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On exit laws for resolvents

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Abstract Let $\mathbb{P} = (P_t)_{t\geq 0}$ be a contraction semigroup of kernels on $L^2(m)$, with resolvent $\mathbb{V} := (V_p)_{p>0}$ and let β be a $\mathcal{K}$-subordinator. Let $\mathbb{P}^\beta$ be the subordinated semigroup of $\mathbb{P}$ by means of β (i.e $P_t^\beta = \int_0^\infty P_s \beta_t(ds)$) with associated resolvent $\mathbb{V}^\beta$. First we give a bijection between a class of $\mathbb{V}$-exit laws and $\mathbb{P}$-exit laws. Next we prove, under some regularity assumptions, that each $\mathbb{V}^\beta$-exit law is subordinated to a $\mathbb{V}$-exit law. As application, we give an integral representation of $\mathbb{P}^\beta$-potentials in terms of $\mathbb{P}$-exit laws and of β.

Keywords Contraction semigroup of kernels · Resolvent · Exit law · Bochner subordination · Potential · Representation

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1 Introduction

Let $(E, \mathcal{E})$ be measurable space, let m be a σ-finite measure and let $\mathbb{P} := (P_t)_{t\geq 0}$ be a contraction semigroup of kernels on $L^2(m)$. A $\mathbb{P}$-exit law is a family $\varphi = (\varphi_t)_{t>0}$ of nonnegative elements of $L^2(m)$ satisfying the functional equation

$$P_s\varphi_t = \varphi_{s+t}, \qquad s, t > 0. \tag{1.1}$$

This notion is introduced by Dynkin [6] in the framework of potential theory without reference measure. Now, let $\mathbb{V}$ be the resolvent of $\mathbb{P}$. A $\mathbb{V}$-exit law is a family $f =$

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$(f_p)_{p>0}$ of positive measurable functions on E satisfying

$$f_p = f_q + (q-p)V_p f_q; \quad V_p f_q = V_q f_p, \qquad 0 < p < q. \tag{1.2}$$

This notion is the object of the ergodic theory, which studies the behavior of pf_p as $p \to \infty$ and which has the advantage to establish a most important result in the potential theory, known under the name of Mokobodski's Theorem (cf [5], p. 110). The second motivation of the present work came from [12] that represent potentials of $\mathbb{P}$ in terms of additive kernels of $\mathbb{P}$, while studying exit laws for resolvent. In the same context, while characterizing exit laws for resolvents in the original and the subordinated structure, we want to give a result of representation of potentials by $\mathbb{P}$-exit law.

Let $(\varphi_t)_{t>0}$ be a $\mathbb{P}$-exit law then the family $(f_p)_{p>0}$ defined by

$$f_p := \int_0^\infty e^{-pt}\,\varphi_t\,dt, \qquad p > 0 \tag{1.3}$$

is a $\mathbb{V}$-exit law. One of the aim of the present paper is to study the converse under some regularity assumptions. More precisely, we suppose that $\mathbb{P}$ satisfies a suitable condition (C) and we prove the following result: let $f = (f_p)_{p>0}$ be a $\mathbb{V}$-exit law satisfying $P_t f_p \in L^2(m)$ and $\lim_{t\to 0} e^{-pt} P_t f_p = f_p$ for each $p > 0$. Then there exists a unique $\mathbb{P}$-exit law $\varphi = (\varphi_t)_{t>0}$ such that (1.3) holds.

Consider a Bochner subordinator $\beta = (\beta_t)_{t>0}$, that is a vaguely continuous convolution semigroup of subprobability measures on $[0,\infty[$. Let $\mathbb{P}^\beta$ be the subordinated semigroup of $\mathbb{P}$ by means of β, i.e.

$$P_t^\beta u := \int_0^\infty P_s u\,\beta_t(ds), \qquad u \in L^2(m). \tag{1.4}$$

Now, we suppose that β is a $\mathcal{K}$-subordinator, that is the associated potential measure $\kappa := \int_0^\infty \beta_s\,ds$ is absolutely continuous with respect to the Lebesgue measure. Let $\mathbb{V}^\beta$ be the resolvent of $\mathbb{P}^\beta$, then

$$V_p^\beta := \int_0^\infty V_s\,\rho_p(ds), \qquad p > 0, \tag{1.5}$$

where ρ_p is a positive measure on $[0,\infty[$ described in (section 4 below).

