$\left| \Gamma(\alpha + 1) \left(\frac{f(y) - f(x)}{(y - x)^\alpha} \right) - {}^cD_a^\alpha f(y) \right| = 0$$

and

$$\left| \Gamma(\alpha + 1) \left(\frac{f(y) - f(x)}{(y - x)^\alpha} \right) - {}^cD_a^\alpha f(x) \right| = 0.$$

Next,

$$\begin{aligned} |{}^cD_a^\alpha f(y) - {}^cD_a^\alpha f(x)| &= \left| {}^cD_a^\alpha f(y) - \Gamma(\alpha + 1) \left(\frac{f(y) - f(x)}{(y - x)^\alpha} \right) + \right. \\ &\qquad \left. \Gamma(\alpha + 1) \left(\frac{f(y) - f(x)}{(y - x)^\alpha} \right) - {}^cD_a^\alpha f(x) \right| \\ &= \left| \Gamma(\alpha + 1) \left(\frac{f(y) - f(x)}{(y - x)^\alpha} \right) - {}^cD_a^\alpha f(y) \right| + \\ &\qquad \left| \Gamma(\alpha + 1) \left(\frac{f(y) - f(x)}{(y - x)^\alpha} \right) - {}^cD_a^\alpha f(x) \right| \\ &= 0 \end{aligned}$$

therefore,

$$ {}^{c}D_{a}^{\alpha}f(y) - {}^{c}D_{a}^{\alpha}f(x) = 0, $$

hence,

$$ {}^{c}D_{a}^{\alpha}f(y) = {}^{c}D_{a}^{\alpha}f(x); \qquad (x, y \in [a, b]) $$

i.e., ${}^{c}D_{a}^{\alpha}f$ is a constant function.

Conversely, we assume that ${}^{c}D_{a}^{\alpha}f$ is a constant function. Consider a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. By the mean value theorem for Caputo fractional derivatives (Theorem 1.2), there exists $\zeta_i \in (x_{i-1}, x_i)$ such that

$$ \Gamma(\alpha+1)\left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^{\alpha}}\right) = {}^{c}D_{a}^{\alpha}f(\zeta_i); \quad (i = 1, 2, \ldots, n). $$

The fact that ${}^{c}D_{a}^{\alpha}f$ is a constant function gives

$$ \Gamma(\alpha+1)\left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^{\alpha}}\right) = {}^{c}D_{a}^{\alpha}f(x_i) = {}^{c}D_{a}^{\alpha}f(x_{i-1}); \quad (i = 1, 2, \ldots, n). $$

For each $i = 1, 2, \ldots, n$, we have

$$ \Gamma(\alpha+1)\left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^{\alpha}}\right) - {}^{c}D_{a}^{\alpha}f(x_i) = 0 $$

and

$$ \Gamma(\alpha+1)\left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^{\alpha}}\right) - {}^{c}D_{a}^{\alpha}f(x_{i-1}) = 0. $$

Consequently,

$$ \sum_{i=1}^{n}\left|\Gamma(\alpha+1)\left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^{\alpha}}\right) - {}^{c}D_{a}^{\alpha}f(x_i)\right| = 0 $$

and

$$ \sum_{i=1}^{n}\left|\Gamma(\alpha+1)\left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})^{\alpha}}\right) - {}^{c}D_{a}^{\alpha}f(x_{i-1})\right| = 0. $$

i.e., $FDV_l(f, P) = 0$ and $FDV_r(f, P) = 0$ for all partitions P of $[a, b]$. Evidently,

$$ FDV_l(f, [a, b]) = 0 \text{ and } FDV_r(f, [a, b]) = 0, $$

hence,

$$ \begin{aligned} FDV(f, [a, b]) &= FDV_l(f, [a, b]) + FDV_r(f, [a, b]) \\ &= 0. \end{aligned} $$

□

3 Completeness of $BFDV[a,b]$

In this section, we proved some results as Lemma 3.1 to Lemma 3.3. These results are useful to prove that $BFDV[a,b]$ is a Banach algebra.

