An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Tomul LXII, 2016, f. 2, vol. 1

A Skorohod problem driven by Clarke subdifferential and applications to SDEs

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Received: 15.VI.2012 / Accepted: 20.IX.2012

Abstract In this paper we consider the following generalized Skorohod problem with singular input:

$$\begin{cases} dx(t) + \partial\varphi(x(t))\,dt + \partial_C\phi(x(t))\,dt + G(t,x(t))dt \ni dm(t), \quad t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{1}$$

where m is a continuous function, $\partial_C\phi$ is the Clarke subdifferential operator associated to a function ϕ, $\partial\varphi$ is the subdifferential of a convex, lower semicontinuous function φ and G is a multivalued operator which is upper semicontinuous with respect to x and measurable with respect to t. We provide existence and uniqueness results for the solution of the deterministic problem, and, as applications, SDEs driven by a Clarke subdifferential are also envisaged.

Keywords Clarke subdifferential · Skorohod problem · Stochastic variational inequalities · Multivalued operators

Mathematics Subject Classification (2010) 47J20 · 60H10 · 49J40

1 Introduction

Since the early sixties, the Skorohod problem was studied, starting with its simplest form, by many authors. We mention, for instance: Skorohod [20], McKean [12], Watanabe [22], Chaleyat-Maurel, El Karoui and Marchal [5]. Roughly speaking, the simplest form of the Skorohod problem was as follows: Given a continuous function $m : \mathbb{R}^+ \to \mathbb{R}$ with $m(0) \in \mathbb{R}^+$, find two continuous functions $x, k : \mathbb{R}^+ \to \mathbb{R}$ such that

i) $x(t) + k(t) = x_0 + m(t), t \geq 0$,
ii) $x(t) \geq 0,\ t \geq 0$,

The work for this paper was supported by founds from the Grant PN-II-ID-PCE-2011-3-0843, *Deterministic and stochastic systems with state constraints*, no. 241/05.10.2011.

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iii) k is a nondecreasing function, $k(0) = 0$ and $k(t) = \int_0^t 1_{\{x(s)=0\}} dk(s)$.

The function k is a feedback which forces the solution x to remain in the positive half space. Later the problem was considered on multidimensional spaces and then for infinite dimensional spaces. Moreover, the complexity of the domain on which the solution is forced to remain was steeply generalized, as we can see in Tanaka [21], Lions and Sznitman [10], Saisho [18], Dupuis and Ishii [7], Cépa [4]. The Skorohod problem proved to be useful in practical applications, becoming important when modeling different phenomena from molecular dynamics, financial mathematics, games theory, queueing networks and control theory.

In [16], Răşcanu considered the Skorohod problem in the following equivalent form

$$\begin{cases} dx(t) + \partial I_{[0,\infty[}(x(t))(dt) \ni dm(t), \\ x(0) = x_0 \in \mathbb{R}^+, \end{cases}$$

where $\partial I_{[0,\infty[}$ denotes the subdifferential operator of $I_{[0,\infty[}$, which is the convex indicator function. Some generalizations were fully analyzed by considering the multivalued Skorohod problem driven by the subdifferential operator of a proper lower semicontinuous function φ, as we can see below

$$\begin{cases} dx(t) + \partial\varphi(x(t))(dt) \ni f(t, x(t))dt + dm(t), \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Moreover, he proved an existence and uniqueness result for the more general problem

$$\begin{cases} dx(t) + Ax(t)(dt) \ni f(t, x(t))dt + dm(t), \ t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1.2}$$

where $A : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a maximal monotone operator, not necessary a subdifferential operator and m, f are as above (see [16]). This short history of the Skorohod problem and its generalizations, considered on convex domains, can not omit the recent results obtained by Răşcanu, Rotenstein [15] and Gassous, Răşcanu, Rotenstein [8]. In the first paper, the authors provided a new approach when dealing with Eq.(1.2), via convex optimization problems. In the second one, the generalization of the Skorohod problem with oblique reflection to the form of a variational inequality was studied, in both the deterministic and the stochastic case. Another interesting generalization have been made by Maticiuc et al. [11] that have proved the existence and uniqueness of a solution for Eq. (1.2) with the singular input being generated by a càdlàg function m.

The non convex set-up for the Skorohod problem was recently studied by Buckdahn et al. in [3] where the generalized Skorohod equation (1.1) is considered with the Fréchet subdifferential $\partial^-\varphi$, and by Qin and Xue in [14], where they considered the classical subdifferential operator with a Clarke subdifferential operator, by considering the following equation:

$$\begin{cases} dx(t) + \partial_C\phi(x(t))(dt) + G(t, x(t))dt \ni 0, \quad t \in [0, T], \\ x(0) = x_0. \end{cases}$$

Our goal is to obtain a generalization of their results, by studying a multivalued equation, driven by a classical and Clarke subdifferential at the same time, as follows:

$$\begin{cases} dx(t) + \partial\varphi(x(t))(dt) + \partial_C\phi(x(t))(dt) \\ \qquad\qquad +G(t,x(t))dt \ni dm(t), t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{1.3}$$

Adding the singular input $dm\,(t)$ allowed us to consider more general entrance data. This will permit us to study a multivalued stochastic differential equation driven by Clarke subdifferential operator.

The paper is structured as follows. In the first section we introduce the main assumptions, hypothesis, notations and also we recall several useful results which will be used in the sequel. In the second section we define the notion of solution for our main equation and we present the theorems of existence and uniqueness, together with complete proofs. In the last section we study, as applications, a multivalued stochastic differential equation driven by the Clarke subdifferential operator.

2 Preliminaries and notations

a) Notations

We will consider on the space $\mathbb{R}^d$, with the norm $|.|$ and the scalar product $\langle .,. \rangle$, the maximal monotone operator $A : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$. We denote by $\|x\|_{s,t} \overset{def}{=} \sup_{s\leq r\leq t} |x\,(r)|$, $\|x\|_T \overset{def}{=} \|x\|_{[0,T]}$, $\updownarrow h \updownarrow_T = \updownarrow h \updownarrow_{[0,T]}$ the total variation of a function $h : [0,T] \to \mathbb{R}^d$; i.e., if we denote by $\Delta\,(s,t)$ the set of subdivisions of the time interval $[s,t]$ of the form $s = t_0 < t_1 < .... < t_n = t$, we have

$$\updownarrow h \updownarrow_{[s,t]} = \sup \left\{ \sum_{i=0}^{n-1} |h(t_{i+1}) - h(t_i)| : (s{=}t_0 < t_1 < \ldots < t_n{=}t) {\in} \Delta(s,t) \right\}.$$

By $BV\left(0,T;\mathbb{R}^d\right)$ we will denote the space of function with finite total variation.

b) Multivalued operators

Since the main equation (1.3) contains a multivalued operator as a drift, we start by recalling some useful definitions and properties concerning those operators.

Definition 2.1 *A multivalued operator G from a topological space $\mathbb{U}$ to another topological space $\mathbb{V}$ is said to be upper semicontinuous if:*

- for all $u \in \mathbb{U}$, Gu is a compact subset of $\mathbb{V}$;

- for all $u \in \mathbb{U}$, for every $V \in \mathcal{V}_{\mathbb{V}}\,(Gu)$, there is a $U \in \mathcal{V}_{\mathbb{U}}\,(u)$ such that for all $z \in U$ we have $Gz \subset V$, where $\mathcal{V}_{\mathbb{X}}\,(x)$ denote the set of neighborhoods of x in $\mathbb{X}$.

Definition 2.2 *A multivalued operator $\Theta : \mathbb{U} \rightrightarrows \mathbb{V}$ is said to be measurable, if $\Theta^- V := \{u \in \mathbb{U} : \Theta\,(u) \cap V \neq \emptyset\}$ is measurable, for every closed $V \subset \mathbb{V}$.*

c) Clarke subdifferential

We recall the definition of the Clarke subdifferential operator, then we present some of its properties. For more details, the interested reader can consult Clarke [6] or Guillaume [9].

Definition 2.3 *Let $\mathbb{V}$ be a Banach space, $f : \mathbb{V} \to \mathbb{R}$ be Lipschitz near a given point $x_0 \in \mathbb{V}$, and v a vector from $\mathbb{V}$. We denote by $f^o(x_0, v)$ the generalized directional derivative of f at x_0 in the direction v, defined as follows $f^o(x_0; v) \overset{def}{=} \limsup_{y \to x_0, \delta \downarrow 0} \frac{f(y+\delta v)-f(y)}{\delta}$. The Clarke subdifferential of f at x_0 is defined by*

$$\partial_C f(x_0) \overset{def}{=} \{\xi \in V^* : \langle \xi, v \rangle \leq f^o(x_0; v) \text{ for all } v \in \mathbb{V}\}.$$

Lemma 2.4 *Let $f : \mathbb{V} \to \mathbb{R}$ be Lipschitz near a given point x. Then*

(a) The function $v \to f^o(x, v)$ is finite, positively homogeneous additive and Lipschitz.

