An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
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On Jordan ideals and semigroup ideals in 3-prime near-rings with multiplicative derivations

Slimane El Amrani · Abderrahmane Raji

Abstract The aim of this article is to prove certain properties of multiplicative derivations verifying certain algebraic identities on Jordan ideals and semigroup ideals of a 3-prime near-ring. The results obtained characterize the relationship between the structure of this type of near-rings admitting this kind of maps. Additionally, we present examples to show that our assumptions imposed on the various theorems cannot be omitted.

Keywords prime near-rings · multiplicative derivations · commutativity

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1 Introduction

In this document, $\mathcal{N}$ will be a left near-ring. For all $x, y \in \mathcal{N}$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ stands for the anti-commutator $xy + yx$. A near-ring $\mathcal{N}$ is called a zero-symmetric if $0.x = 0$ for all $x \in \mathcal{N}$ (recall that left distributive gives $x.0 = 0$). The set $\mathcal{Z}(\mathcal{N}) = \{x \in \mathcal{N} |\ xy = yx \text{ for all } y \in \mathcal{N}\}$ is called a multiplicative center of $\mathcal{N}$. Recall that $\mathcal{N}$ is said to be 2-torsion free if whenever $2x = 0$ implies $x = 0$, and know as 3-prime if $\mathcal{N}$ has the property that for all $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. An additive subgroup $\mathcal{J}$ of $\mathcal{N}$ is said to be a Jordan ideal of $\mathcal{N}$ if $j \circ n \in \mathcal{J}$ and $n \circ j \in \mathcal{J}$ for all $j \in \mathcal{J}$, $n \in \mathcal{N}$; more details about Jordan ideal and its properties can be found in [9]. A nonempty subset $\mathcal{I}$ of $\mathcal{N}$ is said to be a semigroup left ideal (resp. semigroup right ideal) if $\mathcal{N}\mathcal{I} \subseteq \mathcal{I}$ (resp. $\mathcal{I}\mathcal{N} \subseteq \mathcal{I}$); note that $\mathcal{I}$ is said to be semigroup ideal if I is both a semigroup left ideal and a semigroup right ideal of $\mathcal{N}$. A derivation $d : \mathcal{N} \to \mathcal{N}$ is an endomorphism of $(\mathcal{N}, +)$ with the property that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. The concept of derivation has been generalized in different directions by various authors, for example see [1], [4], [6], [7], [11], [13], [16], [17], [18], [19], [20], [21], [22] and [23], where

further references can be found. In [10], the notion of multiplicative derivation on ring was introduced by Daif as follows: A mapping $d : \mathcal{N} \to \mathcal{N}$, which is not assumed to be additive, is called a multiplicative derivation if $d(xy) = xd(y) + d(x)y$ holds for all $x, y \in \mathcal{N}$. Clearly, any derivation is a multiplicative derivation, but the converse is not true in general.
The literature contains many results on commutativity in prime and semi prime rings admitting suitably constrained derivations, and several authors have proved comparable results on near-rings. Historically, the study of the relationship between the commutativity of a 3-prime near-ring $\mathcal{N}$ and a derivation d on $\mathcal{N}$ was initiated in 1987 by Bell and Mason [2]. After this, a lot of work about commutativity of prime near-rings based on the behavior of different additive mappings, as derivations, generalized derivations, semiderivations and generalized semiderivations had been done (see references for a partial bibliography).
In [14], Hongan proved some results by assuming that the commutativity condition is imposed on an ideal rather than on the whole ring. Motivated by this idea, we pursue this research path and study the structure of 3-prime near-rings in which multiplicative derivations satisfy certain identities involving Jordan ideal and semigroup ideal on $\mathcal{N}$. The results obtained are very precise, unify and complement several results and cover others known in the literature while neglecting an important property marked in the hypotheses of various subsequent works, that is the additivity property considered in the different maps used.

2 Preliminary results

We introduce the following lemmas, which are essential for developing the proof of our results.

