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A Barbilian-type semi-distance, the stabilizing distance, and Gromov hyperbolic spaces

Wladimir G. Boskoff · Bogdan D. Suceavă

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Abstract In the present work, we focus on how the geometries determined by metrics defined through logarithmic oscillations, developed originally in Barbilian's work, are naturally connected to the current study of Gromov hyperbolic spaces. We introduce a distance on a subset of the n-dimensional real space and we call it stabilizing metric. We show it is a distance and we prove it generates an example of Gromov hyperbolic distance on the punctured open unit ball. Then, we study its geometric properties, as this distance induces a Riemannian metric conformal to the Euclidean metric. Additionally, we compare the stabilizing metric with a semi-metric of Barbilian type and with Vuorinen's metric j_G, and we discuss an extension to a class of metrics generated by combinations of these remarkable metrics.

Keywords Apollonian metrics · Gromov hyperbolic spaces · Barbilian spaces · Barbilian metrization procedure

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1 Introduction

Barbilian's metrization procedure was introduced in [1] and the original geometric idea was to provide a generalization of the Klein-Beltrami distance in the hyperbolic geometry (see [30]). Pursuing this quest, Barbilian introduced a distance on a planar domain by a metric formula given by the so-called logarithmic oscillation. In 1959, Barbilian generalized this process to domains of a more general form, withstanding not necessarily on planar sets, but in a more abstract setting (see [2–5]). For the full history of the discovery of what is known today as the Apollonian metric, see [9], and also the very interesting description by Klén and Vuorinen in [27].

Wladimir G. Boskoff
Department of Mathematics and Computer Science
Ovidius University, Constanţa, Romania

Bogdan D. Suceavă
Department of Mathematics
California State University at Fullerton
Fullerton, CA, 92834-6850, U.S.A.
E-mail: bsuceava@fullerton.edu

Barbilian's metrization procedure is important for the following reasons: (1) It yields a natural generalization of Poincaré and Beltrami-Klein's hyperbolic geometries; (2) It has been re-discovered naturally in the context of the study of Apollonian metric (see, e.g., [13,17,18,20–24,31]); (3) Provides a large class of examples of Lagrange generalized metrics irreducible to Riemann, Finsler, or Lagrange metrics (see [10]); (4) It represents a unifying process that can generate in one single way the metrics associated with the spherical, Euclidean and hyperbolic geometry (see [10]).

For these reasons it is important to explore and understand the connections between Barbilian's metrization procedure and other fundamental concepts in coarse geometry. Thus, it is natural to bridge the gap between Barbilian's metrization procedure and Gromov hyperbolicity.

Pursuing Gromov's influential work [15], several investigations have shown that some metrics, particularly in the area of geometric function theory, are Gromov hyperbolic (see [7,21,25,26]). In the classical theory, an important class of examples of Gromov hyperbolic spaces are the so-called CAT(κ) spaces, with $\kappa < 0$ (see [12], p.106). Recently Ibragimov [25] considered an interesting new distance and studied its Gromov hyperbolicity. In the present work, we consider a new metric and we study its properties.

We are connecting Barbilian's metrization theory to Gromov hyperbolicity in the following way. In section 2 we revisit the whole framework of the Barbilian's metrization procedure and prepare the construction of the stabilizing distance, which is introduced in section 3. In the last section we explain the geometric interpretations derived from our construction.

To remind the terminology, consider a metric space (M, d) where d satisfies the usual definition of a distance. Given $X, Y, Z \in M$, the quantity $(X|Y)_Z = \frac{1}{2}[d(X, Z) + d(Y, Z) - d(X, Y)]$ is called the *Gromov product* of X and Y with respect to Z. Denote $a \wedge b = \min\{a, b\}$. The metric space (M, d) is called *Gromov hyperbolic* if there exists some constant $\delta \geq 0$ such that $(X|Y)_W \geq (X|Z)_W \wedge (Z|Y)_W - \delta$, for all $X, Y, W, Z \in M$. Using the fact that $a \vee b = \max\{a, b\}$, the Gromov hyperbolic condition can be rewritten in the following way. (M, d) is Gromov hyperbolic if there exists a constant $\delta \geq 0$ such that $d(X, Z) + d(Y, W) \leq [d(X, W) + d(Y, Z)] \vee [d(X, Y) + d(Z, W)] + 2\delta, \forall X, Y, W, Z \in M$.

