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On weakly τ-quasinormal subgroups and p-nilpotency of finite groups

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Abstract In this article, we give some sufficient conditions for a finite group G to be p-nilpotent under the assumption that p-subgroups of G of certain orders are weakly τ-quasinormal in G. In particular, we extend some recent results.

Keywords s-permutable · weakly τ-quasinormal · p-nilpotent

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1 Introduction

All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert (see [6]). G always denotes a finite group, $|G|$ is the order of G and p denotes a fixed prime.

A subgroup H of G is said to be c-normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G (see [15]). In 2003, Guo and Shum first proved the following theorem:

Theorem 1.1 ([4, Theorem 3.1]) *Let p be an odd prime dividing $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

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A subgroup H of G is said to be s-permutable (or π-quasinormal, s-quasinormal) in G, if $HP = PH$ for every Sylow subgroup P of G (see [7]). In 2009, Wei and Guo got the following theorem:

Theorem 1.2 ([18, Theorem 3.1]) *Let p be an odd prime dividing $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ is s-permutable in G, then G is p-nilpotent.*

A subgroup H of G is said to be s-semipermutable in G if $HP = PH$ for any Sylow p-subgroup P of G with $(p, |H|) = 1$ (see [1]). In 2006, Wang and Wang proved the following theorem:

Theorem 1.3 ([16, Theorem 3.1]) *Let p be an odd prime dividing $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is s-semipermutable in G, then G is p-nilpotent.*

In 2010, Han generalized Theorems 1.2 and 1.3 as follows:

Theorem 1.4 ([5, Theorem 3.1]) *Let p be an odd prime dividing $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ is s-semipermutable in G, then G is p-nilpotent.*

A subgroup H of G is said to be weakly s-permutable in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H (see [14]). In [11], Miao proved the following theorem:

Theorem 1.5 ([11, Theorem 3.1]) *Let p be an odd prime dividing $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is weakly s-permutable in G, then G is p-nilpotent.*

In order to generalized above mentioned subgroups, Lukyanenko and Skiba (see [9]) introduced the following concept.

Definition 1.6 *Let H be a subgroup of a group G. Then we say that:*
(1) *H is τ-quasinormal in G if H permutes with every Sylow subgroup Q of G such that $(|H|, |Q|) = 1$ and $(|H|, |Q^G|) \neq 1$;*
(2) *H is weakly τ-quasinormal in G if G has a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of H which are τ-quasinormal in G.*

In [8], the authors got the following theorem:

Theorem 1.7 ([8, Theorem 4]) *Let p be an odd prime dividing $|G|$ and and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is weakly τ-quasinormal in G, then G is p-nilpotent.*

It is clear from the definitions that s-permutable subgroups, s-semipermutable, c-normal subgroups and weakly s-permutable subgroups are weakly τ-quasinormal subgroups. But the converse is not true, as seen in [9, Examples 1.2 and 1.3]. Our main aim here is to use the weakly τ-quasinormal subgroups to strengthen and extend above mentioned Theorems 1.1, 1.2, 1.3, 1.4, 1.5 and 1.7.

2 Preliminaries

Lemma 2.1 ([14, Lemma 2.6]) *Let N be an elementary abelian normal subgroup of G. Assume that there is a subgroup D of N with $1 < |D| < |N|$ such that every subgroup of N with order $|D|$ is weakly s-permutable in G. Then some maximal subgroup of N is normal in G.*

Lemma 2.2 ([9, Lemma 2.4]) *Let $H \leq K \leq G$ and p be a prime.*

(1) *If H is weakly τ-quasinormal in G, then H is weakly τ-quasinormal in K.*

(2) *Suppose that K is a p-group and H is normal in G. If K is weakly τ-quasinormal in G, then K/H is weakly τ-quasinormal in G/H.*

(3) *Suppose that H is normal in G. Then EH/H is weakly τ-quasinormal in G/H for every weakly τ-quasinormal p-subgroup E in G satisfying $(|H|, |E|) = 1$.*

(4) *Suppose H is a p-group and H is not τ-quasinormal in G. Assume that H is weakly τ-quasinormal in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.*