Lemma 2.1 *Let $\mathcal{N}$ be a 3-prime near-ring and $\mathcal{I}$ be a nonzero semigroup ideal of $\mathcal{N}$.*

(i) [3, Lemma 1.4(i)] *If $x, y \in \mathcal{N}$ and $x\mathcal{I}y = \{0\}$, then $x = 0$ or $y = 0$.*
(ii) [3, Lemma 1.3(i)] *If $x \in \mathcal{N}$ and $x\mathcal{I} = \{0\}$ or $\mathcal{I}x = \{0\}$, then $x = 0$.*

Lemma 2.2 [12, Lemma 4] *Let $\mathcal{N}$ be a near-ring and $d : \mathcal{N} \longrightarrow \mathcal{N}$ is a multiplicative derivation of $\mathcal{N}$. Then, $(xd(y) + d(x)y)z = xd(y)z + d(x)yz$ for all $x, y, z \in \mathcal{N}$.*

Lemma 2.3 *Let $\mathcal{N}$ be a 3-prime near-ring and $\mathcal{J}$ a nonzero Jordan ideal of $\mathcal{N}$.*

(i) [9, Lemma 3] *If $\mathcal{N}$ is 2-torsion free and $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*
(ii) *If $x\mathcal{J} = \{0\}$, then $x = 0$.*

For the proof of (ii), we can use the same approach as used in the proof of [9, Lemma 1].

Lemma 2.4 [12, Lemma 5] *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero multiplicative derivation d such that $ad(\mathcal{N}) = \{0\}$ or $d(\mathcal{N})a = \{0\}$, then $a = 0$.*

Lemma 2.5 *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) [3, Lemma 1.2(iii)] *If $z \in \mathcal{Z}(\mathcal{N}) \setminus \{0\}$ and $xz \in \mathcal{Z}(\mathcal{N})$, then $x \in \mathcal{Z}(\mathcal{N})$.*
(ii) [3, Lemma 1.5] *If $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 2.6 [15, Lemma 2.1] *A near-ring $\mathcal{N}$ admits a multiplicative derivation if and only if it is zero-symmetric.*

3 Main results

In this section, we give some new results and examples concerning the commutativity of a 3-prime near-ring which satisfy some identities on semigroup ideals and Jordan ideals involving multiplicative derivations. We begin by recalling an important result due to Bell and Mason published in 1987, see [2], in which the authors showed that a 2-torsion 3-prime near-ring $\mathcal{N}$ is commutative ring if it admits a nonzero derivation d satisfying $d(\mathcal{N}) \subseteq \mathcal{Z}(\mathcal{N})$. In 2017, Bedir and Gölbaşi [12] proved the same result by using the notion of multiplicative derivation. Our objective in the following theorem is to generalize and improve these results in less assumptions by treating the case in which d is a nonzero multiplicative derivation and $\mathcal{J}$ is a Jordan ideal of $\mathcal{N}$ instead of $\mathcal{N}$.

Theorem 3.1 *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{J}$ be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero multiplicative derivation d, then the following assertions are equivalent*

i) $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$ and $d(\mathcal{J}) \subseteq \mathcal{Z}(\mathcal{N})$,
ii) $\mathcal{N}$ is a commutative ring.

Proof. i) $\Rightarrow$ ii) Let z be an element of $\mathcal{Z}(\mathcal{N})$ such that $d(z) \neq 0$. By hypotheses given, we have

$$d((xz) \circ j)t = td((xz) \circ j) \text{ for all } x, t \in \mathcal{N}, j \in \mathcal{J}.$$

So that,

$$d((x \circ j)z)t = td((x \circ j)z) \text{ for all } x, t \in \mathcal{N}, j \in \mathcal{J}. \tag{3.1}$$

In virtue of Lemma 2.2 and after simplification, we get

$$(x \circ j)d(z)t = t(x \circ j)d(z) \text{ for all } x, t \in \mathcal{N}, j \in \mathcal{J}$$

which means that

$$(x \circ j)d(z) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{3.2}$$

Now, returning to (3.1) and using the fact that $z \in \mathcal{Z}(\mathcal{N})$, we infer that

$$d(z(x \circ j))t = td(z(x \circ j)) \text{ for all } x, t \in \mathcal{N}, j \in \mathcal{J}.$$