Lemma 3.1 *If $f\colon [a,b]\to\mathbb{R}$ is a Caputo fractional differentiable function of order α, $0<\alpha\leq 1$, then*

$$V({}^cD_a^\alpha f)\leq FDV(f).$$

Proof. Let $P=\{x_0,x_1,x_2,\ldots,x_n\}$ be a partition of $[a,b]$. Then

$$\begin{aligned}
V({}^cD_a^\alpha f,P)&=\sum_{i=1}^{n}|{}^cD_a^\alpha f(x_i)-{}^cD_a^\alpha f(x_{i-1})|\\
&=\sum_{i=1}^{n}\left|{}^cD_a^\alpha f(x_i)-\Gamma(\alpha+1)\left(\frac{f(x_i)-f(x_{i-1})}{(x_i-x_{i-1})^\alpha}\right)+\right.\\
&\qquad\left.\Gamma(\alpha+1)\left(\frac{f(x_i)-f(x_{i-1})}{(x_i-x_{i-1})^\alpha}\right)-{}^cD_a^\alpha f(x_{i-1})\right|\\
&\leq\sum_{i=1}^{n}\left|{}^cD_a^\alpha f(x_i)-\Gamma(\alpha+1)\left(\frac{f(x_i)-f(x_{i-1})}{(x_i-x_{i-1})^\alpha}\right)\right|+\\
&\qquad\sum_{i=1}^{n}\left|\Gamma(\alpha+1)\left(\frac{f(x_i)-f(x_{i-1})}{(x_i-x_{i-1})^\alpha}\right)-{}^cD_a^\alpha f(x_{i-1})\right|\\
&=FDV_r(f,P)+FDV_l(f,P),
\end{aligned}$$

taking supremum over all partitions P of $[a,b]$ results in

$$\begin{aligned}
V({}^cD_a^\alpha f,[a,b])&\leq\sup_P\Big(FDV_r(f,P)+FDV_l(f,P)\Big)\\
&\leq\sup_P FDV_r(f,P)+\sup_P FDV_l(f,P)\\
&=FDV_r(f)+FDV_l(f)\\
&=FDV(f).
\end{aligned}$$

□

Lemma 3.2 *If $f\colon [a,b]\to\mathbb{R}$ is a Caputo fractional differentiable function of order α, $0<\alpha\leq 1$, then*

$$FDV_l(f)\leq V({}^cD_a^\alpha(f))$$

and

$$FDV_r(f)\leq V({}^cD_a^\alpha(f)).$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. By the mean value theorem for Caputo fractional derivatives ([5,8]), there exists $c_i \in (x_{i-1}, x_i)$ such that

$$\Gamma(\alpha+1)\left(\frac{f(x_i)-f(x_{i-1})}{(x_i-x_{i-1})^\alpha}\right) = {}^cD_a^\alpha(f(c_i)), \qquad (i = 1, 2, \ldots, n).$$

Consider the partition $P_1 = \{x_0, c_1, x_1, c_2, x_2, \ldots, x_{n-1}, c_n, x_n\}$ of $[a, b]$. Then

$$\begin{aligned} FDV_l(f,P) &= \sum_{i=1}^{n}\left|\Gamma(\alpha+1)\left(\frac{f(x_i)-f(x_{i-1})}{(x_i-x_{i-1})^\alpha}\right) - {}^cD_a^\alpha(f(x_{i-1}))\right| \\ &= \sum_{i=1}^{n}|\,{}^cD_a^\alpha(f(c_i)) - {}^cD_a^\alpha(f(x_{i-1}))| \\ &\leq V({}^cD_a^\alpha(f), P_1) \end{aligned}$$

and

$$\begin{aligned} FDV_r(f,P) &= \sum_{i=1}^{n}\left|\Gamma(\alpha+1)\left(\frac{f(x_i)-f(x_{i-1})}{(x_i-x_{i-1})^\alpha}\right) - {}^cD_a^\alpha(f(x_i))\right| \\ &= \sum_{i=1}^{n}|\,{}^cD_a^\alpha(f(c_i)) - {}^cD_a^\alpha(f(x_i))| \\ &\leq V({}^cD_a^\alpha(f), P_1). \end{aligned}$$

