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Dynamics of weighted composition operators on Stein manifolds

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Abstract In the present paper, we investigate the hypercyclicity of weighted composition operators acting on the space of holomorphic functions on a connected finite-dimensional Stein manifold. Let ψ be a holomorphic self-map on a connected Stein n-manifold Ω and $\omega \in \mathbf{H}(\Omega)$ a holomorphic function. We study the hypercyclicity of weighted composition operator $\Pi_{\psi,\omega} : \mathbf{H}(\Omega) \to \mathbf{H}(\Omega)$ defined by $\Pi_{\psi,\omega}(f) := \omega \cdot (f \circ \psi)$ for every $f \in \mathbf{H}(\mathbf{\Omega})$. We prove that $\Pi_{\psi,\omega}$ is hypercyclic if and only if $\omega(p) \neq 0$ at each $p \in \Omega$, ψ is univalent without fixed points in Ω, $\psi(\Omega)$ is a Runge domain and for every compact holomorphically convex set $K \subset \Omega$ there is an integer n such that $K \cap \psi^{[n]}(K) = \emptyset$ and their union is holomorphically convex.

Keywords holomorphic functions · weighted composition · hypercyclic · convexity · Stein manifold

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1 Introduction

The universal families and hypercyclic operators have been noticed in 80s and 90s(see [9–11, 14–17, 19–22, 24, 25, 27]). Among the conducted studies, the hypercyclic, weakly hypercyclic and n-weakly hypercyclic bounded linear operators on Fréchet spaces attract the most attention. By definition, a Fréchet space is a complete locally convex metrizable topological vector space. The hypercyclicity of a bounded linear map L on a Fréchet space $\mathcal{F}$ means that for a vector $v \in \mathcal{F}$, its orbit (i.e. $Orb(L, v) = \{L^n v\}_{n=1}^{\infty}$) is dense in $\mathcal{F}$. Such a vector v is called a hypercyclic vector for L. Similarly, L is called weakly hypercyclic if there is $v \in \mathcal{F}$ such that $Orb(L, v)$ is weakly dense in $\mathcal{F}$. If n is a positive integer, then we say that the operator L is n-weakly hypercyclic if there is a vector $v \in \mathcal{F}$ such that $Orb(L, v)$ is n-weakly dense in $\mathcal{F}$. For more information on dynamics of linear operators we refer the readers to the interesting books [6, 18].

In 2009, Grosse-Erdmann and Mortini ([19]) studied the set of holomorphic self-maps on a complex domain $D \subset \mathbf{C}$ whose composition operators are hypercyclic. The composition operator of a holomorphic self-map ψ on D is defined as $\Pi_\psi(f) = f \circ \psi$ for every $f \in \mathbf{H}(D)$, where, $\mathbf{H}(D)$ is the set of holomorphic complex valued functions on D. Since $\mathbf{H}(D)$ is a Fereshet space equipped with the compact-uniformly convergence metric.

Analogous problem in spaces of real analytic functions was considered in [8] and in kernels of general (non-necessarily Cauchy-Riemann) partial differential equations in [23]. The hypercyclicity of composition operators on Hardy space has been considered by Bourdon and Shapiro in [9,10]. Also, in [26], Shapiro has proved that for a univalent holomorphic self-map ψ on D, the composition operator Π_ψ is hypercyclic if and only if ψ has no fixed point in D.

The second development in this subject was the study of hypercyclicity of composition operators related to self-maps defined on $\mathbf{C}^k$, poly-discs or Euclidean balls (see [7,19]).

Recently, the hypercyclic weighted composition operators associated to a holomorphic self-map has been interested by some researchers. Especially, when a holomorphic self-map on D is elliptic, it is an automorphism and has a fixed point in D. In fact, it is conjugate to a rotation map $z \to \rho z$ for some complex number ρ with $|\rho| = 1$. In the non-elliptic case, one can use the well-known theorem due to Denjoy-Wolff ([25]).

Grosse-Erdmann and Mortini have proved that every holomorphic self-map ψ on a simply connected or infinitely connected domain $D \subset \mathbf{C}$ has hypercyclic composition operator if and only if ψ is injective, $\psi(D)$ is a Runge domain with respect to D and the sequence $\{\psi^{[n]}\}_{n=1}^{\infty}$ is run-away, i.e. for every compact set $K \subset D$ there is some n such that $K \cap \psi^{[n]}(K) = \emptyset$. Also, on a finitely connected domain, no ψ induces hypercyclic Π_ψ. Dynamics of the weighted composition operators on different settings have been studied in [3,4].

In this paper, instead of an arbitrary domain D, we consider the maps and functions on a Stein manifold. In fact, associated to a holomorphic self-map ψ on a connected Stein n-manifold Ω and a holomorphic function $\omega \in \mathbf{H}(\Omega)$, we study the hypercyclicity of weighted composition operator $\Pi_{\psi,\omega} : \mathbf{H}(\Omega) \to \mathbf{H}(\Omega)$ defined by $\Pi_{\psi,\omega}(f) := \omega \cdot (f \circ \psi)$ for every $f \in \mathbf{H}(\Omega)$.