Again invoking Lemma 2.2, we obtain

$$d(z)(x \circ j)t = td(z)(x \circ j) \text{ for all } x, t \in \mathcal{N}, j \in \mathcal{J},$$

and hence

$$d(z)(x \circ j) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{3.3}$$

Replacing x by jx in (3.2), we get

$$j(x \circ j)d(z) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}.$$

Taking to account of Lemma 2.5(i) together with (3.2), the last result yields

$$(x \circ j)d(z) = 0 \text{ or } j \in \mathcal{Z}(\mathcal{N}) \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{3.4}$$

Case 1: Assume that

$$(x \circ j)d(z) = 0 \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{3.5}$$

Left multiplying (3.5) by $md(z)$, where $m \in \mathcal{N}$, we find that

$$md(z)(x \circ j)d(z) = 0 \text{ for all } x, m \in \mathcal{N}, j \in \mathcal{J}.$$

From (3.3), we have $d(z)(x \circ j) \in \mathcal{Z}(\mathcal{N})$, then the previous equation can be written as

$$d(z)(x \circ j)\mathcal{N}d(z) = \{0\} \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}.$$

Using the 3-primeness of $\mathcal{N}$ and the fact that $d(z) \neq 0$, we infer that

$$d(z)(x \circ j) = 0 \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}, \tag{3.6}$$

which implies that

$$d(z)x(-j) = d(z)jx \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{3.7}$$

Substituting xt for x in (3.7), where $t \in \mathcal{N}$, and using it again, we arrive at

$$d(z)xt(-j) = d(z)x(-j)t \text{ for all } x, t \in \mathcal{N}, j \in \mathcal{J}.$$

Putting $-j$ instead of j in the last expression, we obtain $d(z)xtj = d(z)xjt$ for all $x, t \in \mathcal{N}, j \in \mathcal{J}$ which can be written as

$$d(z)\mathcal{N}[t, j] = \{0\} \text{ for all } t \in \mathcal{N}, j \in \mathcal{J}.$$

In view of Lemma 2.1(i) and $d(z) \neq 0$, we find that $j \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}$ and hence $\mathcal{N}$ is a commutative ring by Lemma 2.3(i). From (3.6) and 2-torsion freeness of $\mathcal{N}$, we obtain $d(z)\mathcal{N}j = \{0\}$ for all $j \in \mathcal{J}$. Invoking Lemma 2.1(i) and taking into account that $d(z) \neq 0$, we conclude that $\mathcal{J} = \{0\}$, a contradiction.
Case 2: Suppose that there are $x_0 \in \mathcal{N}$ and $j_0 \in \mathcal{J}$ such that $(x_0 \circ j_0)d(z) \neq$

0, then (3.4) assures that $j_0 \in \mathcal{Z}(\mathcal{N})$ and $j_0 \neq 0$. Replacing j by j_0 in (3.3), we obtain

$$d(z)(x+x)j_0 \in \mathcal{Z}(\mathcal{N}) \text{ for all } x \in \mathcal{N}.$$

In virtue of Lemma 2.5(i) and $j_0 \neq 0$, the previous relation forces

$$d(z)(x+x) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x \in \mathcal{N}. \tag{3.8}$$

Taking $(x_0 \circ j_0)r$ instead of x in (3.8), where $r \in \mathcal{N}$, we find that

$$d(z)(x_0 \circ j_0)(r+r) \in \mathcal{Z}(\mathcal{N}) \text{ for all } r \in \mathcal{N}.$$

Again, applying Lemma 2.5(i), we get

$$d(z)(x_0 \circ j_0) = 0 \text{ or } r + r \in \mathcal{Z}(\mathcal{N}) \text{ for all } r \in \mathcal{N}. \tag{3.9}$$