For a given partition P of $[a, b]$, there is a partition P_1 of $[a, b]$ such that

$$FDV_l(f,P) \leq V({}^cD_a^\alpha(f), P_1),$$

and

$$FDV_r(f,P) \leq V({}^cD_a^\alpha(f), P_1)),$$

taking supremum over all partitions P of $[a, b]$ gives

$$FDV_l(f) \leq V({}^cD_a^\alpha(f))$$

and

$$FDV_r(f) \leq V({}^cD_a^\alpha(f)).$$

□

Example 3.1 Take $\alpha = 1$ and $f(x) = x^2$ with $[a, b] = [0, 1]$. The formula for total variation of a monotone function in conjunction with the fact that $f'(x)$

is a monotone function gives $V(f') = |f'(1) - f'(0)| = 2$ (see proposition 1.3 of [1]). Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[0, 1]$. Now,

$$\begin{aligned}\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} - f'(x_{i-1}) &= \frac{x_i^2 - x_{i-1}^2}{x_i - x_{i-1}} - 2x_{i-1} \\ &= x_i - x_{i-1} \\ &= \frac{f'(x_i) - f'(x_{i-1})}{2}.\end{aligned}$$

Similarly, one can show that

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} - f'(x_i) = \frac{-f'(x_i) + f'(x_{i-1})}{2},$$

Next,

$$\begin{aligned}FDV_l(f, P) &= \sum_{i=1}^{n} \left| \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} - f'(x_{i-1}) \right| \\ &= \frac{1}{2} \sum_{i=1}^{n} |f'(x_i) - f'(x_{i-1})| \\ &= \frac{1}{2} V(f', P).\end{aligned}$$

Similarly,

$$FDV_r(f, P) = \frac{1}{2} V(f', P),$$

taking supremum over all partitions P of $[0, 1]$ yields

$$FDV_l(f) = \frac{1}{2} V(f') \text{ and } FDV_r(f) = \frac{1}{2} V(f').$$

The fact that $V(f') \neq 0$ implies

$$FDV_l(f) < V(f')$$

and

$$FDV_r(f) < V(f').$$

i.e., a strict inequality in Lemma 3.2 for $\alpha = 1$ and $f(x) = x^2$. Next,

$$\begin{aligned}FDV(f) &= FDV_l(f') + FDV_r(f') \\ &= \frac{1}{2} V(f') + \frac{1}{2} V(f') \\ &= V(f')\end{aligned}$$

gives an equality in Lemma 3.1 for $\alpha = 1$ and $f(x) = x^2$.

Remark 3.1 We have $V(f,[0,1]) = 1$ and $FDV(f,[0,1]) = 2$ for $f(x) = x^2$ and $\alpha = 1$. Therefore, the total variation and the total fractional differential variation are different for $f(x) = x^2$ and $[a,b] = [0,1]$.

Lemma 3.3 *Let $\{f_n\}$ be a sequence of Caputo fractional differentiable functions defined on $[a,b]$, $0 < \alpha \leq 1$ such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{{}^cD_a^\alpha f_n\}$ converges to some function g_α, relative to the supremum norm $||\cdot||_\infty$, then $g_\alpha = {}^cD_a^\alpha f$.*

Proof. For $\alpha = 1$, we have ${}^cD_a^\alpha(f_n)(x) = f_n'(x)$. By our assumption, $\{f_n'\}$ converges to g_1 relative to the supremum norm $||\cdot||_\infty$. We see from Theorem 1.7 that $g_1 = f'$, i.e.,

$$\lim_{n\to\infty} f_n' = f'$$

Since $\{f_n'\}$ converges uniformly to f', we interchange limit and integration to obtain

$$\begin{aligned}
\lim_{n\to\infty} {}^cD_a^\alpha(f_n) &= \lim_{n\to\infty} \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f_n'(t)}{(x-t)^\alpha} dt \\
&= \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{\lim_{n\to\infty} f_n'(t)}{(x-t)^\alpha} dt \\
&= \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(t)}{(x-t)^\alpha} dt \\
&= {}^cD_a^\alpha(f)
\end{aligned}$$

and hence,

$$g_\alpha = {}^cD_a^\alpha f.$$

□

Theorem 3.4 *$BFDV[a,b]$ is a Banach algebra relative to the norm*

$$||f||_{BFDV} = ||f||_\infty + ||\,{}^cD_a^\alpha(f)||_\infty + V(f) + FDV(f). \tag{3.1}$$