Lemma 2.3 ([3, Theorem 8.3.1]) *Let P be a Sylow p-subgroup of G, where p is an odd prime divisor of $|G|$. Then G is p-nilpotent if and only if $N_G(Z(J(P)))$ is p-nilpotent, where $J(P)$ is the Thompson subgroup of P.*

Lemma 2.4 ([10, Theorem 1.4]) *Let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Suppose that there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ or 4 (if P is a nonabelian 2-group and $|D| = 2$) is weakly τ-quasinormal in G. Then G is p-nilpotent.*

Lemma 2.5 ([13, Theorem 3.1]) *Suppose that P is a p-group and A is an automorphism p'-group of P and p^k is a fixed number satisfying $1 < p^k < |P|$, where p is an odd prime. If every subgroup of P with order p^k is A-invariant, then A is cyclic.*

Lemma 2.6 ([12, Lemma A]) *If P is an s-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.7 ([17, Lemma 2.8]) *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

Lemma 2.8 ([8, Lemma 2]) *Let $H \leq G$ and p be a prime.*

(1) *If H is τ-quasinormal in G and $H \leq O_p(G)$, then H is s-permutable in G.*

(2) *If H is weakly τ-quasinormal in G and $H \leq O_p(G)$, then H is weakly s-permutable in G.*

3 Main results

Theorem 3.1 *Let p be an odd prime dividing $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ is weakly τ-quasinormal in G, then G is p-nilpotent.*

Proof. Assume that the assertion is false and let G be a counterexample of minimal order. Then we have

Step 1. If $P \leq M < G$, then M is p-nilpotent.

Since $N_M(P) \leq N_G(P)$ and $N_G(P)$ is p-nilpotent, we have that $N_M(P)$ is p-nilpotent. By Lemma 2.2(1), every subgroup H of P with order $|D|$ is weakly τ-quasinormal in M. Therefore, M satisfies the hypothesis of the theorem, and so M is p-nilpotent by the choice of G.

Step 2. $O_p(G) \neq 1$.

We consider the group $Z(J(P))$, where $J(P)$ is the Thompson subgroup of P. If $N_G(Z(J(P))) < G$, then, by Step 1, $N_G(Z(J(P))$ is p-nilpotent. By virtue of Lemma 2.3, G is p-nilpotent, a contradiction. Hence $N_G(Z(J(P))) = G$ and $1 < Z(J(P)) \leq O_p(G) < P$.

Step 3. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then we can consider the factor group $\overline{G} = G/O_{p'}(G)$. Since $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$, it follows that $N_{\overline{G}}(\overline{P})$ is p-nilpotent. For every subgroup $\overline{H}$ of Sylow p-subgroup $PO_{p'}(G)/O_{p'}(G)$ of $G/O_{p'}(G)$ with order $|D|$, we can write $\overline{H} = HO_{p'}(G)/O_{p'}(G)$, where H is a subgroup of P with order $|D|$. Then H is weakly τ-quasinormal in G by hypothesis. Hence $\overline{H}$ is weakly τ-quasinormal in $\overline{G}$ by Lemma 2.2(3). Therefore, $\overline{G}$ satisfies the hypothesis of our theorem. By the minimal choice of G, we get that $G/O_{p'}(G)$ is p-nilpotent and so G is p-nilpotent, a contradiction.

Step 4. $G/O_p(G)$ is p-nilpotent.

Let $\overline{G} = G/O_p(G)$, $\overline{P} = P/O_p(G)$, $\overline{K} = Z(J(\overline{P}))$ and $G_1/O_p(G) = N_{\overline{G}}(Z(J(\overline{P})))$. Since $O_p(\overline{G}) = 1$, we have $N_{\overline{G}}(Z(J(\overline{P})) < \overline{G}$. Thus $G_1 < G$. By Step 1, we have G_1 is p-nilpotent. Then $N_{\overline{G}}(Z(J(\overline{P}))$ is p-nilpotent. Thus $\overline{G}$ is p-nilpotent by Lemma 2.3.