Now, we prove that $d(z)(x_0 \circ j_0) \neq 0$. Indeed, suppose that $d(z)(x_0 \circ j_0) = 0$ and right multiplying this equation by $d(z)m$, and using the fact $(x_0 \circ j_0)d(z) \in \mathcal{Z}(\mathcal{N})$, we find that

$$d(z)m(x_0 \circ j_0)d(z) = 0 \text{ for all } m \in \mathcal{N}.$$

It follows that

$$d(z)\mathcal{N}(x_0 \circ j_0)d(z) = \{0\}.$$

In the light of the 3-primeness of $\mathcal{N}$ and $d(z) \neq 0$, we find that $(x_0 \circ j_0)d(z) = 0$ which contradicts our assumption of the statement. Consequently, (3.9) gives $r + r \in \mathcal{Z}(\mathcal{N})$ for all $r \in \mathcal{N}$. Replacing r by r^2 in the last relation, we obtain $r(r+r) \in \mathcal{Z}(\mathcal{N})$ which, because of Lemma (2.5)(i), means that $r + r = 0$ or $r \in \mathcal{Z}(\mathcal{N})$ for all $r \in \mathcal{N}$. But, the first condition with 2-torsion free yield $r = 0$ and therefore in the both cases, we get $r \in \mathcal{Z}(\mathcal{N})$ for all $r \in \mathcal{N}$, and hence $\mathcal{N}$ is a commutative ring by Lemma (2.5)(ii).
Clearly, we have $ii) \Rightarrow i)$. This completes the proof of our Theorem.

□

Our goal in the following theorems is to study the relationship between the structure of $\mathcal{N}$ and the existence of a nonzero multiplicative derivation d on $\mathcal{N}$. For this, it is interesting to assume that $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$. It is easy to see that if $d(\mathcal{Z}(\mathcal{N})) = \{0\}$, then the multiplicative law of $\mathcal{N}$ cannot be commutative. In fact, suppose that the law " . " is abelian, in this case $\mathcal{N} = \mathcal{Z}(\mathcal{N})$ and therefore $d(\mathcal{N}) = \{0\}$, which means that $d = 0$.

Theorem 3.2 *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, $\mathcal{I}$ be a nonzero semigroup ideal of $\mathcal{N}$ and $\mathcal{J}$ be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a multiplicative derivation d verifying any one of the following conditions:*

(i) $[d(j), x] \in \mathcal{Z}(\mathcal{N})$ for all $x \in \mathcal{I}$, $j \in \mathcal{J}$,
(ii) $[x, d(j)] \in \mathcal{Z}(\mathcal{N})$ for all $x \in \mathcal{I}$, $j \in \mathcal{J}$,

then $\mathcal{N}$ is a commutative ring or $d^2(\mathcal{J}) = \{0\}$.

Proof. (i) Assume that

$$[d(j), x] \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.10}$$

Replacing x by $d(j)x$ in (3.10), where $x \in \mathcal{I}$, we get $d(j)[d(j), x] \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. By application of Lemma 2.5(i), the latter relation shows that

$$[d(j), x] = 0 \text{ or } d(j) \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.11}$$

Suppose that there is an element $j_0 \in \mathcal{J}$ such that $d(j_0) \notin \mathcal{Z}(\mathcal{N})$. In this case, (3.11) assures that $d(j_0)x = xd(j_0)$ for all $x \in \mathcal{I}$. Taking xt instead of x in the preceding equation, where $t \in \mathcal{N}$, we find that $\mathcal{I}[d(j_0), t] = \{0\}$ for all $t \in \mathcal{N}$. According to Lemma 2.1(ii), we conclude that $d(j_0) \in \mathcal{Z}(\mathcal{N})$, a contradiction. Consequently, (3.11) gives $d(\mathcal{J}) \subseteq \mathcal{Z}(\mathcal{N})$. So, if $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$, we conclude that $\mathcal{N}$ is a commutative ring by Theorem 3.1, else we infer that $d^2(\mathcal{J}) = \{0\}$.
(ii) Using similar techniques as above, we get the required result.

□

As an application of Theorem 3.2, we get the following results.