Proof. Prajapati and Kachhia [10] proved that $BFDV[a,b]$ is a normed algebra. Let $\{f_n\}$ be a Cauchy sequence in $BFDV[a,b]$. (3.1) implies

$$||f_n||_{BV} \leq ||f_n||_{BFDV}; \qquad (n = 1,2,3,\ldots),$$

so that $\{f_n\}$ is a Cauchy sequence in $BV[a,b]$. The fact that $BV[a,b]$ is complete relative to the norm $||\cdot||_{BV}$ defined in (1.2) (cf., e.g., [[1], Proposition 1.10]), gives $\{f_n\}$ converges to some f in $BV[a,b]$. (3.1) shows that

$$||\,{}^cD_a^\alpha(f_n)||_\infty + FDV(f_n) \leq ||f_n||_{BFDV}; \qquad (n = 1,2,3,\ldots).$$

Lemma 3.1 gives

$$||\,{}^cD_a^\alpha(f_n)||_{BV} \leq ||f_n||_{BFDV}$$

so that $\{{}^cD_a^\alpha(f_n)\}$ is a Cauchy sequence in $BV[a,b]$. Consequently, the function sequence $\{{}^cD_a^\alpha(f_n)\}$ converges to some $g_\alpha \in BV[a,b]$.

Since $||f||_\infty \leq ||f||_{BV}$, $\{f_n\}$ converges to f and $\{{}^cD_a^\alpha f_n\}$ converges to g_α, relative to the supremum norm $||\cdot||_\infty$. By Lemma 3.3, $g_\alpha = {}^cD_a^\alpha(f)$. ${}^cD_a^\alpha(f) \in BV[a,b]$ because $g_\alpha \in BV[a,b]$. Theorem 1.6 combined with the fact that ${}^cD_a^\alpha(f) \in BV[a,b]$ gives $f \in BFDV[a,b]$.

Let $\epsilon > 0$ be given, then there exists a natural number N_1 such that $n \geq N_1$ implies

$$||f_n - f||_\infty + V(f_n - f) < \frac{\epsilon}{4}$$

There also exists a natural number N_2 such that $n \geq N_2$ implies

$$||\,{}^cD_a^\alpha(f_n) - {}^cD_a^\alpha(f)||_\infty + V\left({}^cD_a^\alpha(f_n) - {}^cD_a^\alpha(f)\right) < \frac{\epsilon}{4}$$

Evidently, $n \geq N_2$ implies

$$||\,{}^cD_a^\alpha(f_n) - {}^cD_a^\alpha(f)||_\infty < \frac{\epsilon}{4} \quad \text{and} \quad V\left({}^cD_a^\alpha(f_n - f)\right) < \frac{\epsilon}{4}$$

We see from Lemma 3.2 that $n \geq N_2$ implies

$$FDV_l(f_n - f) < \frac{\epsilon}{4} \quad \text{and} \quad FDV_r(f_n - f) < \frac{\epsilon}{4}$$

Therefore, $n \geq N_2$ implies

$$\begin{aligned} FDV(f_n - f) &= FDV_l(f_n - f) + FDV_r(f_n - f) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}. \end{aligned}$$

For $N = \max\{N_1, N_2\}$ and $n \geq N$,

$$\begin{aligned} ||f_n - f||_{BFDV} &= ||f_n - f||_\infty + ||\,{}^cD_a^\alpha(f_n - f)||_\infty + \\ &\qquad V(f_n - f) + FDV(f_n - f) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

therefore, $\{f_n\}$ converges to some f in $BFDV[a,b]$. Therefore, $BFDV[a,b]$ is a complete normed algebra. Hence, $BFDV[a,b]$ is a Banach algebra.

Let $f\colon [a,b] \to \mathbb{R}$ be the constant function $f = 1$. Evidently, $||1||_\infty = 1$. The fact that the total variation of a constant function is 0 gives $V(1) = 0$. Since the Caputo fractional derivative of a constant function is 0, we have $||\,{}^cD_a^\alpha 1||_\infty = 0$. Proposition 2.4 combined with the fact that ${}^cD_a^\alpha f$ is a constant function gives $FDV(1) = 0$.

Consequently,

$$\begin{aligned} ||1||_{BFDV} &= |||1||_\infty + ||\,{}^cD_a^\alpha(1)||_\infty + V(1) + FDV(1) \\ &= 1 + 0 + 0 + 0 \\ &= 1, \end{aligned}$$

and hence, $BFDV[a,b]$ is a Banach algebra with identity.

□

Acknowledgements Authors are thankful to all reviewers and Dr. Rekha Mehta, Retired Professor, Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar for their valuable comments and suggestions for the betterment of paper.

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Received: 23.VIII.2024 / Revised: 07.II.2025 / Accepted: 10.II.2025

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