We prove that $\Pi_{\psi,\omega}$ is hypercyclic if and only if for every $p \in \Omega$, $\omega(p) \neq 0$ and ψ is univalent without fixed points in Ω, $\psi(\Omega)$ is a Runge domain and for every compact holomorphically convex set $C \subset \Omega$ there is an integer n such that $C \cap \psi^{[n]}(C) = \emptyset$ and their union is holomorphically convex.

The language of holomorphically convexity seems to be natural for working with hypercyclicity of $\Pi_{\psi,\omega}$, as it is strongly connected with some approximation theorems for that space (e.g. the Runge and Oka-Weil theorems).

2 Preliminaries

First, we introduce a special class of complex analytic manifolds, called Stein manifolds ([20]).

2.1 Stein manifolds

Let Ω be a finite dimensional complex manifold and C be a compact subset of Ω. Usually, $\widehat{C}_\Omega$ stands for the holomorphic hull of C with respect to Ω, defined by

$$\widehat{C}_\Omega := \{p \in \Omega : |f(p)| \leq \sup_C |f|, \forall f \in H(\Omega)\}.$$

C is called holomorphically convex (or shortly, Ω-convex) if $C = \widehat{C}_\Omega$.

In special case $\Omega = \mathbf{C}^n$, $\widehat{C}_\Omega$ is denoted with shorter symbol $\widehat{C}$ and is called the polynomial hull of C. So, C is called polynomially convex if $C = \widehat{C}$.

Definition 2.1 A sequence $(C_r)_{r=1}^\infty$ of compact subsets of Ω is called an *exhaustion* of Ω if $\bigcup_{r=l}^\infty C_r = \Omega$ and for each r, $C_r \subset intC_{r+1}$. Also, Ω is called *countable at infinity* if it has an exhaustion.

For two finite-dimensional complex manifolds Ω, Λ, the notation $\Theta(\Omega, \Lambda)$ denotes the set of all holomorphic maps $\phi : \Omega \to \Lambda$. In the case $\Lambda = \Omega$, we write $\Theta(\Omega)$ instead of $\Theta(\Omega, \Omega)$. When $\Lambda = \mathbf{C}^n$, we equip the linear space $\Theta(\Omega, \mathbf{C}^n)$ with the compact-open topology, which is the topology of uniform convergence on compact subsets. In the case $n = 1$ we shortly write $\mathbf{H}(\Omega)$ instead of $\Theta(\Omega, \mathbf{C})$, which is the class of all holomorphic functions on Ω. A holomorphic function on an open subset of the complex plane is called univalent if it is injective.

We say that a sequence of holomorphic maps $\{\phi_r : \Omega \to \Lambda\}_{r=1}^\infty$ is *compactly divergent* (in $\Theta(\Omega, \Lambda)$) if for each compact subsets $S \subset \Omega$ and $K \subset \Lambda$ there exists a natural number r_0 such that $\phi_r(S) \cap K = \emptyset$ for all natural numbers $r \geq r_0$.

We say that $\{\phi_r\}_{r=1}^\infty$ is *run-away* (in $\mathcal{O}(\Omega, \Lambda)$ if for each compact subsets $S \subset \Omega$ and $K \subset \Lambda$, there is r_0 such that $\phi_{r_0}(S) \cap K = \emptyset$.

In the case $\Omega = \Lambda$, it is always enough to consider the situation when $S = K$. Note that when Ω and Λ are countable at infinity, the sequence $\{\phi_r\}_{r=1}^\infty$ is run-away if and only if it has a compactly divergent subsequence. Also, in this case, $\{\phi_r\}_{r=1}^\infty$ is compactly divergent if and only if each of its subsequences is run-away.

We say that a holomorphic map $\phi : \Omega \to \Lambda$ is *regular* if its derivative is monomorphism at each point of Ω. We say that $\phi : \Omega \to \Lambda$ is *proper* if $\phi^{-1}(K)$ is compact for every compact set $K \subset \Lambda$. Also, $\phi : \Omega \to \Lambda$ is said to be *almost proper* if for every compact set $K \subset \Lambda$, each connected component of $\phi^{-1}(K)$ is compact.

In this paper $\mathbf{D} \subset \mathbf{C}$ denotes the *open unit disc* in complex plane and $\mathbf{D}_* = \mathbf{D}\backslash\{0\}$ and $\mathbf{C}_* = \mathbf{C}\backslash\{0\}$.

For points $p, q \in \Omega$ let

$$d^*_\Omega(p,q) := sup\{|f(p)| : f \in \Theta(\Omega, D), f(q) = 0\}, \tag{2.1}$$

$$d_\Omega(p,q) := \frac{1}{2}\log\frac{1+d^*_\Omega(p,q)}{1-d^*_\Omega(p,q)}. \tag{2.2}$$

In fact d_Ω is the Carathéodory pseudodistance and d^*_Ω is the Möbius pseudodistance in Ω. For more information we refer the reader to [22].