The geometric idea is best captured in Mikhail Gromov's explanation from [16] at p. 19, where he writes:

> *It is hardly possible to find a convincing definition of the curvature (tensor) for an arbitrary metric space X, but one can distinguish certain classes of metric spaces corresponding to Riemannian manifolds with curvatures of a given type. This can be done, for example, by imposing inequalities between mutual distances of finite configurations of points in X.*

In the present work, we focus on how the geometries determined by metrics defined through logarithmic oscillations, developed originally in Barbilian's works [1,2], are naturally connected and are merging towards the current study of Gromov hyperbolic spaces.

2 Barbilian's logarithmic oscillation on the punctured n- dimensional real space yields a Gromov hyperbolic space

Consider G an arbitrary subset in $\mathbb{R}^n$. For any x and y in $\mathbb{R}^n$ denote by $|x-y|$ the Euclidean distance between x and y. Let $M \subset \mathbb{R}^n$ such that $M \cap G = \emptyset$. Denote $\delta_x = \min_{z\in M} |x-z|$, with $x \in G$.

Theorem 2.1 (Barbilian**, 1934, 1959)** *Consider $G = \mathbb{R}^n - \{0\}$ and $M = \{0\}$. Denote by $\delta_x = |x-0| = |x|$, the Euclidean distance between x and the origin. Then*

$$d_B(x,y) = \ln \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \tag{2.1}$$

is a semi-distance on $G = \mathbb{R}^n - \{0\}$.

Although a more general fact was proved by Barbilian in [2], we'll present here the argument in the context specified in the Theorem, since some of these steps are pursued in the proofs in Section 3, and these results are the main facts in this work. For the historical importance of Barbilian's distance (see [9]). For the original theory, as developed by Barbilian, see [2–5]. Barbilian's distance was rediscovered later in [6] and studied under the name of Apollonian metric by many other authors e.g. [8,9, 13,17,21]. The point-set topological context in which the distance could be endowed is studied in [29] (see also [28]).

Proof. Clearly $d_B(x,y) = 0$ if $x = y$. However, if $\delta_x = \delta_y$ we get $d_B(x,y) = 0$. That's why it is a semi-distance. Further: $d_B(x,y) = d_B(y,x)$, $\forall x, y \in \mathbb{R} - \{0\}$. To prove the triangle inequality, we must show that

$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \cdot \frac{\delta_z \vee \delta_y}{\delta_z \wedge \delta_y} \geq \frac{\delta_x \vee \delta_z}{\delta_x \wedge \delta_z}, \qquad \forall x, y, z \in \mathbb{R}^n - \{0\}. \tag{2.2}$$

Due to the symmetry in x and z, out of the six cases that need to be discussed, there will be three left, namely:

(i) $\delta_x > \delta_y > \delta_z$, which reduce the inequality (2.2) to $\frac{\delta_x}{\delta_y} \cdot \frac{\delta_y}{\delta_z} \geq \frac{\delta_x}{\delta_z}$, which holds true.