Step 5. G is p-soluble and $C_G(O_p(G)) \leq O_p(G)$. It follows easily G is p-soluble by Step 4. Since $O_{p'}(G) = 1$, we have that $C_G(O_p(G)) \leq O_p(G)$ by [3, Theorem 6.3.2].

Step 6. $G = PQ$, where Q is a Sylow q-subgroup of G with $q < p$.

Since G is p-soluble, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G ([3, Theorem 6.3.5]). If $PQ < G$, then PQ is p-nilpotent by Step 1 and so $Q \leq C_G(O_p(G)) \leq O_p(G)$. This contradiction shows that $G = PQ$. If $p < q$, then, by virtue of Lemma 2.4, G is p-nilpotent, a contradiction. Hence Step 6 holds.

Step 7. $|P : D| > p$.

This follows from Theorem 1.7.

Step 8. If G has a normal subgroup B such that $|G : B| = p$, then $P \cap B = O_p(G)$.

Obviously, $P \cap B$ is a maximal subgroup of P and also a Sylow p-subgroup of B. If $N_G(P \cap B) < G$, then, by Step 1, $N_G(P \cap B)$ is p-nilpotent and so is $N_B(P \cap B)$. From Step 7 and Lemma 2.2(1), every subgroup of $P \cap B$ of order $|D|$ is weakly τ-quasinormal in B. Consequently, B satisfies the hypothesis of our theorem. Hence B is p-nilpotent by the minimal choice of G. Consequently, G is p-nilpotent. This contradiction shows that $P \cap B$ is a normal subgroup of G and so $P \cap B \leq O_p(G)$. Since $O_p(G) < P$, it follows that $P \cap B = O_p(G)$.

Step 9. $|D| < |O_p(G)|$. Since $G/O_p(G)$ is p-nilpotent, it follows from $G = PQ$ that $QO_p(G)/O_p(G)$ is the normal p-complement of $P/O_p(G)$ in $G/O_p(G)$. Write $L = QO_p(G)$. Then L is normal in G. Clearly, G/L is a p-group. Pick a maximal subgroup M/L of G/L. Then M is a normal subgroup of G such that $|G : M| = p$. By Step 8,

$P \cap M = O_p(G)$. It is easy to see that $O_p(G)$ is a maximal subgroup of P. By Step 7, $|D| < |O_p(G)|$.

Step 10. Every subgroup H of $O_p(G)$ with order $|D|$ is s-permutable in G.

Let H be any subgroup of $O_p(G)$ with order $|D|$. Then H is weakly τ-quasinormal in G by hypothesis. In view of Lemma 2.8, H is weakly s-permutable in G. If H is not s-permutable in G, then, by [14, Lemma 2.10(5)], G has a normal subgroup T such that $|G : T| = p$ and $G = HT$. By Step 8, $P = P \cap HT = H(P \cap T) = HO_p(G) = O_p(G)$, a contradiction. Therefore, H is s-permutable in G.

Step 11. The final contradiction. By Step 5, $C_Q(O_p(G)) = 1$ and so $Q = Q/C_Q(O_p(G)) \lesssim Aut(O_p(G))$. Let H be any subgroup of $O_p(G)$ with order $|D|$. Since H is s-permutable in G by Step 10, it follows from Lemma 2.6 that H is Q-invariant. Hence Q is cyclic by Lemma 2.5. Now [6, IV, Satz 2.8] implies that $G = PQ$ is q-nilpotent and so P is normal in G. By hypothesis, $G = N_G(P)$ is p-nilpotent, a contradiction. □

Corollary 3.2 *Let p be an odd prime dividing $|G|$ and $\mathcal{F}$ a saturated formation containing $\mathcal{N}_p$, the class of all p-nilpotent groups. Suppose that G has a normal subgroup N such that $G/N \in \mathcal{F}$ and P is a Sylow p-subgroup of N. If $N_G(P)$ is p-nilpotent and there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ is weakly τ-quasinormal in G, then $G \in \mathcal{F}$.*