Definition 2.2 An complex manifold Ω of dimension n is said to be a *Stein manifold* if it satisfies the following conditions:

1. Ω is countable at infinity,
2. the set $\widehat{C}_\Omega$ is compact for every compact subset $C \subset \Omega$,
3. $\mathbf{H}(\Omega)$ separates points in Ω, i.e. for each two distinct points $p, q \in \Omega$, there exists $f \in \mathbf{H}(\Omega)$ with $f(p) \neq f(q)$,
4. For each $p \in \Omega$ there exists a map $L \in \Theta(\Omega, \mathbf{C}^n)$ such that its derivative at p is an isomorphism.

Here, we deal with (connected) Stein manifolds for which all balls w.r.t. d_Ω are relatively compact in the topology of Ω showing that they have some good hypercyclicity properties.

2.2 Hypercyclic operators

First, we present the definition of hypercyclic and weakly hypercyclic operators on a given topological vector space X over the complex numbers filed $\mathbf{C}$.

For a self-map $T : X \to X$ and integer n, $T^{[n]} = T \circ T \circ \ldots \circ T$ (n times) denotes nth iteration of T.

Definition 2.3 Let $\{L_r\}_{r=1}^\infty$ be a sequence of continuous self-maps of X.

1. $\{L_r\}_{r=1}^\infty$ is called *topologically transitive* if for every non-empty open subsets $U, V \subset X$ there exists r_0 such that $L_{r_0}(U) \cap V \neq \emptyset$.
2. A point $p \in X$ is said to be an *universal element* for $\{L_r\}_{r=1}^\infty$ if the sequence $\{L_r(p)\}_{r=1}^\infty$ of points is dense in X.
3. The sequence $\{L_r\}_{r=1}^\infty$ is said to be *universal* if it admits a universal element.
4. We say that $\{L_r\}_{r=1}^\infty$ is called *hereditarily universal* if each of its subsequences is universal.

For more information one can see [5,6,18]. The characterization of the topological transitivity and universality for the weighted translation operators may be found in [1,2].

Definition 2.4 Let X and Y be two topological spaces, and let $\mathcal{C}(X,Y)$ denote the set of all continuous maps between X and Y. Given a compact subset $C \subset X$ and an open subset $U \subset Y$, let $\mathcal{V}(C,U)$ denote the set of all functions $f \in \mathcal{C}(X,Y)$ such that $f(C) \subset U$. Then the collection of all such $\mathcal{V}(C,U)$ is a subbase for the compact-open topology on $\mathcal{C}(X,Y)$. (This collection does not always form a base for a topology on $\mathcal{C}(X,Y)$.

Definition 2.5 Let $T : X \to X$ be a continuous linear self-map. We say that T is hypercyclic with respect to an increasing sequence $\{n_l\}_{l=1}^{\infty} \subset N$ if the sequence $\{T^{[n_l]}\}_{l=1}^{\infty}$ is universal. We call T shortly hypercyclic if it is hypercyclic w.r.t. the full sequence $\{n\}_{n=1}^{\infty}$. If T is hypercyclic, then any universal element of $\{T^{[n]}\}_{n=1}^{\infty}$ is called a hypercyclic vector.

Lemma 2.6 *If T is weakly hypercyclic, then the adjoint operator T^* does not have any eigenvectors.*

Proof. By assumption, T is weakly hypercyclic and then it is 1-weakly hypercyclic. Therefore, T^* does not have any eigenvectors by Proposition 3.2 in [12].

□

Now, we recall two essential theorems for our considerations. For the first one see Theorem 1 and for the second one see Proposition 1 in [16].

Theorem 2.7 *Every sequence $\{T_n\}_{n=1}^{\infty}$ of continuous self-maps on a separable Fréchet space $\mathcal{F}$ is topologically transitive if and only if the set of its universal elements is dense in $\mathcal{F}$. Moreover, in this case the set of universal elements for $\{T_n\}_{n=1}^{\infty}$ is a dense G_δ-subset of $\mathcal{F}$.*

Theorem 2.8 *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of continuous self-maps on a separable Fréchet space $\mathcal{F}$. If T_n has dense range in $\mathcal{F}$ for each $n \in N$ and the sequence $\{T_n\}_{n=1}^{\infty}$ is commuting (i.e. for every natural numbers n and k, we have $T_k \circ T_n = T_n \circ T_k$), then the set of universal elements of $\{T_n\}_{n=1}^{\infty}$ is empty or dense in $\mathcal{F}$.*

By these theorems we get a corollary that allows us to investigate topological transitivity instead of hypercyclicity. Also, Theorem 3 in [16] has a similar argument.