(ii) $\delta_x > \delta_z > \delta_y$, which reduce the inequality (2.2) to $\frac{\delta_x}{\delta_y} \cdot \frac{\delta_z}{\delta_y} \geq \frac{\delta_x}{\delta_z}$, which holds true since

$$\delta_x \left(\frac{\delta_z^2 - \delta_y^2}{\delta_z} \right) \geq 0, \qquad \forall x, y, z \in \mathbb{R}^n - \{0\}.$$

(iii) $\delta_y > \delta_x > \delta_z$, reduces the inequality (2.2) to $\frac{\delta_y}{\delta_x} \cdot \frac{\delta_y}{\delta_z} \geq 1$. This turns out to be true since $(\delta_y^2 - \delta_x^2) \cdot \frac{1}{\delta_x \cdot \delta_z} \geq 0, \forall x, y, z \in \mathbb{R}^n - \{0\}$. □

Remark 2.1 We should point out why we called $d_B(x,y)$ a semi-distance of Barbilian type. Consider $x, y \in \mathbb{R}^n - \{0\}$ with the property that $|x| = |x-0| = \delta_x > \delta_y = |y-0| = |y|$.

Consider a set $K \subset \mathbb{R}^n$ and $p \in K$. The influence function (for terminology, see [2]) $f : K \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ defined by $f(p,x) = |x-p|$ for the particular case when $K = \{0\}$ leads us to

$$M_1 = \max_{p\in K} \frac{f(p,x)}{f(p,y)} = \max_{p\in K} \frac{|x-p|}{|y-p|} = \frac{|x-0|}{|y-0|} = \frac{|x|}{|y|} = \frac{\delta_x}{\delta_y},$$

$$m_1 = \min_{p\in K} \frac{f(p,x)}{f(p,y)} = \min_{p\in K} \frac{|x-p|}{|y-p|} = \frac{|x-0|}{|y-0|} = \frac{|x|}{|y|} = \frac{\delta_x}{\delta_y}.$$

Since we are in the framework of the Barbilian's work presented in [2] and consistently used in works like e.g. [8,9], the logarithmic oscillation is:

$$\ln \frac{M_1}{m_1} = \ln \left(\frac{\delta_x}{\delta_y}\right)^2 = 2\ln \frac{\delta_x}{\delta_y} = 2\ln \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} = 2d_B(x,y).$$

We obtain that $d_B(x,y)$ is actually half of Barbilian's classical distance induced by logarithmic oscillation.

We present next the result that inspires the title of the present section.

Theorem 2.2 *Consider $G = \mathbb{R}^n - \{0\}$ and $M = \{0\}$. Denote by $\delta_x = |x-0| = |x|$, the Euclidean distance between x and the origin. Then the distance given by* (2.1) *is Gromov hyperbolic on G, with $\delta = 0$.*

Proof. We need to prove there is a $\delta \geq 0$ such that $d_B(x,y) + d_B(z,w) \leq (d_B(x,z) + d_B(y,w)) \vee (d_B(x,w) + d_B(y,z)) + 2\delta, \forall x,y,z \in \mathbb{R}^n - \{0\}$. We prove that this condition is satisfied for $\delta = 0$. The condition we have to show rephrases as:

$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \cdot \frac{\delta_z \vee \delta_w}{\delta_z \wedge \delta_w} \leq \left(\frac{\delta_x \vee \delta_z}{\delta_x \wedge \delta_z} \cdot \frac{\delta_y \vee \delta_w}{\delta_y \wedge \delta_w}\right) \vee \left(\frac{\delta_x \vee \delta_w}{\delta_x \wedge \delta_w} \cdot \frac{\delta_y \vee \delta_z}{\delta_y \wedge \delta_z}\right). \tag{2.3}$$

Without any loss of generality, assume that $\delta_x \leq \delta_y \vee \delta_z \vee \delta_w$. Under this assumption, (2.3) becomes:

$$\frac{\delta_y}{\delta_x} \cdot \frac{\delta_z \vee \delta_w}{\delta_z \wedge \delta_w} \leq \left(\frac{\delta_z}{\delta_x} \cdot \frac{\delta_y \vee \delta_w}{\delta_y \wedge \delta_w}\right) \vee \left(\frac{\delta_w}{\delta_x} \cdot \frac{\delta_y \vee \delta_z}{\delta_y \wedge \delta_z}\right). \tag{2.4}$$