Theorem 3.3 *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, $\mathcal{I}$ be a nonzero semigroup ideal of $\mathcal{N}$ and $\mathcal{J}$ be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a multiplicative derivation d verifying any one of the following assertions:*

(i) $[d(j), x] \circ t \in \mathcal{Z}(\mathcal{N})$ for all $t \in \mathcal{N}, x \in \mathcal{I}, j \in \mathcal{J}$,
(ii) $[x, d(j)] \circ t \in \mathcal{Z}(\mathcal{N})$ for all $t \in \mathcal{N}, x \in \mathcal{I}, j \in \mathcal{J}$,

then $\mathcal{N}$ is a commutative ring or $d^2(\mathcal{J}) = \{0\}$.

Proof. Suppose that the condition (i) holds, then

$$[d(j), x] \circ t \in \mathcal{Z}(\mathcal{N}) \text{ for all } t \in \mathcal{N}, x \in \mathcal{I}, j \in \mathcal{J}. \tag{3.12}$$

If $\mathcal{Z}(\mathcal{N}) = \{0\}$, it is straightforward to verify that $d^2(\mathcal{J}) = \{0\}$. Now, suppose that $\mathcal{Z}(\mathcal{N}) \neq \{0\}$ and let z an element of $\mathcal{Z}(\mathcal{N}) \setminus \{0\}$. Replacing t by z in (3.12), we obtain

$$z([d(j), x] + [d(j), x]) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x \in \mathcal{I}, j \in \mathcal{J}.$$

Since $z \neq 0$, the last relation together Lemma 2.5(i) prove that

$$[d(j), x] + [d(j), x] \in \mathcal{Z}(\mathcal{N}) \text{ for all } x \in \mathcal{I}, j \in \mathcal{J}. \tag{3.13}$$

Now, taking $t = [d(j), x]$ in (3.12), we find that

$$[d(j), x]([d(j), x] + [d(j), x]) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x \in \mathcal{I}, j \in \mathcal{J}. \tag{3.14}$$

In virtue of Lemma 2.5(i) and 2-torsion freeness of $\mathcal{N}$, from (3.13) and (3.14) it follows that

$$[d(j), x] \in \mathcal{Z}(\mathcal{N}) \text{ or } [d(j), x] = 0 \text{ for all } x \in \mathcal{I}, j \in \mathcal{J},$$

which leads to $[d(j), x] \in \mathcal{Z}(\mathcal{N})$ for all $x \in \mathcal{I}, j \in \mathcal{J}$. And therefore $\mathcal{N}$ is a commutative ring or $d^2(\mathcal{J}) = \{0\}$ by Theorem 3.2(i).
For the second condition, we can using similar approach as above with suitable changes.

□

Theorem 3.4 *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, $\mathcal{J}$ a nonzero Jordan ideal of $\mathcal{N}$ and d a multiplicative derivation of $\mathcal{N}$. If $\mathcal{J}$ has any one of the following properties:*

(i) $d(j) \circ x \in \mathcal{Z}(\mathcal{N})$ *for all* $j \in \mathcal{J}$, $x \in \mathcal{N}$,
(ii) $x \circ d(j) \in \mathcal{Z}(\mathcal{N})$ *for all* $j \in \mathcal{J}$, $x \in \mathcal{N}$,
(iii) $(d(j) \circ x) \circ t \in \mathcal{Z}(\mathcal{N})$ *for all* $j \in \mathcal{J}$, $x, t \in \mathcal{N}$,
(iv) $(x \circ d(j)) \circ t \in \mathcal{Z}(\mathcal{N})$ *for all* $j \in \mathcal{J}$, $x, t \in \mathcal{N}$,

then $\mathcal{N}$ is a commutative ring or $d^2(\mathcal{J}) = \{0\}$.