Corollary 2.9 *Let $\mathcal{F}$ be a separable Fréchet space, let $T : \mathcal{F} \to \mathcal{F}$ be a continuous map, and let $\{n_l\}_{l=1}^{\infty}$ be an increasing sequence of natural numbers. Then, T is hypercyclic w.r.t. $\{n_l\}_{l=1}^{\infty}$ if and only if the sequence $\{T^{[n_l]}\}_{l=1}^{\infty}$ is topologically transitive.*

Remark 2.1 the compact-open topology is a topology defined on the set of continuous maps between two topological spaces. The compact-open topology is one of the commonly used topologies on function spaces, and is applied in homotopy theory and functional analysis. It was introduced by Ralph Fox in 1945 [13].

2.3 Runge domain in Stein manifold

In this section we try to provide a clear idea about Runge domains in Stein manifolds. A domain U in a Stein Manifold Ω is said to be a *Runge domain* with respect to Ω if every function $f \in \mathbf{H}(U)$ can be approximated uniformly on U by a sequence of members of $\mathbf{H}(\Omega)$. The well-known Oka-Weil theorem says that on every compact Ω-convex subset $C \subset \Omega$, every holomorphic function (i.e. holomorphic on a neighborhood of C) can be approximated uniformly by functions from $\mathbf{H}(\Omega)$. This theorem turns out to be a very nice tool for showing topological transitivity of sequence $\{\Pi_{\psi}^{[n_l]}\}_{l=1}^{\infty}$, because - as we shall see - the question of topological transitivity of that sequence may be translated to a question of approximation of some functions by elements of $\mathbf{H}(\Omega)$. Its one-dimensional version, known as Runge theorem, was used in [19].

Remark 2.2 By definition, a Stein manifold Ω has an exhaustion $\{C_r\}_{r=1}^{\infty}$ of compact subsets of Ω such that $\bigcup_{r=l}^{\infty} C_r = \Omega$ and for each r, $C_r \subset intC_{r+1}$. So, we can take a sequence of semi-norms $\{p_r : \mathbf{H}(\Omega) \to R\}$ defined by $p_r(f) := \sup\{|f(p)| p \in C_r\}$, which gives the topology of $\mathbf{H}(\Omega)$. Then, $\mathbf{H}(\Omega)$ with this topology is a Fréchet space. Also, by this topology, $\mathbf{H}(\Omega)$ is separable. This observation allows us to use Corollary 2.9 for the space $X = \mathbf{H}(\Omega)$, with Ω being a connected Stein manifold.

The following well-known theorem ([28]) characterizes Runge domains in a Stein manifold Ω in the language of holomorphic hulls:

Theorem 2.10 Let U be a Stein manifold which is a domain of a connected Stein manifold Ω. Then, the following conditions are equivalent:

1. The domain U is a Runge domain in Ω.
2. $\widehat{C}_{\Omega} = \widehat{C}_U$ for every compact subset $C \subset U$.
3. $\widehat{C}_{\Omega} \cap U = \widehat{C}_U$ for every compact subset $C \subset U$.

Lemma 2.11 Let S and D be two compact subsets of a connected Stein manifold Ω. Then the following conditions are equivalent:

1. S and D are separable in Ω.
2. There exist open and disjoint subsets $U, V \subset \Omega$ such that $\widehat{S}_{\Omega} \subset U$, $\widehat{D}_{\Omega} \subset V$ and $\widehat{(S \cup D)}_{\Omega} \subset U \cup V$.
3. $\widehat{S}_{\Omega} \cap \widehat{D}_{\Omega} = \emptyset$ and $\widehat{(S \cup D)}_{\Omega} = \widehat{S}_{\Omega} \cup \widehat{D}_{\Omega}$.

In particular, if S and D are disjoint and Ω-convex, then $S \cup D$ is Ω-convex if and only if S and D are separable in Ω.

Lemma 2.12 Let S and D be two disjoint compact subsets of a connected Stein manifold Ω such that $S \cup D$ is Ω-convex. Then S and D are both Ω-convex.

3 Main results

Proposition 3.1 Let Ω be a connected Stein n-manifold, $\omega \in \mathbf{H}(\Omega)$ and $\psi \in \Theta(\Omega)$. Suppose that the weighted composition map $\Pi_{\psi,\omega}$ is hypercyclic. Then:

1. $\omega \neq 0$ on Ω and ψ has no fixed point.
2. ψ is injective.
3. $\psi(\Omega)$ is a Runge domain w.r.t. Ω.
4. $\{\psi^r\}_{r=1}^{\infty}$ is run-away.

Proof. 1. We consider the point evaluation linear functional $T_p : \mathbf{H}(\Omega) \to \mathbf{C}$ at each point $p \in \Omega$ by rule $T_p(g) := g(p)$. It is well known that T_p is continuous on $\mathbf{H}(\Omega)$. The adjoint operator $\Pi^*_{\psi,\omega}$ satisfies the following equations

$$\Pi^*_{\psi,\omega}(T_p)(g) = T_p \circ \Pi_{\psi,\omega}(g) = T_p(\omega \cdot (g \circ \psi)) = \omega(p) \cdot (g \circ \psi)(p).$$

Now, if $\omega(p) = 0$ or $\psi(p) = p$, then $\Pi^*_{\psi,\omega}$ has a eigenvalue and then $\Pi_{\psi,\omega}$ is not hypercyclic.