By using

$$\frac{\delta_z \vee \delta_w}{\delta_z \wedge \delta_w} = \frac{\delta_z}{\delta_w} \vee \frac{\delta_w}{\delta_z},$$

and after a simplification both sides by $\delta_x > 0$, the inequality (2.4) turns into:

$$\frac{\delta_y \cdot \delta_z}{\delta_w} \vee \frac{\delta_y \cdot \delta_w}{\delta_z} \leq \left(\frac{\delta_y \cdot \delta_z}{\delta_w} \vee \frac{\delta_w \cdot \delta_z}{\delta_y}\right) \vee \left(\frac{\delta_y \cdot \delta_w}{\delta_z} \vee \frac{\delta_w \cdot \delta_z}{\delta_y}\right),$$

which is certainly true. □

3 The stabilizing distance

Consider G an arbitrary subset in $\mathbb{R}^n$. For any x and y in $\mathbb{R}^n$ denote by $|x-y|$ the Euclidean distance between x and y. Let $M \in \mathbb{R}^n$ such that $M \cap G = \emptyset$. Denote $\delta_x = \min_{z\in M} |x-z|$, with $x \in G$. We introduce *the stabilizing distance* on G by the following expression:

$$d_{G,M}(x,y) = \ln \left(\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x-y|\right). \tag{3.1}$$

The set M is called *the supporting set*. Note that this distance is different than the j_G and $\tilde{j}_G$ metrics introduced by GEHRING and OSGOOD [14], and by VUORINEN [31], respectively, and whose Gromov hyperbolicity is studied by HÄSTÖ in [19]. We are proposing this terminology since in the expression of the logarithmic oscillation there is added further information inherited from the Euclidean metric; this thought led us to this terminology.

Theorem 3.1 *With the notations specified above, $d_{G,M}$ is a distance on G.*

Proof. Since $\delta_x \vee \delta_y \geq \delta_x \wedge \delta_y$, clearly $\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x-y| \geq 1$, therefore $d_{G,M}(x,y) = 0$ if and only if $x = y$. For the same reason, $d_{G,M}(x,y) \geq 0$, $\forall x, y \in \mathbb{R}^n$. Obviously $d_{G,M}(x,y)$ is symmetric.

To complete the argument, we have to prove that

$$d_{G,M}(x,y) + d_{G,M}(y,z) \geq d_{G,M}(x,z), \qquad \forall x, y, z \in G.$$

This inequality can be rephrased as

$$\left(\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x-y|\right) \cdot \left(\frac{\delta_y \vee \delta_z}{\delta_y \wedge \delta_z} + |y-z|\right) \geq \left(\frac{\delta_x \vee \delta_z}{\delta_x \wedge \delta_z} + |x-z|\right), \forall x, y, z \in G.$$

Since this inequality is symmetric in x and z, in order to prove this assertion we distinguish three cases:

(i) $\delta_x \geq \delta_y \geq \delta_z$;
(ii) $\delta_x \geq \delta_z \geq \delta_y$;
(iii) $\delta_y \geq \delta_z \geq \delta_x$.

In the first case, the inequality reduces to:

$$\frac{\delta_x}{\delta_y} \cdot |y-z| + \frac{\delta_y}{\delta_z} \cdot |x-y| + |x-y| \cdot |y-z| \geq |x-z|,$$

which is true by the assumptions corresponding to case (i) and by triangle inequality in $\mathbb{R}^n$. In the second case, the inequality reduces to

$$\frac{\delta_x}{\delta_y^2 \cdot \delta_z}(\delta_z^2 - \delta_y^2) + \frac{\delta_z}{\delta_y} \cdot |x-y| + \frac{\delta_x}{\delta_y} \cdot |y-z| + |x-y| \cdot |y-z| \geq |x-z|.$$