(b) The function $f^o(.,.)$ is upper semicontinuous.

(c) $\partial_C f(x)$ is nonempty, convex, weak-compact subset of $\mathbb{V}^*$ and $\|\xi\|_* \leq K$ for every $\xi \in \partial_C f(x)$, where K is the Lipschitz constant of f near the point x.*

(d) For every $v \in \mathbb{V}$, we have $f^o(x_0, v) = \max\{\langle \xi, v \rangle : \xi \in \partial_C f(x_0)\}$.

Remark 2.1 When f is smooth (continuously differentiable), the Clarke subdifferential $\partial_C f(x)$ reduces to the singleton $\{\nabla f(x)\}$, and, when f is convex, $\partial_C f(x)$ coincides with the classical $\partial f(x)$.

Definition 2.5 *We say that a function f is regular at x, if*

(i) for all v, $f_x'(x; v)$ exists;

(ii) for all v, $f_x'(x; v) = f^o(x, v)$, where $f_x'(x; v) = \lim_{t \to 0} \frac{f(x+tv)-f(x)}{t}$ is the usual directional derivative.

Remark 2.2 Any locally Lipschitz and convex function is regular, and if f_1 and f_2 are regular at x, then $\partial_C(f_1 + f_2)(x) = \partial_C f_1(x) + \partial_C f_2(x)$.

We remind the following Lemma from Schirotzek ([19], Proposition 7.3.8).

Lemma 2.6 *The Clarke subdifferential mapping $\partial_C f(x) : \mathbb{V} \rightrightarrows \mathbb{V}^*$ is norm-to-weak* upper semicontinuous, that is for any $x \in \mathbb{V}$ and U a weak* open subset of $\mathbb{V}^*$ containing $\partial_C f(x)$, and x_n a sequence of $\mathbb{V}$, with $x_n \longrightarrow x$ as $n \longrightarrow \infty$, one has $\partial_C f(x_n) \subseteq U$, for n large enough.*

d) Variational inequalities in Skorohod problem

We will recall a result of existence and uniqueness for the generalized Skorohod problem below, written under the form of a variational inequality, for details we invite the reader to see the papers of Barbu, Răşcanu [2] or Răşcanu [16]:

$$\begin{cases} dx(t) + \partial\varphi(x(t))(dt) \ni dm(t), \ t \in [0, T], \\ x(0) = x_0 \in \overline{D(\partial\varphi)} = \overline{D(\varphi)}, \end{cases} \tag{2.1}$$

where $m : [0, T] \to \mathbb{R}^d$ is a continuous function, $\varphi : \mathbb{R}^d \to]-\infty, +\infty]$ is a convex lower semi-continuous function, with $\partial\varphi : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ its subdifferential operator, defined by $\partial\varphi \overset{def}{=} \left\{x^* \in \mathbb{R}^d : \langle x^*, z - x \rangle + \varphi(x) \leq \varphi(z)\right\}$. Denote $Dom(\varphi) = D(\varphi) = \{x \in \mathbb{R}^d : \varphi(x) < +\infty\}$ and $Dom(\partial\varphi) = D(\partial\varphi) = \{x \in \mathbb{R}^d : \partial\varphi(x) \neq \emptyset\}$. We have $\overline{Dom(\varphi)} = \overline{Dom(\partial\varphi)}$ and $int(D(\varphi)) = int(D(\partial\varphi))$.

Definition 2.7 *A pair (x,j) is a solution of the generalized Skorohod problem (2.1) (and we denote it by $(x,j)=\mathcal{GS}(\partial\varphi;x_0,m)$) if*

$$\begin{array}{ll}(a) & x,\ j:[0,T]\to\mathbb{R}^d \text{ are continuous,}\\ (b) & x(t)\in\overline{D(\varphi)},\quad j\in BV_{loc}\left(0,T;\mathbb{R}^d\right),\ j(0)=0,\\ (c) & x(t)+j(t)=x_0+m(t),\quad t\in[0,T],\\ (d) & \displaystyle\int_s^t\langle y(r)-x(r),dj(r)\rangle+\int_s^t\varphi(x(r))\,dr\le\int_s^t\varphi(y(r))\,dr,\\ & \qquad\text{for all } 0\le s\le t\le T\quad \text{and all } y\in C\left([0,T];\mathbb{R}^d\right).\end{array}$$

Proposition 2.8 *Let φ be a convex function as above and we suppose that, in addition, it satisfies $H(\varphi):int(D(\varphi))\neq\emptyset$. Then the variational inequality (2.1) has a unique solution (x,j). Moreover, if $\mathcal{M}$ is a bounded and equicontinuous subset of $C\left([0,T];\mathbb{R}^d\right)$, then one can find a positive constant $C_{\mathcal{M}}$ (for explicit form of this constants see [13], Proposition 4.16) such that:*
1. *If $m\in\mathcal{M}$ and (x,j)=$\mathcal{GS}(\partial\varphi;x_0,m)$, then*

$$\|x\|_T^2+\updownarrow j\updownarrow_T\le C_{\mathcal{M}}. \tag{2.2}$$

2. *If $m,\hat{m}\in\mathcal{M}$, $(x,j)=\mathcal{GS}(\partial\varphi;x_0,m)$ and $(\hat{x},\hat{j})=\mathcal{GS}(\partial\varphi;\hat{x}_0,\hat{m})$, then $\|x-\hat{x}\|_T^2\le C_{\mathcal{M}}(|x_0-\widehat{x}_0|^2+\|m-\hat{m}\|_T)$.*
3. *The mapping $m\mapsto x\overset{def}{=}F_{x_0}(m)$ is a continuous one from $C([0,T];\mathbb{R}^d)$ into $C([0,T];\overline{D(\varphi)})$.*

3 Generalized Skorohod problem with Clarke subdifferential operator

In this section we recall the main multivalued differential equation (1.3), we give the definition of its solution and we present the assumptions on the function ϕ and on the multivalued operator G, then we will provide the existence and uniqueness results.

Consider the multivalued equation

$$\begin{cases}dx(t)+\partial\varphi(x(t))dt+\partial_C\phi(x(t))dt+G(t,x(t))dt\ni dm(t),t\in[0,T],\\ x(0)=x_0\in\overline{D(\partial_C\phi)}\cap\overline{D(\varphi)},\end{cases} \tag{3.1}$$

where $\partial_C\phi$ is the Clarke subdifferential operator associated to the function ϕ, $\partial\varphi$ is the classical subdifferential operator associated to a convex, l.s.c function φ, $G(t,x)$ is a time dependent multivalued operator on $\mathbb{R}^d$ and $m:[0,T]\to\mathbb{R}^d$ is a continuous function.

We suppose that there exist positive constants c and M such that

$$\begin{cases}i) & |\partial_C\phi(x)|\le c(1+|x|),\quad \text{for all } x\in\mathbb{R}^d;\\ ii) & \forall x_1,x_2\in\mathbb{R}^d,\ \forall y_1\in\partial_C\phi(x_1),\ \forall y_2\in\partial_C\phi(x_2),\\ & \langle x_1-x_2,y_1-y_2\rangle\ge-M|x_1-x_2|^2\end{cases} \tag{3.2}$$

and

$$\begin{cases}i) & \varphi:\mathbb{R}^d\to]-\infty,+\infty] \text{ is a proper convex l.s.c. function}\\ ii) & int(Dom(\varphi))\neq\emptyset.\end{cases} \tag{3.3}$$

Definition 3.1 A multivalued mapping $G : [0,T] \times \overline{D(\varphi)} \rightrightarrows \mathbb{R}^d$ is said to satisfy conditions (R), if the following conditions hold:
(a) for almost all t in $[0,T]$, the mapping $x \to G(t,x)$ is multivalued u.s.c. from $\overline{D(\varphi)}$ to $\mathbb{R}^d$, with convex values.
(b) for all $x \in \overline{D(\varphi)}$, the mapping $t \to G(t,x)$ is multivalued measurable from $[0,T]$ to $\mathbb{R}^d$ (i.e., for all $\chi \in \mathbb{R}^d$ and all $x \in \overline{D(\varphi)}$, the function $\upsilon_{x,\chi} : t \to \sup\{\langle y, \chi\rangle : y \in G(t, x)\}$ is measurable on $[0,T]$).
(c) there exist two functions γ and δ from $L^p_{loc}(0,T)$, with $1 \leq p \leq \infty$, such that for almost all $t \in [0,T]$ and all $x \in \overline{D(\varphi)}$, $\sup_{y \in G(t,x)} |y| \leq \gamma(t)|x| + \delta(t)$.