2. Let $f \in \mathbf{H}(\Omega)$ be a hypercyclic vector for $\Pi_{\psi,\omega}$. For every $g \in Orb(\Pi_{\psi,\omega}, f)$ there is a nonnegative integer k such that at each point $p \in \Omega$ we have

$$T_p(g) = g(p) = (\Pi^k_{\psi,\omega}(f))(p) = \omega(p) \prod_{j=1}^{k-1} (\omega \circ \psi^j)(p) \cdot (f \circ \psi^k)(p),$$

and then

$$\frac{1}{\omega(p)} T_p(g) = \prod_{j=1}^{k-1} (\omega \circ \psi^j)(p) \cdot (f \circ \psi^k)(p).$$

Now, assume that $\psi(p) = \psi(q)$ for some distinct points $p, q \in \Omega$. So, the linear functionals $\frac{1}{\omega(p)} T_p$ and $\frac{1}{\omega(q)} T_q$ have the same value at each function $g \in Orb(\Pi_{\psi,\omega}, f)$. By the continuity of linear functionals, this property can be extended to every function in the closure of $Orb(\Pi_{\psi,\omega}, f)$ (i.e. for every $g \in \overline{Orb(\Pi_{\psi,\omega}, f)}$). Therefore, we have $\frac{1}{\omega(p)} T_p = \frac{1}{\omega(q)} T_q$ on $\mathbf{H}(\Omega)$ and then, putting $g = 1$, we get $\frac{1}{\omega(p)} T_p(1) = \frac{1}{\omega(q)} T_q(1)$ which gives $\omega(p) = \omega(q)$. Therefore, the equality $g(p) = g(q)$ holds for every $g \in \mathbf{H}(\Omega)$, which by condition (3) in Definition 2.2, implies that $p = q$. So, ψ is injective.

3. It is enough to prove that the subset of restrictions $\{h|_{\psi(\Omega)} : h \in \mathbf{H}(\Omega)\}$ is dense in $\mathbf{H}(\psi(\Omega))$. If $g \in \mathbf{H}(\psi(\Omega))$, then $g \circ \psi$ is holomorphic on Ω, so there is an increasing sequence of natural numbers $(l_k)_{k=1}^{\infty}$ such that $f \circ \psi^{[l_k]} \to g \circ \psi$ on Ω (where, $f \in \mathbf{H}(\Omega)$ is a hypercyclic vector for $C_{\psi,\omega}$). Hence $f \circ \psi^{[l_k - 1]} \to g$ on $\psi(\Omega)$, as the mapping ψ is a biholomorphism on its image.

4. Assume that $f \in \mathbf{H}(\Omega)$ is a hypercyclic vector for $\Pi_{\psi,\omega}$ and S is a compact subset of Ω. By part (3), for each positive integer k, there exists a positive integer l_k such that $|f \circ \psi^{[l_k]} - k| \leq \frac{1}{k}$ on S. By the principle of maximum and minimum values on compact sets, $\mu := \sup\{|f(p)| : p \in S\}$ and

$$\lambda_k := \inf\{|f(p)| : p \in \psi^{[l_k]}(S)\} = \inf\{|(f \circ \psi^{[l_k]})(p)| : p \in S\}$$

are finite real numbers for each $k \in N$.
Clearly, by choosing a big k satisfying the inequality $k - \frac{1}{k} > \mu$, we have

$$\lambda_k \geq k - \frac{1}{k} > \mu.$$

But, for every point $p_0 \in \psi^{[l_k]}(S) \cap S$, we have $\lambda_k \leq |f(p_0)|$ and $|f(p_0)| \leq \mu$ and then $\lambda_k \leq \mu$ which is a contradicts. Hence, $\psi^{[l_k]}(S) \cap S = \emptyset$.

□

The second part of Proposition 3.1 gives that ψ is a biholomorphism on its image, which is a classical result may be found in [21]. Therefore, $\psi(\Omega)$ is a Stein manifold.

Proposition 3.2 Let Ω be a connected Stein n-manifold, $\omega \in \mathbf{H}(\Omega)$ and $\psi \in \Theta(\Omega)$. If the weighted composition operator $\Pi_{\psi,\omega}$ is weakly hypercyclic on $\mathbf{H}(\Omega)$, then we have $\omega(p) \neq 0$ for every $p \in \Omega$ and ψ is an injective self-map without any fixed point.

Proof. Let f be a weakly hypercyclic vector for $\Pi_{\psi,\omega}$ in $\mathbf{H}(\Omega)$. Since $\Pi_{\psi,\omega}$ is weakly hypercyclic, then by Lemma 2.6 the adjoint operator $\Pi^*_{\psi,\omega}$ does not have any eigenvectors.