This is also true, for similar reasons. In the third case, the inequality that needs to be established reduces to:

$$\frac{1}{\delta_z \cdot \delta_x}(\delta_y^2 - \delta_x^2) + \frac{\delta_y}{\delta_x} \cdot |y-z| + \frac{\delta_y}{\delta_x} \cdot |x-y| + |x-y| \cdot |y-z| \geq |x-z|,$$

which is also true by the assumptions in case (iii) and by the standard triangle inequality. This concludes the proof. □

Theorem 3.2 *Consider as G the punctured open unit ball*

$$D = \{x \in \mathbb{R}^n | x = (a_1, a_2, ..., a_n), a_1^2 + a_2^2 + ... + a_n^2 < 1\} - \{(0,0,...,0)\}.$$

Let M=$\{(0,0,...,0)\}$ be the supporting set. Denote by $\delta_x = |x - (0,0, ..., 0)| = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}$, the Euclidean distance between the point $x = (a_1, a_2, ..., a_n)$ and the origin. Then the stabilizing distance $d_{D,M}$ defined by (3.1) *is Gromov hyperbolic, with $\delta = \frac{1}{2} \ln 9$.*

For the proof of this Theorem, we need the following.

Lemma 3.3 *In the conditions of Theorem 3.2, we have:*

$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x-y| \leq 3 \cdot \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \tag{3.2}$$

Proof. The triangle inequality yields $|x-y| < \delta_x \vee \delta_y + \delta_x \wedge \delta_y$. Hence:

$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x-y| \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + \delta_x \vee \delta_y + \delta_x \wedge \delta_y,$$

which immediately yields

$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x-y| \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + 2 \cdot \delta_x \vee \delta_y.$$

Since $0 < \delta_x \wedge \delta_y < 1$, then $\frac{1}{\delta_x \wedge \delta_y} > 1$. Therefore, $\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} > \delta_x \vee \delta_y$, which gives (3.2). □

We are ready now to prove Theorem 3.2.

Proof. We need to show that the stabilizing distance satisfies on D the Gromov syperbolicity condition with $2\delta = \ln 9$, when the supporting set M is just the singleton consisting of the origin. We have to prove that for any $x, y, z, w \in D$ the following inequality holds true:

$$\begin{aligned} &\frac{1}{9}\left(\frac{\delta_x \vee \delta_z}{\delta_x \wedge \delta_z} + |x-z|\right) \cdot \left(\frac{\delta_w \vee \delta_y}{\delta_w \wedge \delta_y} + |w-y|\right) \\ &\quad \leq \left(\frac{\delta_x \vee \delta_w}{\delta_x \wedge \delta_w} + |x-w|\right) \cdot \left(\frac{\delta_y \vee \delta_z}{\delta_y \wedge \delta_z} + |y-z|\right) \\ &\quad \vee \left(\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x-y|\right) \cdot \left(\frac{\delta_w \vee \delta_z}{\delta_w \wedge \delta_z} + |w-z|\right). \end{aligned} \tag{3.3}$$

Without loss of generality we can assume

$$\delta_x \leq \delta_y \wedge \delta_z \wedge \delta_w \tag{3.4}$$

We also use in our argument

$$\frac{\delta_y \vee \delta_z}{\delta_y \wedge \delta_z} = \frac{\delta_y}{\delta_z} \vee \frac{\delta_z}{\delta_y} \tag{3.5}$$

The inequality (3.3) is certainly true if when we maximize the left hand side and we minimize the right hand side we obtain a true statement. By using (3.2) we need to prove:

$$\frac{\delta_x \vee \delta_z}{\delta_x \wedge \delta_z} \cdot \frac{\delta_y \vee \delta_w}{\delta_y \wedge \delta_w} \leq \frac{\delta_x \vee \delta_w}{\delta_x \wedge \delta_w} \cdot \frac{\delta_y \vee \delta_z}{\delta_y \wedge \delta_z} \vee \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \cdot \frac{\delta_w \vee \delta_z}{\delta_w \wedge \delta_z}.$$