We suppose that the multifunction $G : I \times \overline{D(\varphi)} \to \mathbb{R}^d$ satisfies both the condition (R) and the following one-sided Lipschitz continuous condition: there exists $\eta \in L^1([0,T], \mathbb{R}^+)$ such that, for every $x, y \in \mathbb{R}^d$, $t \in [0,T]$ and $u \in G(t,x)$, $v \in G(t,y)$ we have

$$\langle v - u, y - x\rangle \geq -\eta(t)\,|x - y|^2 . \tag{3.4}$$

We introduce now the notion of solution for our equation.

Definition 3.2 *Given two functions $x, l : [0,T] \to \mathbb{R}^d$ and a locally Lipschitz function f, we say that $dl(t) \in \partial_C f(x(t))(dt)$, if for all $T \geq 0$,*

$$\begin{array}{ll} (a) & x, l : [0,T] \to \mathbb{R}^d \text{ are continuous,} \\ (b) & l \in BV_{loc}\left(\mathbb{R}_+; \mathbb{R}^d\right), \quad l(0) = 0, \\ (c) & \displaystyle\int_s^t \langle v(r), dl(r)\rangle \leq \int_s^t f^o(x(r), v(r))\, dr, \\ & \qquad \text{for all } 0 \leq s \leq t \leq T \quad \text{and } v \in C\left([0,T]; \mathbb{R}^d\right). \end{array} \tag{3.5}$$

Definition 3.3 *Given two functions $x, g : [0,+\infty) \to \mathbb{R}^d$ we say that $dg(t) \in \partial\varphi(x(t))(dt)$ on $\mathbb{R}_+$ if, for all $T > 0$,*

$$\begin{array}{l} (a)\ x \in C([0,T]; \mathbb{R}^d), g \in BV([0;T]; \mathbb{R}^d), g(0) = 0; \\ (b)\ \int_0^T \varphi(x(t))dt < \infty, \text{ for all } T \geq 0, \\ \\ (c)\ \int_s^t \langle y(r) - x(r), dg(r)\rangle + \int_s^t \varphi(x(r))dr \leq \int_s^t \varphi(y(r))dr, \\ \quad \text{for all } 0 \leq s \leq t \leq T \text{ and } y \in C([0,T]; \mathbb{R}^d). \end{array}$$

By $f^o(x(r), v(r))$ we denoted the generalized directional derivative of f at $x(r)$ in the direction $v(r)$, that is

$$f^o(x(r); v(r)) = \limsup_{y \to x(r), \delta \downarrow 0} \frac{f(y + \delta v(r)) - f(y)}{\delta}.$$

Definition 3.4 *A triplet of function (x, j, l) is called a solution of the evolution inclusion with Clarke subdifferential (3.1) if $x, j, l : [0,T] \to \mathbb{R}^d$ are continuous, and*

i) $x(t) \in \overline{D(\varphi)}, \forall t \geq 0$,
ii) $j, l \in BV\left([0,T]; \mathbb{R}^d\right), \quad$ *with* $j(0) = l(0) = 0$,

iii) $x(t)+j(t)+l(t)+\int_0^t \beta(s)\,ds = x_0+m(t)$,
iv) $dj(t)\in\partial\varphi(x(t))(dt)$, $dl(t)\in\partial_C\phi(x(t))(dt)$ *and* $\beta(t)\in G(t,x(t))$.

Remark 3.1 Assertion $(3.5-c)$ is equivalent to

$$\langle z, l(t)-l(s)\rangle \le \int_s^t f^o(x(r),z)\,dr, \quad \text{for all } z\in\mathbb{R}^d,$$

because $\int_s^t\langle v(r), dl(r)\rangle = \lim_{n\to\infty}\int_s^t\langle v(\frac{\lfloor nr\rfloor}{n}),\, dl(r)\rangle$ and $\int_s^t f^o(x(r),$ $v(r))dr = \lim_{n\to\infty}\int_s^t f^o(x(r), v(\frac{\lfloor nr\rfloor}{n}))dr$, where $\lfloor x\rfloor$ represents the integer part of x.

In order to prove the existence and uniqueness of the solution for the multivalued equation (3.1), we first prove the following auxiliary theorem.

Theorem 3.5 *Let G be a time dependent multivalued operator on $\mathbb{R}^d$ satisfying conditions (R), φ satisfying* (3.3) *and $m\in\mathcal{M}$ ($\mathcal{M}$ defined in Proposition 2.8). Then the following variational inequality admits a solution (x,j) on $[0,T]$:*

$$dx(t)+\partial\varphi(x(t))\,dt + G(t,x(t))\,dt \ni dm(t), \quad x(0)=x_0. \tag{3.6}$$

More precisely:
(1) *there exists a measurable selection $\beta:[0,T]\to\mathbb{R}^d$ such that $\beta(t)\in G(t,x(t))$ almost everywhere in $[0,T]$;*
(2) *(x,j) is a solution of $dx(t)+\partial\varphi(x(t))\,dt\ni dm(t)-\beta(t)\,dt$, $x(0)=x_0$, in the sense of Definition* 2.7.

Proof. We give just the main lines of the proof since it is very similar to the one from the smooth case found in Attouch and Damlamian [1].

We define the operator $\mathbb{G}_p : L^p\left(0,T;\mathbb{R}^d\right)\rightrightarrows L^p\left(0,T;\mathbb{R}^d\right)$ with $y\in\mathbb{G}_p(x)$, meaning that $x(t)\in\overline{D(\varphi)}$ a.e. on $(0,T)$ and $y(t)\in G(t,x(t))$.

Step 1. First we suppose in the conditions (R), $\gamma\equiv 0$. The main idea consists in proving that the multivalued operator $\mathbb{G}_p$ is upper semi-continuous, with nonempty convex compact values from $L^p\left(0,T;\mathbb{R}^d\right)$ into $w-L^p\left(0,T;\mathbb{R}^d\right)$, which allowed us to apply the Kakutani-Glicksberg-Fan fixed point theorem (see the Annex) for an adequate multivalued equation. First we obviously see that $\mathbb{G}_p$ takes its values on $X_\delta=\{f\in L^p\left(0,T;\mathbb{R}^d\right) : |f(t)|\le\delta(t) \text{ a.e.}\}$, which is a bounded closed convex subset of $L^p\left(0,T;\mathbb{R}^d\right)$. Consequently, it is compact in $w-L^p\left(0,T;\mathbb{R}^d\right)$ for all $p>1$. For $p=1$, the Dunford-Pettis criterion affirms also that X_δ is compact in $w-L^1\left(0,T;\mathbb{R}^d\right)$. To prove that the operator $\mathbb{G}_p$ is nonempty and upper semi-continuous we adopt the same idea from Attouch and Damlamian [1], Proposition 3.5. Since $\mathbb{R}^d$ is separable we need to prove only that its graph is closed in $L^p(0,T;\mathbb{R}^d)\times w-L^p(0,T;\mathbb{R}^d)$ (the interested reader can consult the proof of Attouch and Damlamian [1], Proposition 3.4).

We remark that a solution of the inequality (3.6) is equivalent to a solution of the inequality $\beta\in\mathbb{G}_1(F_{x_0}(m-\int_0^t\beta(s)ds))$. Since the operator $\mathbb{G}_1$ is a multivalued u.s.c one from X_δ into itself and it has nonempty convex compact values, we can apply the Kakutani-Glicksberg-Fan fixed point theorem for multivalued upper-semicontinuous mappings in order to get the result.