For each $p_0 \in \Omega$, the point evaluation linear functional $T_{p_0} : \mathbf{H}(\Omega) \to \mathbf{C}$ defined by $T_{p_0}(f) = f(p_0)$ is continuous on $\mathbf{H}(\Omega)$ and

$$(\Pi_{\psi,\omega})^* T_{p_0}(f) = T_{p_0} \Pi_{\psi,\omega}(f) = \omega(p_0) f(\psi(p_0)) = \omega(p_0) T_{\psi(p_0)}(f).$$

Thus, if $\psi(p_0) = p_0$ or $\omega(p_0) = 0$ for some $p_0 \in \Omega$, then adjoint of $\Pi_{\psi,\omega}$ has an eigenvector which gives a contradiction with Lemma 2.6. Hence, ω is nonzero at each point of Ω and ψ has no fix point in Ω.

Now we prove that ψ is injective. Assuming $\psi(p) = \psi(q)$, from equality $\Pi^n_{\psi,\omega} f = \prod_{j=0}^{n-1} \Pi^j_\psi(\omega) \Pi^n_\psi f$ we get

$$\begin{aligned} \frac{\Pi^n_{\psi,\omega}(f(p))}{\omega(p)} &= \prod_{j=1}^{n-1} \Pi^j_\psi(\omega)(p) \Pi^n_\psi(f)(p) \\ &= \prod_{j=1}^{n-1} \Pi^j_\psi(\omega)(q) \Pi^n_\psi(f)(q) = \frac{\Pi^n_{\psi,\omega}(f(q))}{\omega(q)}. \end{aligned} \tag{3.1}$$

The linear functionals T_p and T_q is weakly continuous on $\mathbf{H}(\Omega)$ and then both $\frac{1}{\omega(p)}T_p$ and $\frac{1}{\omega(q)}T_q$ are also weakly continuous.

Choosing a sequence $\{\Pi_{\psi,\omega}^{n_k}\}_{k=1}^{\infty}$ in $Orb(\Pi_{\psi,\omega}f)$ which is weakly convergent to 1, we have $\frac{1}{\omega(p)}T_p\Pi_{\psi,\omega}^{n_k}(f) \to \frac{1}{\omega(p)}T_p 1$ and $\frac{1}{\omega(q)}T_q\Pi_{\psi,\omega}^{n_k}(f) \to \frac{1}{\omega(q)}T_q 1$. Since the left sides are the same, we have $\omega(p) = \omega(q)$.

So, by a similar way we can see that $g(p) = g(q)$ for every $g \in \mathbf{H}(\Omega)$, which by condition (3) in Definition 2.2 gives $p = q$. Hence, ψ in injective. □

Here, we prefer to re-describe the topology of $\mathbf{H}(\Omega)$ and the concept of topologically transitivity of weighted composition operators.

For every function $f_0 \in \mathbf{H}(\Omega)$, compact subset $C \subset \Omega$ and positive real number ϵ, the ϵ-neighborhood of f_0 is defined by

$$N_\epsilon^C(f_0) := \{f \in \mathbf{H}(\Omega) : \forall y \in C, |f(y) - f_0(y)| < \epsilon\}.$$

The family of all such a neighborhoods forms a basis for the topology of $\mathbf{H}(\Omega)$.

Also, we need to clear the concept of topological transitivity of the sequence $(\Pi_{\psi,\omega}^{[n_l]})_{l=1}^{\infty}$, where $\{n_l\}_{l=1}^{\infty}$ is an increasing sequence of natural numbers, $\psi \in \mathbf{H}(\Omega)$ is an injective holomorphic self-map with no fix point and $0 \neq \omega \in \mathbf{H}(\Omega)$. The sequence $(\Pi_{\psi,\omega}^{[n_l]})_{l=1}^{\infty}$ is topologically transitive if and only if for given $\epsilon > 0$, $g, h \in \mathbf{H}(\Omega)$ and compact $S \subset \Omega$ there is a natural number k and a function $f \in \mathbf{H}(\Omega)$ such that $|f - g| < \epsilon$ and $|\Pi_{\psi,\omega}^{[n_k]}(f) - h| < \epsilon$ on S.

Since the mapping ψ is injective and ω is nonzero, the above condition has another form:

$$|f-g| < \epsilon \ \texttt{on}\ S \quad and \quad |f-[\prod_{j=0}^{k-1} \Pi_\psi^j(\omega)]^{-1}\cdot h\circ\psi^{[-n_k]}| < \epsilon \ \texttt{on}\ \psi^{[n_k]}(S). \quad (3.2)$$

Since Ω is a Stein manifold, we can restrict to considering only Ω-convex sets.

Remark 3.1 It follows from the equivalence of conditions (1) and (2) in Theorem 2.10 that ψ maps every Ω-convex compact $S \subset \Omega$ onto an Ω-convex compact set. Also, it implies that for any integer number n the set $\psi^{[n]}(C)$ is Ω-convex.