By using (3.4) and (3.5) all that's left to prove it

$$\frac{\delta_z}{\delta_x} \cdot \left(\frac{\delta_y}{\delta_w} \vee \frac{\delta_w}{\delta_y}\right) \leq \frac{\delta_w}{\delta_x} \cdot \left(\frac{\delta_y}{\delta_z} \vee \frac{\delta_z}{\delta_y}\right) \vee \frac{\delta_y}{\delta_x} \cdot \left(\frac{\delta_w}{\delta_z} \vee \frac{\delta_z}{\delta_w}\right).$$

By simplifying $\delta_x > 0$ both sides, we obtain

$$\frac{\delta_z \cdot \delta_y}{\delta_w} \vee \frac{\delta_z \cdot \delta_w}{\delta_y} \leq \left(\frac{\delta_w \cdot \delta_y}{\delta_z} \vee \frac{\delta_w \cdot \delta_z}{\delta_y} \right) \vee \left(\frac{\delta_z \cdot \delta_y}{\delta_w} \vee \frac{\delta_w \cdot \delta_y}{\delta_z} \right).$$

This last inequality is clearly true. □

4 Geometric interpretations

Remark 4.1 Consider now $D \subset \mathbb{R}^2$, the punctured open unit disk centered at the origin (same notation as in the previous section, taking now $n = 2$). We compute the Riemannian metric corresponding to the stabilizing distance. To this goal, let $x = (x_1, y_1)$, $y = (x_2, y_2) = (x_1 + dx, y_1 + dy)$. We have $|x - y| = \sqrt{dx^2 + dy^2}$. Bearing in mind that $\frac{\delta_x}{\delta_y} \vee \frac{\delta_y}{\delta_x} = \frac{|y|}{|x|}$ and the stabilizing distance

$$\begin{aligned} d_{D,M}(x, y) \approx ds &= \ln \left(1 + \frac{|y|}{|x|} - 1 + |x - y| \right) \\ &= \ln \left(1 + \frac{|y| - |x| + |x| \cdot |x - y|}{|x|} \right) \\ &\approx \frac{|y| - |x| + |x| \cdot |x - y|}{|x|} \approx \frac{1 + |x|}{|x|} \cdot |x - y|, \end{aligned}$$

where in the last step we have used $|y| - |x| \approx |y - x|$. We derive the expression of the Riemannian metric $ds^2 = \frac{(1+\sqrt{x^2+y^2})^2}{x^2+y^2} \cdot (dx^2 + dy^2)$.

Remark 4.2 As one may intuitively expect, the Gaussian curvature of the Riemannian metric corresponding to the stabilizing distance is everywhere negative on the punctured disk:

$$K(x, y) = -\frac{\sqrt{x^2 + y^2}}{1 + 6(x^2 + y^2) + x^4 + y^4 + 2x^2y^2 + 4\sqrt{x^2 + y^2} + 4(x^2 + y^2)^{3/2}}.$$

We conclude this work with a comparison between several metrics. We have discussed here Barbilian's semi-metric d_B, the stabilizing distance $d_{M,D}$ and we have mentioned Vuorinen's metric [31], $j_G(x, y) = \ln(1 + \frac{|x-y|}{\delta_x \wedge \delta_y})$. It's natural to ask how are these metrics related.

Remark 4.3 With the notations specified in the previous paragraph, for M formed by the singleton set with origin as its only element, on $D - \{0\}$ we have $d_B(x, y) - \ln 3 \leq d_{D,M}(x, y) \leq d_B(x, y) + \ln 3$. The argument to support this double inequality is the following:

$$\frac{1}{3} \cdot \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x - y| \leq 3 \cdot \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y}.$$

The sequence of inequalities provides an immediate argument to see that $d_{D,M}$ is Gromov hyperbolic, as a consequence of Hästö's argument from [19], pp. 1137-1138.