Proof. a) Assume that (i) is verified, i.e,

$$(d(j) \circ x) \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}, x \in \mathcal{N}, \tag{3.15}$$

and proving that $\mathcal{N}$ is a commutative ring or $d^2(\mathcal{J}) = \{0\}$.
We dividing the proof into cases which exhaust all the possibilities.
Case 1: Suppose that $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$. Let $z \in \mathcal{Z}(\mathcal{N})$ such that $z \neq 0$ and substituting z for x in (3.15), we obtain $z(d(j)+d(j)) \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}$. As $z \neq 0$, by application of Lemma 2.5(i), the previous relation shows that $(d(j) + d(j)) \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}$. Now, taking $x = d(j)$ in (3.15), we find that $d(j)(d(j)+d(j)) \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}$. By application of Lemma 2.5(i) and 2-torsion freeness of $\mathcal{N}$, the last expression shows that

$$d(j) = 0 \text{ or } d(j) \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J},$$

which reduces to $d(\mathcal{J}) \subseteq \mathcal{Z}(\mathcal{N})$ and hence $\mathcal{N}$ must be a commutative ring by Theorem 3.1.
Case 2: Assume that $d(\mathcal{Z}(\mathcal{N})) = \{0\}$. In this case, putting $d(j)x$ instead of x in (3.15) and applying Lemma 2.5(i), we get

$$d(j) \circ x = 0 \text{ or } d(j) \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}, x \in \mathcal{N}. \tag{3.16}$$

Let j a fixed element in $\mathcal{J}$. If there exists $x_0 \in \mathcal{N}$ such that $d(j) \circ x_0 \neq 0$, then from (3.16) we obtain $d(j) \in \mathcal{Z}(\mathcal{N})$. Now, suppose that

$$d(j) \circ x = 0 \text{ for all } x \in \mathcal{N}. \tag{3.17}$$

Substituting xt for x in (3.17), where $t \in \mathcal{N}$, we conclude that $\mathcal{N}[-d(j), t] = \{0\}$ for all $t \in \mathcal{N}$. According to Lemma 2.1(ii), we conclude that $-d(j) \in \mathcal{Z}(\mathcal{N})$. Replacing x by $-d(j)$ in (3.17), we obtain $(-d(j))(d(j) + d(j)) = 0$. Left multiplying by m, where $m \in \mathcal{N}$, and in view of 3-primeness together 2-torsion freeness of $\mathcal{N}$, we get $d(j) = 0$. Accordingly, (3.16) shows that $d(j) = 0$ or $d(j) \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}$. But, both conditions give $d^2(\mathcal{J}) =$

$\{0\}$.
b) Suppose that the condition (iii) is satisfied, so that

$$(d(j) \circ x) \circ t \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}, t, x \in \mathcal{N}. \tag{3.18}$$

If $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$, there exists an element $z \in \mathcal{Z}(\mathcal{N})$ such that $z \neq 0$. Putting z instead of t in (3.18), we obtain $z(d(j) \circ x + d(j) \circ x) \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}, x \in \mathcal{N}$ which, in view of Lemma 2.5(i) and $z \neq 0$, implies that

$$d(j) \circ x + d(j) \circ x \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}, x \in \mathcal{N}.$$

Now, substituting $d(j) \circ x$ for t in (3.18), we find that $(d(j) \circ x)(d(j) \circ x + d(j) \circ x) \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}, x \in \mathcal{N}$. Invoking Lemma 2.5$(i)$ and using 2-torsion freeness of $\mathcal{N}$, we find that

$$d(j) \circ x \in \mathcal{Z}(\mathcal{N}) \text{ or } d(j) \circ x = 0 \text{ for all } j \in \mathcal{J}, x \in \mathcal{N}.$$

It follows that $d(j) \circ x \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}, x \in \mathcal{N}$, and so from the part (a) we conclude that $\mathcal{N}$ is a commutative ring or $d^2(\mathcal{J}) = \{0\}$.
If $d(\mathcal{Z}(\mathcal{N})) = \{0\}$, reasoning as in the case of (a) with the appropriate changes, we obtain $d^2(\mathcal{J}) = \{0\}$.
For conditions (ii) and (iv), arguing in the same way as in the cases (a) and (b), we find the desired result.

□

The following example shows that the 3-primeness imposed in the hypotheses of Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4 is necessary.