Proof. By Theorem 3.1, N is p-nilpotent. Let T be the normal p-complement of N. Then T is normal in G. First assume that $T \neq 1$. Since G/T satisfies the hypothesis of theorem, $G/T \in \mathcal{F}$ by induction. Let $F_i(i = 1, 2)$ be the full and integrated formation function such that $\mathcal{N}_p = LF(F_1)$ and $\mathcal{F} = LF(F_2)$, respectively. Then $G/C_G(A/B) \in F_1(r)$ for every G-chief factor A/B under T and every prime divisor r of $|A/B|$. By [2, Proposition IV 3.11], $G/C_G(A/B) \in F_2(r)$ for every G-chief factor A/B under T and every prime divisor r of $|A/B|$. This proves that $G \in \mathcal{F}$. Now assume that $T = 1$ and $N = P$. In this case, by our hypothesis, $G = N_G(P)$ is p-nilpotent and therefore $G \in \mathcal{F}$. □

Theorem 3.3 *Let p be an odd prime dividing $|G|$ and P a Sylow p-subgroup of G. Suppose that there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ is weakly τ-quasinormal in G and $N_G(H)$ is p-nilpotent. Then G is p-nilpotent.*

Proof. Assume that the assertion is false and let G be a counterexample of minimal order. Then we have

Step 1. If $P \leq M < G$, then M is p-nilpotent.

By Lemma 2.2(1), every subgroup H of P with order $|D|$ is weakly τ-quasinormal in M. Since $N_M(H) \leq N_G(H)$ and $N_G(H)$ is p-nilpotent, $N_M(H)$ is p-nilpotent. Therefore, M satisfies the hypothesis of the theorem, and so M is p-nilpotent by the choice of G.

Step 2. P is normal in G.

If $N_G(P) < G$, then $N_G(P)$ is p-nilpotent by Step 1. Applying Theorem 3.1, G is p-nilpotent, a contradiction. Hence $N_G(P) = G$, i.e., P is normal in G.

Step 3. Every subgroup H of P with order $|D|$ is weakly s-permutable in G.

This follows from Lemma 2.8.

Step 4. $|N| \leq |D|$.

This follows from Lemma 2.1.

Step 5. $|N| < |D|$.

If $|N| = |D|$, then $G = N_G(N)$ is p-nilpotent by hypothesis, a contradiction.

Step 6. G has a unique minimal normal subgroup N such that $N \leq P$ and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. In view of Step 2, G is p-soluble. Consequently, N is a p-subgroup or p'-subgroup. By Lemma 2.2, it is easy to see that G/N satisfies the hypothesis of the theorem. Hence G/N is p-nilpotent by the choice of G. If N is a p'-subgroup, then G is p-nilpotent, a contradiction. Hence $N \leq P$. Since the class of all p-nilpotent groups is a saturated formation, the uniqueness of N and $\Phi(G) = 1$ is obvious.

Step 7. The final contradiction.

Noticing $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. By virtue of Lemma 2.7, $P \cap M$ is normal in G. Consequently, $P \cap M = 1$ by the uniqueness of N. It follows that $N = P$, which contradicts Step 5. □

Corollary 3.4 *Let p be an odd prime dividing $|G|$ and $\mathcal{F}$ a saturated formation containing $\mathcal{N}_p$, the class of all p-nilpotent groups. Suppose that G has a normal subgroup N such that $G/N \in \mathcal{F}$ and P is a Sylow p-subgroup of N. If there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ is weakly τ-quasinormal in G and $N_G(H)$ is p-nilpotent, then $G \in \mathcal{F}$.*

Proof. By Theorem 3.3, N is p-nilpotent. Let T be the normal p-complement of N. Then T is normal in G. By using the arguments as in the proof of Corollary 3.2, we may assume that $T = 1$ and $N = P$ is a p-group. For any prime q dividing $|G|$ with $q \neq p$ and any Sylow q-subgroup Q of G, it is clear that PQ is a subgroup of G and so PQ is p-nilpotent by Theorem 3.3, and therefore we have $PQ = P \times Q$. It follows that $G/C_G(A/B)$ is a p-group for every G-chief factor A/B under P. Now by using [2, Proposition IV 3.11], we see that $G \in \mathcal{F}$. □

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