Remark 4.4 To compare Vuorinen's metric j_G with Barbilian's semi-metric d_B, by a similar argument we obtain the following double inequality $d_B(x,y)-\ln 3 \leq j_G(x,y) \leq d_B(x,y)+\ln 3$. This is due to the following argument $\frac{1}{3}\frac{\delta_x\vee\delta_y}{\delta_x\wedge\delta_y} \leq \frac{\delta_x\vee\delta_y}{\delta_x\wedge\delta_y} \leq 1+\frac{|x-y|}{\delta_x\wedge\delta_y} \leq 3\cdot\frac{\delta_x\vee\delta_y}{\delta_x\wedge\delta_y}$. Along these lines one can immediately prove that j_G is Gromov hyperbolic, see HÄSTÖ'S Theorem 1, [19], p. 1138.

Example 4.1 For this remark, we work on $D-\{0\}\subset\mathbb{R}^n$. Let $M=\{0\}$. Clearly the sum of two metrics is another metric. Consider the metric $d_C(x,y)=j_G(x.y)+d_{D,M}(x,y)$ on $D-\{0\}$. It satisfies

$$\frac{1}{6}\cdot\frac{\delta_x\vee\delta_y}{\delta_x\wedge\delta_y} \leq 2\cdot\frac{\delta_x\vee\delta_y}{\delta_x\wedge\delta_y} \leq d_C(x,y) \leq 6\cdot\frac{\delta_x\vee\delta_y}{\delta_x\wedge\delta_y}, \qquad \forall x,y\in D-\{0\}.$$

Hence $d_B(x,y)-\ln 6 \leq d_C(x,y) \leq d_B(x,y)+\ln 6, \forall x,y\in D-\{0\}$, and this proves the fact that d_C on $D-\{0\}$ is another Gromov hyperbolic space.

To get a better understanding of d_C, we elaborate on its form:

$$\begin{aligned} d_C(x,y) &= \ln\left(1+\frac{|x-y|}{\delta_x\wedge\delta_y}\right)+\ln\left(\frac{\delta_x\vee\delta_y}{\delta_x\wedge\delta_y}+|x-y|\right) \\ &= \ln\left(1+\frac{|x-y|}{\delta_x\wedge\delta_y}\right)\left(\frac{\delta_x\vee\delta_y}{\delta_x\wedge\delta_y}+|x-y|\right) \\ &= \ln\left(\frac{\delta_x\vee\delta_y}{\delta_x\wedge\delta_y}+|x-y|+\frac{\delta_x\vee\delta_y}{(\delta_x\wedge\delta_y)^2}\cdot|x-y|+\frac{|x-y|}{\delta_x\wedge\delta_y}\right). \end{aligned}$$

Remark 4.5 Similar remarks can be pursued for a whole class of metrics of form $\alpha j_G(x,y)+\beta d_{D,M}(x,y)$.

Remark 4.6 Since in Remark 4.2 we've discussed the Gaussian curvature of the Riemannian metric generated by the stabilizing metric, we should see how much this is for Vuorinen's metric j_G, as well as for Barbilian's semi-distance d_B. For both d_B and j_G we obtain the following $ds^2=\frac{1}{x^2+y^2}(dx^2+dy^2)$ on $D-\{0\}\subset\mathbb{R}^2$. This metric has Gaussian curvature everywhere vanishing $K\equiv 0$, since if one switches to polar coordinates $x=r\cos\theta$, $y=r\sin\theta$, we get $ds^2=\frac{1}{r^2}dr^2+d\theta^2$. If in this last form we swap $r\to\frac{1}{r_1}$, we get $ds^2=dr_1^2+d\theta^2$, which is the flat Euclidean metric.

Remark 4.7 It will always be interesting to study new approachable and intuitive examples of Gromov hyperbolic metric spaces (see, e.g., [11]).

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