Example 3.1 Let $\mathcal{S}$ be an unitary 2-torsion free zero-symmetric left near-ring. Define $\mathcal{N}$, $\mathcal{J}$, $\mathcal{I}$ and $d : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y, z \in \mathcal{S} \right\}, \quad \mathcal{J} = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x \in \mathcal{S} \right\},$$

$$\mathcal{I} = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x \in \mathcal{S} \right\} \quad \text{and} \quad d \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathcal{N}$ is a 2-torsion free non 3-prime left near-ring, $\mathcal{I}$ is a nonzero semigroup ideal of $\mathcal{N}$, $\mathcal{J}$ is a nonzero Jordan ideal of $\mathcal{N}$ and d is a nonzero multiplicative derivation of $\mathcal{N}$ for which $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$. Further, we can also verify that d satisfies the following properties:

1. $d(\mathcal{J}) \subset \mathcal{Z}(\mathcal{N})$,
2. $[d(J), A] \in \mathcal{Z}(\mathcal{N})$,
3. $[A, d(J)] \in \mathcal{Z}(\mathcal{N})$,
4. $[d(J), A] \circ B \in \mathcal{Z}(\mathcal{N})$,
5. $[A, d(J)] \circ B \in \mathcal{Z}(\mathcal{N})$,
6. $d(J) \circ C \in \mathcal{Z}(\mathcal{N})$,
7. $C \circ d(J) \in \mathcal{Z}(\mathcal{N})$,
8. $(d(J) \circ C) \circ B \in \mathcal{Z}(\mathcal{N})$,
9. $(C \circ d(J)) \circ B \in \mathcal{Z}(\mathcal{N})$,

for all $A \in \mathcal{I}, J \in \mathcal{J}, B, C \in \mathcal{N}$. However, neither $d^2(\mathcal{J}) = \{0\}$ nor $\mathcal{N}$ is a commutative ring because the additive law of $\mathcal{N}$ is not commutative.

Theorem 3.5 *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{J}$ be a nonzero Jordan ideal of $\mathcal{N}$. There is no multiplicative derivation d satisfies any one of the following conditions:*
(i) $d(j) = \pm j$ for all $j \in \mathcal{J}$,
(ii) $d(ij) = \pm ij$ for all $i, j \in \mathcal{J}$.

Proof. (i) Suppose that $\mathcal{N}$ admits a multiplicative derivation d such that

$$d(j) = j \text{ for all } j \in \mathcal{J}. \tag{3.19}$$

Substituting $jx \circ j$ for j in (3.19) and using it again, we arrive at

$$jxj + jjx = 0 \text{ for all } j \in \mathcal{J}, x \in \mathcal{N}. \tag{3.20}$$

Taking xt instead of x in (3.20), where $t \in \mathcal{N}$, and using (3.20), we find that $jxt(-j) = jx(-j)t$ for all $j \in \mathcal{J}, x, t \in \mathcal{N}$. Putting $-j$ instead of j in the last equation, we obtain

$$(-j)\mathcal{N}[t, j] = \{0\} \text{ for all } j \in \mathcal{J}, t \in \mathcal{N}.$$

By 3-primeness of $\mathcal{N}$, we get the conclusion $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{N})$ and therefore $\mathcal{N}$ is a commutative ring by Lemma 2.3(i). In this case, returning to (3.20) and using the 2-torsion freeness and 3-primeness of $\mathcal{N}$, we conclude that $\mathcal{J} = \{0\}$, which contradicts our original assumption that $\mathcal{J} \neq \{0\}$.
(ii) Assume that $\mathcal{N}$ has a multiplicative derivation d such that

$$d(ij) = ij \text{ for all } i, j \in \mathcal{J}. \tag{3.21}$$

Substituting $j \circ j$ for j in (3.21) and noting that $j + j \in \mathcal{J}$, we arrive at $2d(i)j^2 = 0$ for all $i, j \in \mathcal{J}$. By 2-torsion freeness of $\mathcal{N}$, it follows that

$$d(i)j^2 = 0 \text{ for all } i, j \in \mathcal{J}. \tag{3.22}$$

Writing $jx \circ j$ instead of j in (3.21), where $x \in \mathcal{N}$, we find that

$$id(j(x \circ j)) + d(i)j(x \circ j) = ij(x \circ j) \text{ for all } i, j \in \mathcal{J}, x \in \mathcal{N}.$$