Theorem 3.3 Let Ω be a connected Stein manifold, $\psi \in \Theta(\Omega)$, $\omega \in \mathbf{H}(\Omega)$ and the weighted composition operator $\Pi_{\psi,\omega}$ is hypercyclic on $\mathbf{H}(\Omega)$. Then for every Ω-convex compact subset $S \subset \Omega$, there exists positive integer n such that $S \cap \psi^{[n]}(S) = \emptyset$ and the set $S \cup \psi^{[n]}(S)$ is Ω-convex.

Proof. Suppose that $\Pi_{\psi,\omega}$ is hypercyclic. In view of Corollary 2.9, the condition 3.2 holds. Fix an Ω-convex compact set $S \subset \Omega$. By Remark 3.1 we get that the set $\psi^{[n]}(S)$ is Ω-convex. Using the condition 3.2 for $g = 0$, $h = 1$ and $\epsilon = \frac{1}{2}$, we get that there are $f \in \mathbf{H}(\Omega)$ and natural number k such that $f(S) \subset \frac{1}{2}D$ and $\frac{\lambda}{2}(\psi^{[k]}(S)) \subset (1 + \frac{1}{2}D)$ where $\lambda = sup_S[\prod_{j=0}^{k-1} \Pi_{\psi}^{j}(\omega)]$. This implies that S and $\frac{\lambda}{2}\psi^{[k]}(S)$ are separable in Ω, so by Lemma 2.11 the sum $C \cup \frac{\lambda}{2}\psi^{[k]}(S)$ is Ω-convex.

□

Theorem 3.4 Let Ω be a connected Stein manifold, $\psi \in \mathcal{O}(\Omega)$, $\omega \in \mathbf{H}(\Omega)$ and the following conditions hold:

1. for every $p \in \Omega$, $\omega(p) \neq 0$ and ψ is an injective self-map without fixed point in Ω.
2. for every Ω-convex compact subset $S \subset \Omega$, there exists positive integer n such that $S \cap \psi^{[n]}(S) = \emptyset$ and the set $S \cup \psi^{[n]}(S)$ is Ω-convex.

Then, the weighted composition operator $\Pi_{\psi,\omega}$ is hypercyclic on $\mathbf{H}(\Omega)$.

Proof. Assume that $\{K_n\}_{n=1}^{\infty}$ be an exhaustion of Ω. Without lose of generality, we can assume that every K_n is Ω-convex. Since the compact-open topology on $\mathbf{H}(\Omega)$ is independent of the chosen exhaustion, we can endow $\mathbf{H}(\Omega)$ with the topology induced by the semi-norms on $\mathbf{H}(\Omega)$ defined by $p_n(f) := \sup\{|f(p)| : p \in K_n\}$. Let $U, V \subset \mathbf{H}(\Omega)$ be non-empty open sets and fix $f \in U$ and $g \in V$. By definition of compact-open topology of $\mathbf{H}(\Omega)$, there is a closed ball B centered at 0 and a positive real number ϵ such that, every $h_1 \in U$ satisfies $\sup_{p\in B} |f(p) - h_1(p)| < \epsilon$ and similarly every $h_2 \in V$ satisfies $\sup_{p\in B} |g(p) - h_2(p)| < \epsilon$.

Now, assume that D be another closed ball in Ω such that $B \subset D^{\circ}$. Since ψ is an injective self-map without fixed point on Ω, then the function f is holomorphic on some neighborhood of D, and the function $\frac{g\circ(\psi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0-1}(\omega\circ(\psi^{[k]})^{-1})}$ is holomorphic on some neighborhood of $\psi^{[n_0]}(D)$.

By assumption (2), there exists n_0 such that $D \cap \psi^{[n_0]}(D) = \emptyset$ and the compact set $K := D \cup \psi^{[n_0]}(D)$ is Ω-convex (by Oka-Weil theorem), there exists a holomorphic function $h \in H(\Omega)$ such that $\sup_{z\in D} |f(z) - h(z)| < \epsilon$ and

$$\sup_{y\in\psi^{[n_0]}(D)} |\frac{g \circ (\psi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})}(y) - h(y)| < \frac{\epsilon}{m}.$$

where $m := \max_{y\in\psi^{[n_0]}(D)} |\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})(y)|$.

Hence $\sup_{z\in K} |f(z) - h(z)| < \epsilon$ and

$$\begin{aligned} &\sup_{z\in K} |g(z) - (\omega\Pi_{\psi})^{n_0} h(z)| \\ &= \sup_{z\in K} |\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})(y)(\frac{g \circ (\psi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})}(y) - h(y))| < \epsilon, \end{aligned} \tag{3.3}$$

where $y := \psi^{[n_0]}(z)$. This shows that $h \in U$ and $(\Pi_{\psi,\omega})^{n_0} h \in V$, so that $\Pi_{\psi,\omega}$ is topologically transitive. Since $\mathbf{H}(\Omega)$ is a separable Fréchet space, $\Pi_{\psi,\omega}$ is hypercyclic.