Step 2. In the general case $\gamma \not\equiv 0$ we consider $\mathcal{O}$ a bounded closed convex neighborhood of x_0. Clearly $int(\mathcal{O}) \cap D(\partial\varphi) \neq \emptyset$, so, using the result of Rockafellar for maximal monotone operators (see ROCKAFELLAR [17]), we have that $(\partial\varphi + I_{\mathcal{O}})$ is again a maximal monotone operator and $|G(t,x(t))| \leq \gamma(t) \times \sup\{|x| : x \in \mathcal{O}\} + \delta(t) \overset{def}{=} \hat{\delta}(t) \in L^1_{loc}(0,T)$. According to the first part of the proof, we obtain a solution x for the following equation $dx(t) + (\partial\varphi + \partial I_{\mathcal{O}})(x(t))dt + G(t,x(t))dt \ni dm(t),\ x(0) = x_0$, for $t \in [0,T]$. From the continuity of x there exists a $T_1 \leq T$, such that for every $t \in [0,T_1]$, $x(t) \in int\mathcal{O}$ and x is a solution on $[0,T_1]$ of $dx(t) + \partial\varphi(x(t))dt + G(t,x(t))dt \ni dm(t), x(0) = x_0$. Using (2.2), one can extend the solution to the right side of T_1, which concludes the proof. □

Theorem 3.6 *If ϕ satisfies the assumptions* (3.2), *φ satisfies* (3.3) *and G verifies condition* (R) *and* (3.4), *then there exists at least one solution (x,j,l) for the multivalued equation* (3.1). *Moreover, if $\mathcal{M}$ is a bounded and equicontinuous subset of $C([0,T];\mathbb{R}^d)$, then there exists a positive constant $C_{0,\mathcal{M}} = C_0(\|\mathcal{M}\|_T, n_0, u_0, r_0)$ (the constants are explained in the proof) such that:*

1. *If $m \in \mathcal{M}$ and (x,j,l) is a solution of* (3.1), *then*

$$\|x\|_T^2 + \updownarrow j \updownarrow_T + \updownarrow l \updownarrow_T \leq C_{0,\mathcal{M}}(1 + |x_0|^2). \tag{3.7}$$

2. *If $(x_1,j_1,l_1),(x_2,j_2,l_2)$ are two solutions of* (3.1) *corresponding respectively to the singular inputs $m_1,\ m_2$ and, respectively to the initial data $x_{1,0},\ x_{2,0}$, then*

$$\|x_1 - x_2\| \leq C_{0,\mathcal{M}}\left(1 + |x_{1,0}| + |x_{2,0}|\right)\left(|x_{1,0} - x_{2,0}| + \|m_1 - m_2\|_T^{1/2}\right). \tag{3.8}$$

In particular, if $x_{1,0} = x_{2,0}$ and $m_1 = m_2$, then we obtain the uniqueness of the solution.

3. *For every solution (x,j,l) with initial data $x_0 \in \overline{D(\varphi)}$ and singular input $dm(t)$, the mapping $(x_0,m) \to x : \overline{D(\varphi)} \times C\left([0,T];\mathbb{R}^d\right) \to C([0,T];\overline{D(\varphi)})$ is continuous.*

Proof. From Lemma 2.6 and the assumptions (3.2), the multivalued operator $\partial_C\phi(x) + G(t,x)$ satisfies conditions (R). Therefore we can use Theorem 3.5 to obtain the existence result.

Let's now prove the estimates (3.7). Letting (x,j,l) be a solution of (3.1) and $u_0 \in int(D(\varphi))$ fixed, one has

$$\begin{aligned} |x(t) - m(t) - u_0|^2 + 2\int_0^t \langle x(r) - u_0, dj(r) + dl(r) + \beta(r)\,dr\rangle \\ = |x_0 - u_0|^2 + 2\int_0^t \langle m(r), dj(r) + dl(r) + \beta(r)\,dr\rangle. \end{aligned} \tag{3.9}$$

Also, there exists $r_0 \in]0,1]$ such that $\overline{B}(u_0,r_0) \subset Dom(\varphi)$ and $r_0 \updownarrow j \updownarrow_t \leq \int_0^t \langle x(r) - u_0, dj(r)\rangle + \int_0^t [b_0 + b_0|x(r) - u_0|]\,dr$, where $b_0 \overset{def}{=} \sup\{|\hat{u}| : \hat{u} \in \partial\varphi(u_0 + u), |u| \leq r_0\}$ (see PARDOUX and RĂŞCANU [13], Remark 4.15).

Using assumptions (3.4) for G, we have that, for all $|v| \leq 1$, $\langle \hat{\beta}(t) - \beta(t), x - u_0 - r_0 v\rangle \leq \eta(t)|x - u_0 - r_0 v|^2$, where $\hat{\beta} \in G(t, u_0 + r_0 v(t))$, and we have $|\hat{\beta}| \leq$

$\gamma(t)|u_0 + r_0 v| + \delta(t)$. As consequence

$$\begin{aligned}
&r_0\langle\beta(t), v\rangle + \langle x - u_0, -\beta(t)\rangle \\
&\leq \eta(t)|x - u_0 - r_0 v|^2 + (\gamma(t)|u_0 + r_0 v| + \delta(t))|x - u_0 - r_0 v| \\
&\leq c(\eta(t) + 1)(|x - u_0|^2 + |r_0|^2) + c_{u_0,r_0}(\gamma^2(t) + \delta^2(t)) \\
&\leq c(\eta(t) + 1)(|x - u_0|^2) + c_{u_0,r_0}(\gamma^2(t) + \delta^2(t) + \eta(t)).
\end{aligned}$$

Taking the supremum over $|v| \leq 1$, we get that

$$\begin{aligned}
&r_0|\beta(t)| + \langle x - u_0, -\beta(t)\rangle \leq c(\eta(t) + 1)(|x - u_0|^2) \\
&+ c_{u_0,r_0}(\gamma^2(t) + \delta^2(t) + \eta(t)).
\end{aligned}$$

On the other hand, from Lemma 2.4$-(d)$ and the hypotheses $((H_\phi) - i)$, we get $|f^0(x; v)| \leq c|v|(1 + |x|)$. Therefore,

$$\begin{aligned}
\updownarrow l \updownarrow_t &= \sup_{||v||_T \leq 1} \int_0^t \langle v(r), dl(r)\rangle \leq \sup_{||v||_T \leq 1} \int_0^t f^0(x(r), v(r))\, dr \\
&\leq \int_0^t c(1 + |x(r)|)\, dr,
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_0^t \langle x(r) - u_0, -dl(r)\rangle \leq \|x - u_0\|_t \int_0^t c(1 + |x(r)|)dr \\
&\int_0^t c[\|x - u_0\|_t^2 + (1 + |u_0|)\|x(r) - u_0\|_t]dr.
\end{aligned}$$

Replacing in the last equation (equation 3.9), we obtain

$$\begin{aligned}
&|x(t) - m(t) - u_0|^2 + 2r_0 \updownarrow j \updownarrow_t + 2r_0 \updownarrow l \updownarrow_t + 2r_0 \int_0^t |\beta(r)|\, dr \\
&\leq |x_0 - u_0|^2 + 2\int_0^t \langle m(r), dj(r) + dl(r) + \beta(r)\, dr\rangle \\
&+ C\int_0^t (\eta(r) + 1)\|x - u_0\|_r^2 dr,
\end{aligned}$$

where C is a positive constant depending on u_0, r_0, b_0.

We consider now the following subdivision of the interval $[0, T]$, $0 = t_0 < t_1 < \cdots < t_{n_0} = T$, $t_{i+1} - t_i \leq \frac{T}{n_0 - 1}$, such that $t_k = t$, and $\frac{T}{n_0} + \mathbf{m}_{\mathcal{M}}(\frac{T}{n_0}) \leq \frac{r_0}{4}$ ($\mathbf{m}_{\mathcal{M}}(\varepsilon) = \sup\{\mathbf{m}_m(\varepsilon) : m \in \mathcal{M}\}$), with $\mathbf{m}$ is the modulus of continuity of the continuous function

m. We have

$$\begin{aligned}
&\int_0^t \langle m(r), dj(r) + dl(r) + \beta(r)\, dr\rangle \\
&= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \langle m(r) - m(t_i), dj(r) + dl(r) + \beta(r)\, dr\rangle \\
&+ \sum_{i=0}^{k-1} \left\langle m(t_i), j(t_{i+1}) - j(t_i) + l(t_{i+1}) - l(t_i) + \int_{t_i}^{t_{i+1}} \beta(r)\, dr \right\rangle \\
&\leq \mathbf{m}_{\mathcal{M}}\left(\frac{T}{n_0}\right)\left(\updownarrow j \updownarrow_t + \updownarrow l \updownarrow_t + \int_0^t |\beta(r)|\, dr\right) \\
&+ \sum_{i=0}^{k-1} \langle m(t_i), m(t_{i+1}) - x(t_{i+1}) + u_0 - m(t_i) + x(t_i) - u_0\rangle \\
&\leq \frac{r_0}{2} \updownarrow j \updownarrow_t + \frac{r_0}{2} \updownarrow l \updownarrow_t + \frac{r_0}{2} \int_0^t |\beta(s)|\, ds + 2n_0 \|m\|_t \|x - m - u_0\|_t .
\end{aligned}$$

We use the fact that $\frac{1}{2}|x(t) - u_0|^2 - \|m\|_t^2 \leq |x(t) - m(t) - u_0|^2$, to get that

$$\begin{aligned}
&\frac{1}{2}|x(t) - u_0|^2 + r_0 \updownarrow j \updownarrow_t + r_0 \updownarrow l \updownarrow_t + r_0 \int_0^t |\beta(r)|\, dr \leq |x_0 - u_0|^2 + \|m\|_t^2 \\
&+ 4n_0 \|m\|_t \|x - m - u_0\|_t + C\int_0^t (\eta(r) + 1)\|x - u_0\|_r^2 dr.
\end{aligned}$$

It follows $\frac{1}{2}|x(t) - u_0|^2 + r_0 \updownarrow j \updownarrow_t + r_0 \updownarrow l \updownarrow_t \leq |x_0 - u_0|^2 + C(n_0)\|\mathcal{M}\|_T^2 + C\int_0^t (\eta(r) + 1)\|x - u_0\|_r^2 dr$, where $\|\mathcal{M}\|_T \overset{def}{=} \sup\{\|y\|_T,\ y \in \mathcal{M}\}$. Using Gronwall's inequality we obtain the estimates (3.7).