Using (3.21) together (3.22) and solving the preceding equation, we infer that

$$d(i)jxj = 0 \text{ for all } i, j \in \mathcal{J}, x \in \mathcal{N},$$

that is $d(i)j\mathcal{N}j = \{0\}$ for all $i, j \in \mathcal{J}$. By 3-primeness of $\mathcal{N}$, the last relation shows that $d(i)j = 0$ or $j = 0$ for all $i, j \in \mathcal{J}$ which reduces to $d(i)\mathcal{J} = \{0\}$ for all $i \in \mathcal{J}$ and hence $d(\mathcal{J}) = \{0\}$ by Lemma 2.3(ii). Now, returning to (3.21) and applying the property defining of d, we find that $i\mathcal{J} = \{0\}$ for all $i \in \mathcal{J}$; again in view of Lemma 2.3(ii), we conclude that $\mathcal{J} = \{0\}$, a contradiction.
For the two cases $d(j) = -j$ and $d(ij) = -ij$ for all $i, j \in \mathcal{J}$, we can us the same arguments as used above.

□

The following example demonstrates that the 3-primeness of $\mathcal{N}$ in Theorem 3.5 is crucial.

Example 3.2 Let $(\mathbb{Z},+)$ and $(M_2(\mathbb{Z}),+)$ be the group of integers and the usual additive group of 2×2 matrix, respectively. Let $\mathcal{N}=\mathbb{Z}\times M_2(\mathbb{Z})$, then $(\mathcal{N},+)$ is a group. Define the multiplicative law, denoted $\star$, on $\mathcal{N}$ by: $(x_1,y_1)\star(x_2,y_2)=(0,det(y_1)y_2)$ for all $(x_1,y_1),(x_2,y_2)\in\mathcal{N}$. It is obvious to prove that $(\mathcal{N},+,\star)$ is 2-torsion free, non 3-prime left near-ring. We define $\mathcal{J}$, d_1 and d_2 by:

$$\mathcal{J}=\mathbb{Z}\times\{0\},\quad d_1(x,y)=(x,0)\quad\text{and}\quad d_2(x,y)=(-x,0).$$

So that, $\mathcal{J}$ is a nonzero Jordan ideal of $\mathcal{N}$, d_1 and d_2 are nonzero multiplicative derivations of $\mathcal{N}$. However, the following situations hold:

1. $d_1(J)=J$ for all $J\in\mathcal{J}$,
2. $d_2(J)=-J$ for all $J\in\mathcal{J}$,
3. $d_1(AB)=AB$ for all $A,B\in\mathcal{J}$,
4. $d_1(AB)=-AB$ for all $A,B\in\mathcal{J}$.

Open Question: In [8, Corollary 3], it was proven that if a 2-torsion free 3-prime near-ring $\mathcal{N}$ admits a nonzero Jordan ideal J and a derivation d such that $d(J)=\{0\}$, then $d=0$ or the elements of J commute under the multiplication of $\mathcal{N}$. Also, as indicated in [5, Lemma 2.7], the second previous condition brings us back to J is commutative. Does these results remain true when d is a multiplicative derivation of $\mathcal{N}$ satisfying $d^2(J)=\{0\}$?.

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Authors

Slimane El Amrani,
LMACS Laboratory,
Faculty of Sciences and Technology,
Sultan Moulay Slimane University,
Beni Mellal, Morocco,
E-mail: slimaneelamrani1994@gmail.com

Abderrahmane Raji (Corresponding author),
LMACS Laboratory,
Faculty of Sciences and Technology,
Sultan Moulay Slimane University,
Beni Mellal, Morocco,
E-mail: rajiabd2@gmail.com