□

Theorem 3.5 Let Ω be a connected Stein manifold, $\psi \in \Theta(\Omega)$ and $\omega \in \mathbf{H}(\Omega)$ and $\{n_l\}_{l=1}^{\infty}$ be an increasing sequence of positive integer numbers. Then the operator $\Pi_{\psi,\omega}$ is hypercyclic w.r.t. $\{n_l\}_{l=1}^{\infty}$ if and only if

1. for every $p \in \Omega$, $\omega(p) \neq 0$ and ψ is injective without fixed points in Ω,
2. $\psi(\Omega)$ is a Runge domain w.r.t. Ω,
3. for every Ω-convex compact subset $S \subset \Omega$ there is a positive integer number k such that the sets S and $\psi^{[k]}(S)$ are separable in Ω.

Proof. The sufficiency of two first parts follow from Theorem 3.3 and Theorem 3.4. For the third part, since $\psi(\Omega)$ is a Runge domain in Ω and S is Ω-convex, if the sets S and $\psi^{[k]}(S)$ are separable in Ω, then $\psi^{[k]}(S)$ is Ω-convex and by Lemma 2.11 their sum is Ω-convex. Necessity in both parts follows directly from Theorem 3.3 and Theorem 3.4, Lemma 2.11 and the necessary conditions.

□

4 Examples

In this section, the following examples illustrate how the obtained results work in practice.

Example 4.1 Let $\Omega = \mathbf{C}$ and $w \in \Omega$ be a fixed nonzero point. We take the translation map $\psi : \Omega \to \Omega$ by rule

$$z \mapsto z + w$$

and the function $\omega : \Omega \to \mathbf{C}$ by rule

$$z \mapsto exp(z).$$

Clearly, we have $\omega \neq 0$ on Ω and ψ has no fixed point, ψ is injective, $\psi(\Omega)$ is a Runge domain w.r.t. Ω and $\{\psi^r\}_{r=1}^{\infty}$ is run-away.

Hence, the weighted composition operator $\Pi_{\psi,\omega} : \mathbf{H}(\Omega) \to \mathbf{H}(\Omega)$ maps every $f \in \mathbf{H}(\Omega)$ to $\Pi_{\psi,\omega}(f) := \omega \cdot (f \circ \psi)$. At each $p \in \Omega$, we have

$$\Omega_{\psi,\omega}(f)(z) = \omega(z) \cdot (f \circ \psi)(z) = exp(z) \cdot f(z + w)$$

and in general form

$$\Pi^n_{\psi,\omega}(f)(z) = \omega(z) \cdot (\omega \circ \psi)(z) \cdots (\omega \circ \psi^{n-1})(z) \cdot (f \circ \psi^n)(z)$$

which means that

$$\Pi^n_{\psi,\omega}(f)(z) = exp(z) \cdot exp(z + w) \cdots exp(z + (n-1)w) \cdot f(z + nw).$$

By Theorem 3.4, the weighted composition operator $\Pi_{\psi,\omega}$ is hypercyclic on $\mathbf{H}(\Omega)$.

Example 4.2 Let $\Omega = C^k$ and $w = (w_1, w_2, \cdots, w_k)$ be a fixed nonzero point in Ω. We Take the translation map $\psi : \Omega \to \Omega$ by rule

$$z = (z_1, z_2, \cdots, z_k) \mapsto z + w = (z_1 + w_1, z_2 + w_2, \cdots, z_k + w_k)$$

and the function $\omega : \Omega \to \mathbf{C}$ by rule

$$z = (z_1, z_2, \cdots, z_k) \mapsto exp(z) := exp(z_1) \cdots exp(z_k).$$

Clearly we have $\omega \neq 0$ on Ω and ψ has no fixed point, ψ is injective, $\psi(\Omega)$ is a Runge domain w.r.t. Ω and $\{\psi^r\}_{r=1}^{\infty}$ is run-away.

Hence, the weighted composition operator $\Pi_{\psi,\omega} : \mathbf{H}(\Omega) \to \mathbf{H}(\Omega)$ maps every $f \in \mathbf{H}(\Omega)$ to $\Pi_{\psi,\omega}(f) := \omega \cdot (f \circ \psi)$. At each $p \in \Omega$, we have

$$\Pi_{\psi,\omega}(f)(z) = \omega(z) \cdot (f \circ \psi)(z) = exp(z) \cdot f(z + w)$$

and in general form

$$\Pi^n_{\psi,\omega}(f)(z) = \omega(z) \cdot (\omega \circ \psi)(z) \cdots (\omega \circ \psi^{n-1})(z) \cdot (f \circ \psi^n)(z)$$

which means that

$$\Pi^n_{\psi,\omega}(f)(z) = exp(z) \cdot exp(z + w) \cdots exp(z + (n-1)w) \cdot f(z + nw).$$

Theorem 3.4 guarantees that $\Pi_{\psi,\omega}$ is hypercyclic on $\mathbf{H}(\Omega)$.

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