Let's prove now the uniqueness of the solution. Let (x_1, j_1, l_1) and (x_2, j_2, l_2) be two solutions of (3.1), with $dj_i(t) \in \partial\varphi(x_i)(dt)$ and $dl_i(t) \in \partial_C\phi(x_i)(dt)$ such that $dx_i(t) + dj_i(t) + dl_i(t) + \beta_i(t)\, dt = dm_i(t)$, $i = 1, 2$.

Then we have:

$$\begin{aligned}
&|x_1(t) - m_1(t) - x_2(t) + m_2(t)|^2 = |x_{1,0} - x_{2,0}|^2 \\
&- 2\int_0^t \langle x_1(r) - x_2(r), dj_1(r) - dj_2(r)\rangle \\
&- 2\int_0^t \langle x_1(r) - x_2(r), (dl_1(r) - dl_2(r)) + (\beta_1(r) - \beta_2(r))\, dr\rangle \\
&+ 2\int_0^t \langle m_1(r) - m_2(r), +(dj_1(r) - dj_2(r)) + (dl_1(r) - dl_2(r)) + (\beta_1 - \beta_2)dr\rangle \\
&\leq |x_{1,0} - x_{2,0}|^2 + 2\int_0^t (M + \eta(r))|x_1(r) - x_2(r)|^2\, dr + 2\|m_1 - m_2\|_T \times \\
&\left(\updownarrow j_1 \updownarrow_T + \updownarrow j_2 \updownarrow_T + \updownarrow l_1 \updownarrow_T + \updownarrow l_2 \updownarrow_T + \int_0^t (\gamma(r) + \delta(r))dr\,(1 + \|x_1\|_T + \|x_2\|_T)\right).
\end{aligned}$$

Using the inequality

$$|x_1(t) - m_1(t) - x_2(t) + m_2(t)|^2 \geq \frac{1}{2}|x_1(t) - x_2(t)|^2 - \|m_1 - m_2\|_T^2$$

and the estimates (3.7), we conclude, as in [16], (3.8) and the uniqueness clearly follows.

Now let's prove the assertion 3. For $n \in \mathbb{N}^*$, we take $x_{0,n} \in \overline{D(\varphi)}$, $m_n \in C([0,T]; \mathbb{R}^d)$ and (x_n, j_n, l_n) the corresponding solution of the multivalued equation (3.1) with initial data $x_{0,n}$, and singular input m_n, such that

$$x_{0,n} \to x_0 \text{ in } \mathbb{R}^d,\ m_n \to m \quad \text{in } C\left([0,T];\mathbb{R}^d\right).$$

Since the set $\mathcal{M} = \{m, m_1, m_2, \ldots\}$ is a relatively compact subset of $C\left([0,T];\mathbb{R}^d\right)$, then, by (3.7), we have

$$\|x_n\|_T^2 + \updownarrow j_n \updownarrow_T + \updownarrow l_n \updownarrow_T \leq R = C_{0,\mathcal{M}}\left(1 + |x_0|^2\right),$$

and by the estimates (3.8), it follows

$$\|x_n - x_j\|_T \leq C_{0,\mathcal{M}}\left(|x_{0,n} - x_{0,j}| + \|m_n - m_j\|_T^{1/2}\right).$$

Hence there exists $x \in C\left([0,T];\mathbb{R}^d\right)$ such that $x_n \underset{n\to\infty}{\longrightarrow} x$ in $C([0,T]; \overline{D(\varphi)})$.

Since $\updownarrow l_n \updownarrow_T \leq R$, there exists a function $l \in BV\left(0,T;\mathbb{R}^d\right)$ such that $l_n \overset{w*}{\to} l$ in $BV\left(0,T;\mathbb{R}^d\right)$ and we have, for all $v \in C([0,T];H)$,

$$\begin{aligned}\int_0^t \langle v(r), dl(r)\rangle &= \lim_{n\to\infty}\int_0^t \langle v(r), dl_n(r)\rangle \leq \lim_{n\to\infty}\int_0^t \phi^o(x_n(r), v(r))\, dr \\ &\leq \int_0^t \phi^o(x(r), v(r))\, dr.\end{aligned}$$

The last inequality derive from the Lebesgue dominated convergence theorem and the fact that $\phi^o(.,v)$ is upper-semicontinuous. Therefore, we get

$$dl(t) \in \partial_C \phi(x(t))(dt).$$

The fact that the multivalued operator G is demi-closed in $L^1\left(0,T;\mathbb{R}^d\right)$ (see for instance Attouch and Damlamian: 72 [1], Proposition 3.4) and the Lebesgue dominated convergence theorem give us that

$$\int_0^t \beta_n(s)\, ds \to \int_0^t \beta(s)\, ds, \quad \forall t \in [0,T],$$

where $\beta_n(t) \in G(t, x_n)$.

Let define now

$$j(t) = x_0 - \int_0^t \beta(s)\, ds + m(t) - l(t) - x(t).$$

We have

$$j_n \to j \quad \text{in} \quad C\left([0,T];\mathbb{R}^d\right),$$

where

$$j_n = x_0 - \int_0^{\cdot} \beta_n(s)\,ds + m_n - l_n - x_n.$$

By Helly-Bray theorem, we conclude that $j \in BV\left([0,T];\mathbb{R}^d\right)$ and

$$0 \leq \int_s^t \langle x_n(r) - y, dj_n(r) - y^* dr\rangle \to \int_s^t \langle x(r) - y, dj(r) - y^* dr\rangle,$$

for all $(y,y^*) \in \partial\varphi$ and $0 \leq s \leq t \leq T$, which means that $dj(t) \in \partial\varphi(dt)$. As consequence, the limit (x,j,l) is a solution to the multivalued equation (3.1), which completes our proof. □

4 Application: stochastic variational inequality with Clarke subdifferential

In this section we formulate the result of existence and uniqueness of the solution for the SVI (4.1). In the first theorem we prove the uniqueness, the main idea consist of applying Proposition 5.6 for the difference of two solutions. In the second theorem, we prove the existence of the solution, by making appeal to the result from the deterministic framework (Theorem 3.6).

Let $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0})$ be a complete stochastic basis and $\{B_t, t\geq 0\}$ a $k-$ dimensional Brownian motion. We define, for $p \geq 0$, $S_d^p[0,T]$, the space of progressively measurable and continuous stochastic processes $X : \Omega \times [0,T] \to \mathbb{R}^d$ such that

$$\mathbb{E}\sup_{t\in[0,T]} |X_t|^p < +\infty, \quad \text{if } p > 0,$$

and $\Lambda_d^p(0,T)$ the space of progressively measurable processes $X : \Omega \times [0,T] \to \mathbb{R}^d$, such that $\int_0^T |X_t|^2\,dt < +\infty$, $\mathbb{P}-a.s.\ \omega \in \Omega$, if $p = 0$ and $\mathbb{E}(\int_0^T |X_t|^2\,dt)^{p/2} < +\infty$, if $p > 0$.

We consider now the stochastic variational inequality

$$\begin{cases} dX_t + \partial\varphi(X_t)dt + \partial_C\phi(X_t)dt + G(t,X_t)dt \ni Q(t,X_t)dB_t, t\in[0,T], \\ X_0 = \xi \in \overline{D(\varphi)}. \end{cases} \tag{4.1}$$

where ϕ satisfies the assumptions (3.2) and G satisfies condition (3.4). We assume also that $Q(.,.,x) : \Omega \times [0,T] \to \mathbb{R}^{d\times k}$ is a Carathéodory function and satisfies the following conditions: there exists $\ell \in L^2_{loc}(\mathbb{R}_+)$ such that

$$\begin{cases} |Q(t,x) - Q(t,y)| \leq \ell(t)|x-y|, \forall x,y \in \mathbb{R}^d \text{ (Lipschitz condition)}, \\ \displaystyle\int_0^T |Q(t,0)|^2 dt < \infty,\ \mathbb{P}-a.s.\ \omega\in\Omega \text{ (boundedness condition)}. \end{cases} \tag{4.2}$$

We introduce below the notion of solution for the stochastic variational inequality (4.1).

Definition 4.1 A quadruple (X, J, K, β) of $\mathbb{R}^d-$valued stochastic processes is a solution of (4.1) if the following conditions are satisfied, $\mathbb{P}-a.s.\ \omega \in \Omega$, for every $T>0$:

$d_1)$ $X, J, K, \beta \in S_d^0[0,T], J_0 = K_0 = 0$,

$d_2)$ $X_t \in \overline{D(\varphi)}$, a.e., $\updownarrow J \updownarrow_T < \infty$ and $\updownarrow K \updownarrow_T < \infty$,

$d_3)$ $\beta_t \in G(t, X_t)$, a.e.,

$d_4)$ $X_t + J_t + K_t + \beta_t = \xi + \int_0^t Q(s, X_s)\, dB_s,\ \forall t \geq 0$,

$d_5)$ $\int_s^t \phi^o(X_r, v(r))\, dr \geq \int_s^t \langle v(r), dK_r \rangle, \forall v \in C\left([0,T], \mathbb{R}^d\right), \forall 0 \leq s \leq t \leq T$,

$d_6)$ $\int_s^t \langle y(r) - X_r, dJ_r \rangle + \int_s^t \varphi(X_r)\, dr \leq \int_s^t \varphi(y(r))\, dr,\ \forall y \in C\left([0,T], \mathbb{R}^d\right),\ \forall 0 \leq s \leq t \leq T$.

Theorem 4.2 (Uniqueness) *Let the assumptions* (3.2), (3.4) *and* (4.2) *be satisfied and suppose that* (X, J, K, β), $(\hat{X}, \hat{J}, \hat{K}, \hat{\beta})$ *are two solutions of the SVI* (4.1), *corresponding, respectively, to the initial conditions* $\xi, \hat{\xi} \in L^0(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{D(\varphi)})$. *Letting* $p \geq 1$, *if* $\xi = \hat{\xi}$, $\mathbb{P}-a.s.\ \omega \in \Omega$, *then* $\mathbb{P}$*–a.s.* $\omega \in \Omega$

$$X_s = \hat{X}_s,\ J_s = \hat{J}_s,\ K_s = \hat{K}_s,\ \beta_s = \hat{\beta}_s, \quad \textit{for all } s \in [0,T].$$

Proof. Taking in account assumptions (3.4), the stochastic differential inequality (4.1) can be written in the following form:

$$dX_t + dJ_t + dK_t + \beta_t dt \ni Q(t, X_t) dB_t,$$

where J_t, K_t are two bounded variation stochastic processes such that $dJ_t \in \partial\varphi(X_t)(dt)$, $dK_t \in \partial_C \phi(X_t)(dt)$ and $\beta_t \in G(t, X_t), \forall t \in [0,T], \mathbb{P}-a.s. \omega \in\ \Omega$.

Consider (X, J, K, β) and $(\hat{X}, \hat{J}, \hat{K}, \hat{\beta})$ two solutions of the SDE (4.1) and we can write

$$X_t - \hat{X}_t = \xi - \hat{\xi} + \int_0^t d\mathcal{K}_r + \int_0^t Q_r dB_r,$$

where

$$\mathcal{K}_t = -(J_t - \hat{J}_t) - (K_t - \hat{K}_t) - \int_0^t \left[\beta_r - \hat{\beta}_r\right] dr \quad \text{and}$$
$$Q_r = Q(r, X_r) - Q(r, \hat{X}_r).$$

Using the assumptions (3.2), (3.4) and (4.2), we obtain

$$\left\langle X_r - \hat{X}_r, d\mathcal{K}_r \right\rangle + \frac{1}{2} |Q_r|^2\, dr \leq |X_r - \hat{X}_r|^2 dV_r,$$

where

$$V_t = Mt + \int_0^t \left(\eta^+(r) + \frac{1}{2}\ell^2(r)\right) dr.$$

Finally, using Proposition 5.6, we infer that

$$\mathbb{E}\frac{e^{-2V_s}|X_r - \hat{X}_r|^2}{1 + e^{-2V_s}|U_s|^2} \leq \mathbb{E}\frac{e^{-2V_0}|\xi - \hat{\xi}|^2}{1 + e^{-2V_0}|\xi - \hat{\xi}|^2},$$

and, by the continuity of X and $\hat{X}$, we conclude that, $\mathbb{P}-a.s.\ \omega \in \Omega$,

$$X_t = \hat{X}_t, \quad \text{for all } t \in [0,T],$$

which yields also that $\beta_t = \hat{\beta}_t$, $J_t = \hat{J}_t$ and $K_t = \hat{K}_t$. $\square$

Proposition 4.3 *Let $\xi \in L^0(\Omega, \mathcal{F}_0, P; \overline{D(\varphi)})$. If $M \in S_d^0[0,T]$, $M_0 = 0$, then the stochastic differential equation*

$$\begin{cases} dX_t + \partial\varphi(X_t)\,dt + \partial_C\phi(X_t)\,dt + G(t,X_t)dt \ni dM_t, \quad t \in [0,T], \\ X_0 = \xi. \end{cases} \tag{4.3}$$

admits a unique solution $(X,J,K,\beta) \in S_d^0[0,T] \times S_d^0[0,T] \times S_d^0[0,T] \times L^0(\Omega; L^1([0,T]; \mathbb{R}^d))$. In the particular case when M is an Itô integral (i.e. $M_t = \int_0^t Q(s,X_s)dB_s := \int_0^t Q_s dB_s$), if there exist $p \geq 2$ and $u_0 \in int(Dom(\partial(\varphi)))$ satisfying

$$\mathbb{E}|\xi|^p + \mathbb{E}\left(\int_0^T |\beta_{t,u_0}|\,dt\right)^p + \mathbb{E}\left(\int_0^T |Q(t,u_0)|^2\,dt\right)^{p/2} < +\infty, \tag{4.4}$$

with $\beta_{t,u_0} \in G(t,u_0)$, then $(X,K,\beta) \in S_d^p[0,T] \times (S_d^{p/2}[0,T] \cap L^{p/2}(\Omega; BV([0,T], \mathbb{R}^d))) \times L^p(\Omega; L^1([0,T]; \mathbb{R}^d))$.

Proof. For $\omega \in \Omega$ fixed, we conclude, using Theorem 3.6, that there exists a unique solution $(X.(\omega), J.(\omega), K.(\omega), \beta.(\omega)) \in C([0,T]; \mathbb{R}^d) \times \left(C([0,T]; \mathbb{R}^d)\right)^2 \times L^1([0,T]; \mathbb{R}^d)$. Since the process M is progressively measurable and the mapping $(\xi, M) \to X : \overline{D(\varphi)} \times C([0,t]; \mathbb{R}^d) \to C([0,t]; \overline{D(\varphi)})$ is continuous, for each $t \leq T$, then X is progressively measurable. Therefore $X \in S_d^0[0,T]$ and $J, K \in (S_d^0[0,T])^2$.

If $M_t = \int_0^t Q_s dB_s$ then we can write the equation (4.3) under the following equivalent form

$$X_t = \xi + \mathcal{K}_t + \int_0^t Q_r dB_r, \quad \text{where} \quad \mathcal{K}_t = -J_t - K_t - \int_0^t \beta_r dr.$$

According to Proposition 5.3, one can find $0 < r_0 \leq 1$, with $B(u_0, r_0) \subset Dom(\partial(\varphi))$ such that

$$r_0 d\updownarrow J\updownarrow_t \leq \langle X_t - u_0, dJ_t\rangle + b_0|X_t - u_0| + b_0.$$

Also, from assumptions (3.4) we have

$$\begin{aligned} \langle X_t - u_0, -\beta_t dt\rangle &= \langle X_t - u_0, \beta_{t,u_0} - \beta_t\rangle\,dt + \langle X_t - u_0, -\beta_{t,u_0}\rangle\,dt \\ &\leq \eta(t)|X_t - u_0|^2\,dt + |\beta_{t,u_0}|\,|X_t - u_0|\,dt, \end{aligned}$$

and finally, similar to the proof of Theorem 3.6, we can prove

$$\langle X_t - u_0, -dK_t\rangle \leq c|X_t - u_0|^2\,dt + c(1 + |u_0|)|X_t - u_0|\,dt.$$

We have, for all $p \geq 2$ and $\lambda > 1$,

$$dD_r + \langle X_r - u_0, d\mathcal{K}_r\rangle + \left(\frac{p-1}{2} + 9p\lambda\right)|Q_r|^2\,dr \leq dR_r + |X_r - u_0|\,dN_r,$$

where

$$\begin{aligned} D_t &= r_0\updownarrow J\updownarrow_t, \quad R_t = \int_0^t (b_0 + \left(\tfrac{p-1}{2} + 9p\lambda\right)|Q(r,u_0)|^2)dr, \\ L_t &= \int_0^t (b_0 + |\beta_{t,u_0}| + c(1 + |u_0|))\,dr \quad \text{and} \\ V_t &= \int_0^t \left(c + \eta(r) + \left(\tfrac{p-1}{2} + 9p\lambda\right)\ell^2(r)\right)dr. \end{aligned}$$

Finally, using Proposition 5.7, with $\lambda = 2$, we prove that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}|X_t-u_0|^p + r_0^{p/2}\mathbb{E}\updownarrow J\updownarrow_T^{p/2}\\
&\leq C\left(p,\eta\left(r\right),\ell\right)\left[\mathbb{E}\left|\xi-u_0\right|^p+\mathbb{E}\left(\int_0^T\left(b_0+\left|\beta_{t,u_0}\right|+c\left(1+\left|u_0\right|\right)\right)dt\right)^p\right.\\
&\left.+\mathbb{E}\left(\int_0^T\left(b_0+\left(\tfrac{p-1}{2}+9p\lambda\right)\left|Q\left(t,u_0\right)\right|^2\right)dt\right)^{p/2}\right].
\end{aligned}$$

Taking into account assumption (4.4) it follows $X \in S_d^p[0,T]$, and $K \in S_d^{p/2}[0,T] \cap L^{p/2}(\Omega; BV([0,T],\mathbb{R}^d))$. □

Theorem 4.4 *If assumptions* (3.2), (3.4) *and* (4.2) *are satisfied, then the SVI* (4.1) *has a unique solution* $(X,J,K,\beta) \in S_d^0[0,T]\times(S_d^0[0,T])^2\times L^0(\Omega;L^1([0,T];\mathbb{R}^d))$. *Moreover, if there exist* $p \geq 2$ *and* $u_0 \in int(Dom(\partial\varphi))$ *such that, for all* $T \geq 0$,

$$\mathbb{E}\left|\xi\right|^p+\mathbb{E}\left(\int_0^T\left|\beta_{t,u_0}\right|dt\right)^p+\mathbb{E}\left(\int_0^T\left|Q_t\right|^2dt\right)^{p/2}<+\infty, \tag{4.5}$$

then $(X,K)\in S_d^p\left[0,T\right]\times\left(S_d^{p/2}\left[0,T\right]\cap L^{p/2}\left(\Omega;BV\left(\left[0,T\right],\mathbb{R}^d\right)\right)\right)$.

Proof. The uniqueness was proved in Theorem 4.2. For the existence of the solution we will split the proof in two steps.

Step 1. We assume that condition (4.5) holds. Let $U \in S_d^p[0,T]$. Clearly we have $Q(\cdot,U.) \in \Lambda_d^p(0,T)$. Considering the previous Proposition with $M_\cdot = \int_0^\cdot Q(s,U_s)\,dB_s$, the equation

$$\begin{cases} X_t+J_t+K_t+\displaystyle\int_0^t\beta_r dr=\xi+\int_0^t Q(r,U_r)dB_r,\ t\in[0,T],\\ dJ_t\in\partial\varphi\left(X_t\right)dt,\ dK_t\in\partial_C\phi\left(X_t\right)dt,\ \beta_r\in G(t,X_t). \end{cases} \tag{4.6}$$

admits a unique solution $(X,J,K,\beta) \in S_d^p[0,T]\times(S_d^{p/2}[0,T]\cap L^{p/2}(\Omega;BV([0,T],\mathbb{R}^d)))^2\times L^p(\Omega;L^1([0,T];\mathbb{R}^d))$.

Define $\Gamma : S_d^p[0,T]\to S_d^p[0,T]$, by $\Gamma(U)=X$, where $X\in S_d^p[0,T]$ is the solution of (4.6). We will prove that Γ is a contraction on $S_d^p[0,T]$. Letting $U,\hat{U}\in S_d^p[0,T]$, we take $X=\Gamma(U)$ and $\hat{X}=\Gamma(\hat{U})$ and we have $X-\hat{X}=\mathcal{K}_t+\int_0^t Q_r dB_r$, where $\mathcal{K}_t=-(J_t-\hat{J}_t)-(K_t-\hat{K}_t)-\int_0^t[\beta_r-\hat{\beta}_r]dr$ and $Q_r=Q(r,U_r)-Q(r,\hat{U}_r)$. We can write now $\langle X_r-\hat{X}_r,d\mathcal{K}_r\rangle+\frac{1}{2}|Q_r|^2dr\leq dR_t+|X_r-\hat{X}_r|^2dV_r$, with $V_t=Mt+\int_0^t\eta^+(r)\,dr$,

and $R_t = \int_0^t \frac{1}{2}\ell^2(r)\,|U_r - \hat{U}_r|^2 dr$. Let $a > 0$. Using Proposition 5.7, we obtain

$$\begin{aligned}
&e^{-pat} \times \mathbb{E}\left\|e^{-V}(X - \hat{X})\right\|_t^p \\
&\leq e^{-pat} \times C_p \mathbb{E}\left(\int_0^t e^{-2V(r)} l^2(r)\,|U_r - \hat{U}_r|^2 dr\right)^{p/2} \\
&\leq e^{-pat} \times C_p \mathbb{E}\left(\int_0^t l^2(r)\,||e^{-V}(U - \hat{U})||_r^2 dr\right)^{p/2} \\
&\leq e^{-pat} \times C_p \left(\int_0^t e^{2ar} e^{-2ar} l^2(r)\left(\mathbb{E}||e^{-V}(U - \hat{U})||_r^p\right)^{2/p} dr\right)^{p/2} \\
&\leq \rho(a) \sup_{r\in[0,T]}\left(e^{-pat}\mathbb{E}||e^{-V}(X - \hat{X})||_r^p\right),
\end{aligned}$$

where

$$\rho(a) = C_p \sup_{t\in[0,T]}\left(e^{-2at}\int_0^t e^{2ar} l^2(r)\,dr\right)^{p/2}.$$

Consider the supremum over $t \in [0,T]$ and we get

$$|||\Gamma(U) - \Gamma(\hat{U})||| \leq (\rho(a))^{1/p}\,|||U - \hat{U}|||,$$

where the norm $|||.|||$ is defined by $|||X||| = \sup_{t\in[0,T]} e^{-at}[(\mathbb{E}||e^{-V}X||_t^p)^{1/p}]$.

Since $\lim_{a\to\infty}\rho(a) = 0$ then Γ is a strict contraction for a large enough. Using the Banach's fixed point theorem, we conclude that there exists a unique solution for the equation (4.1) such that $(X, J, K, \beta) \in S_d^p[0,T] \times (S_d^{p/2}[0,T] \cap L^{p/2}(\Omega;\, BV([0,T],\, \mathbb{R}^d)))^2 \times L^p(\Omega;\, L^1([0,T];\, \mathbb{R}^d))$.

Step 2. We will prove the existence on $S_d^0[0,T] \times (S_d^0[0,T])^2 \times L^p(\Omega;\, L^1([0,T];\mathbb{R}^d))$. For $n \in \mathbb{N}^*$, we consider the stopping time given by

$$\theta_n = \inf\left\{t \geq 0 : |\xi| + \int_0^t \left[|\beta_{r,u_0}| + |Q_r|^2\right] dr \geq n\right\}.$$

It is obvious that θ_n go increasingly to infinity, so accordingly the first step there exists a unique solution (X^n, K^n, β^n) for the following approximating equation

$$\begin{cases}
X_t^n + J_t^n + K_t^n + \displaystyle\int_0^t 1_{[0,\theta_n]}(r)\,\beta_r^n dr = \xi 1_{\theta_n>0} + \int_0^t 1_{[0,\theta_n]}(r)\,Q(r, X_r^n) dB_r, \\
dJ_t^n \in \partial\varphi(X_t^n)\,dt,\ dK_t^n \in \partial_C\phi(X_t^n)\,dt,\ \beta_r^n \in G(t, X_t^n), \quad t \in [0,T].
\end{cases}$$

We have that, for all $l \in \mathbb{N}^*$,$(X^{n+l}, J^{n+l}, K^{n+l},\ \beta^{n+l})1_{\theta_n>0}1_{[0,\theta_n]} = (X^n\ , J^n,\ K^n,\ \beta^n)1_{[0,\theta_n]}1_{\theta_n>0}$, which concludes that the processes given, for $0 \leq t < \theta_n(\omega)$, by $(X_t(\omega),\ J_t(\omega), K_t(\omega),\ \beta_t(\omega))$=$1_{\theta_n>0}(X_t^n(\omega),\ J_t^n(\omega), K_t^n(\omega), \beta_t^n(\omega))$ represents a solution of equation (4.1) and it belong to the space $S_d^0[0,T]\times(S_d^0[0,T])^2\times L^p(\Omega; L^1([0,T];\, \mathbb{R}^d))$. □

5 Annex

Theorem 5.1 (Kakutani-Glicksberg-Fan fixed point theorem) *Let K be a non-empty, compact and convex subset of a locally convex topological vector space. Let $\varphi : K \rightrightarrows K$ be a multivalued upper-semicontinuous function. If for all $x \in K$, $\varphi(x)$ is non-empty, convex and compact. Then φ has a fixed point.*

Theorem 5.2 *If A_1 and A_2 are two maximal monotone operators defined on a real Hilbert space H and if $int(D(A_1)) \cap D(A_2) \neq \emptyset$, then $A_1 + A_2$ is also a maximal monotone operator.*

Proposition 5.3 *Let $\varphi : \mathbb{R}^d \to]-\infty, +\infty]$ be a proper convex l.s.c. function such that $int(Dom(\varphi)) \neq \emptyset$. Let $(u_0, \hat{u}_0) \in \partial\varphi$, $r_0 \geq 0$ and $b_0 \overset{def}{=} \sup\{\varphi(u_0 + r_0 v) : |v| \leq 1\}$. Then, for every continuous functions $x, k : \mathbb{R}_+ \to \mathbb{R}^d$, with $k \in BV\left(\mathbb{R}_+, \mathbb{R}^d\right)$, such that $dk(t) \in \partial\varphi(x(t))\, dt$, we have, for all $0 \leq s \leq t$,*

$$r_0 \left(\updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s\right) + \int_s^t \varphi(x(r)) dr \leq \int_s^t \langle x(r) - u_0, dk(r) \rangle + (t-s)\, b_0, \tag{5.1}$$

and

$$\begin{aligned} r_0 \left(\updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s\right) + \int_s^t |\varphi(x(r)) - \varphi(u_0)|\, dr \leq \int_s^t \langle x(r) - u_0, dk(r) \rangle \\ + \int_s^t \left(2\,|\hat{u}_0|\,|x(r) - u_0| + b_0 - \varphi(u_0)\right) dr. \end{aligned} \tag{5.2}$$

Proposition 5.4 *Let $k \in \mathbb{N}^*$, consider x_k, x continuous function from $[0,T]$ into $\mathbb{R}^d$ and let $n_k \in BV\left([0,T]; \mathbb{R}^d\right)$, $n : [0,T] \to \mathbb{R}^d$. If $x_k \to x \in C([0,T]; \mathbb{R}^d)$, $n_k(t) \to n(t), \forall t \in [0,T]$ and $\sup \updownarrow n_k \updownarrow_T = R < \infty$, then $n \in BV\left([0,T]; \mathbb{R}^d\right)$, $\updownarrow n \updownarrow_T \leq R$, and*

$$\int_s^t \langle x_k(r), dn_k(r) \rangle \to \int_s^t \langle x(r), dn(r) \rangle, \quad \text{as } k \longrightarrow \infty,$$

for all $0 \leq s \leq t \leq T$.

5.1 Useful inequalities

Proposition 5.5 *Let $x \in BV_{loc}\left([0,\infty[; \mathbb{R}^d\right)$ and $V \in BV_{loc}([0,\infty[; \mathbb{R})$ be two continuous functions. Consider R, $N : [0,\infty[\to [0,\infty[$ two continuous increasing functions. If $\langle x(t), dx(t) \rangle \leq dR(t) + |x(t)|\, dN(t) + |x(t)|^2\, dV(t)$ as signed measures on $[0,\infty[$, then for all $0 \leq t \leq T$,*

$$\begin{aligned} ||e^{-V} x||_{[t,T]} \leq 2 \Bigg[|e^{-V(t)} x(t)| + \left(\int_t^T e^{-2V(s)} dR(s) \right)^{1/2} \\ + \int_t^T e^{-V(s)} dN(s) \Bigg]. \end{aligned} \tag{5.3}$$

If $R = 0$ then, for all $0 \leq t \leq s$,

$$|x(s)| \leq e^{V(s)-V(t)} |x(t)| + \int_t^s e^{V(s)-V(r)} dN(r). \tag{5.4}$$

Remind from PARDOUX and RĂŞCANU [13] an estimate on the local semimartingale $X \in S_d^0$ of the form

$$X_t = X_0 + K_t + \int_0^t G_s dB_s,\ t \geq 0,\ \mathbb{P} - a.s.\ \omega \in \Omega, \tag{5.5}$$

where $K \in S_d^0$, $K_\cdot \in BV_{loc}([0,\infty[;\mathbb{R}^d)$, $K_0 = 0, \mathbb{P}$-a.s. $\omega \in \Omega$ and $G \in \Lambda_{d\times k}^0$. For $p \geq 1$ denote $m_p \overset{def}{=} 1 \vee (p-1)$ and we have the following important result.

Proposition 5.6 *Let $X \in S_d^0$ be a local semimartingale of the form (5.5). Assume there exist $p \geq 1$ and V a $\mathcal{P}$-measurable, with bounded variation, continuous stochastic process, $V_0 = 0$, such that, as signed measures on $[0,\infty[$,*

$$\langle X_t, dK_t\rangle + \frac{1}{2} m_p |G_t|^2 dt \leq |X_t|^2 dV_t, \quad \mathbb{P} - a.s.\ \omega \in \Omega. \tag{5.6}$$

Then, for all $\delta \geq 0$, $0 \leq t \leq s$, we have

$$\mathbb{E}^{\mathcal{F}_t} \frac{|e^{-V_s} X_s|^p}{(1+\delta|e^{-V_s}X_s|^2)^{p/2}} \leq \frac{|e^{-V_t} X_t|^p}{(1+\delta|e^{-V_t}X_t|^2)^{p/2}}, \quad \mathbb{P} - a.s.\ \omega \in \Omega. \tag{5.7}$$

The next proposition is a generalization of the previous one (for more details see also PARDOUX and RĂŞCANU [13]).

Proposition 5.7 *Let $X \in S_d^0$ be a local semimartingale of the form (5.5). Assume there exist three $\mathcal{P}-$measurable, increasing, continuous, stochastic processes D, R, N, $p \geq 1$, $\lambda \geq 0$ and a $\mathcal{P}-$measurable, with bounded variation, continuous stochastic process V, $D_0 = R_0 = N_0 = V_0 = 0$, such that, as signed measures on $[0,\infty[$, $\mathbb{P} -$ a.s. $\omega \in \Omega$,*

$$dD_t + \langle X_t, dK_t\rangle + (\frac{1}{2} m_p + 9p\lambda) |G_t|^2 dt \leq \mathbf{1}_{p\geq 2} dR_t + |X_t| dJ_t + |X_t|^2 dV_t.$$

Then, for all $0 \leq t \leq s$, we have, $\mathbb{P} - a.s.\ \omega \in \Omega$,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \|e^{-V} X\|_{[t,s]}^p + \mathbb{E}^{\mathcal{F}_t} \int_t^s e^{-pV_r} |X_r|^{p-2} dD_r \\
&\quad + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} (dD_r + |G_r|^2 dr) \right)^{\frac{p}{2}} \\
&\quad \leq C_{p,\lambda} \left[e^{-pV_t} |X_t|^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} \mathbf{1}_{p\geq 2} dR_t \right)^{\frac{p}{2}} \right. \\
&\quad \left. + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-V_r} dN_r \right)^p \right].
\end{aligned} \tag{5.8}$$

Acknowledgements The author would like to express his sincere gratitude to the anonymous referees for their valuable comments and suggestions, which have resulted in considerable improvement of the results and presentation of this article.

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