If $f = (f_p)_{p>0}$ is a $\mathbb{V}$-exit law, then family $f^\beta = (f_p^\beta)_{p>0}$ given by

$$f_p^\beta := \int_0^\infty f_s\,\rho_p(ds), \qquad p > 0 \tag{1.6}$$

is a $\mathbb{V}^\beta$-exit law whenever the kernel V^β satisfies the unicity property explained in (subsection 2.1 below). The second main goal of the present paper is to investigate the converse under some appropriate assumptions. In fact, we suppose that V^β satisfies the unicity property and we establish the following result: if $g = (g_p)_{p>0}$ is a $\mathbb{V}^\beta$-exit law verifying $\sup_{p>0} g_p$ is finite $m.a.e$ and $V_1 g_1 \in D(A^\beta)$, then there exists some $\mathbb{V}$-exit law $f = (f_p)_{p>0}$ such that $g_p = f_p^\beta$.

An analogous problem was considered in [2], [14] and [19] in the framework of subordination of $\mathbb{P}$-exit laws.

By applying the two previous results we deduce an integral representation of potentials in terms of exit laws. Namely, let h be a $\mathbb{P}^\beta$-potential, under some sufficient conditions on h there exists a $\mathbb{P}$-exit law $\varphi = (\varphi_t)_{t>0}$ such that

$$h = \int_0^\infty \varphi_s \kappa(ds). \tag{1.7}$$

This problem was investigated in [11] and [15]. The same problem was studied in [1],[7],[8],[16],[17] and [18] for the original semigroup i.e. in the case when β is the trivial subordinator.

2 Preliminaries

Let $(E, \mathcal{E})$ be a mesurable space and let m be a σ-finite positive measure on $(E, \mathcal{E})$. We denote by $\mathcal{F}$ the set of nonnegative measurable functions, by $L^2(m)$ the Banach space of the square integrable (classes of) functions defined on E and by $\|\cdot\|_2$ the associated norm. $L^2_+(m)$ denotes the positive elements of $L^2(m)$. We denote also by $<\cdot,\cdot>$ the inner product of $L^2(m)$. Moreover, in the sequel, equality and inequality holds *m.a.e.* (i.e. almost everywhere with respect to m).

In this section we summarize some known results, we refer the reader to ([3], p. 76-84), ([5], VII and VIII p. 85-95), ([20], 3.9) and ([21], VII-IX).

2.1 Contraction semigroup

A *kernel* on E is a mapping $N : E \times \mathcal{E} \to [0, \infty[$ such that:
(a) $x \to N(x, A)$ is measurable for each $A \in \mathcal{E}$,
(b) $A \to N(x, A)$ is a measure on $(E, \mathcal{E})$ for each $x \in E$. In this case, we may define $Nu(x) := \int_E u(y)\, N(x, dy)$ for $u \in \mathcal{F}$ and $x \in E$. If $N(L^2(m)) \subset L^2(m)$, we say that N is a kernel acting on $L^2(m)$.

A *contraction semigroup* on E is a family $\mathbb{P} := (P_t)_{t\geq 0}$ of kernels acting on $L^2(m)$ such that $P_0 = I$ the identity operator on E,

1. $P_s P_t = P_{s+t}$ for $s, t \geq 0$ (*semigroup property*),
2. $\|P_t u\|_2 \leq \|u\|_2$ *for* $u \in L^2(m)$ *and* $t \geq 0$ (*contraction property*),
3. $\lim_{t\to 0} \|P_t u - u\|_2 = 0$
   for $u \in L^2(m)$ (*strong continuity*).

In this case, the associated *generator* A is defined by $Au := \lim_{t\to 0} \frac{1}{t}(P_t u - u)$ on its domain $D(A)$ which is the set of all functions $u \in L^2(m)$ for which this limit exists in $L^2(m)$. It is known that (cf. [21], p. 237-239)

1. A is closed and $D(A)$ is dense in $L^2(m)$.
2. If $u \in D(A)$ then $P_t u \in D(A)$ and $A(P_t u) = P_t A u$, for each $t > 0$.

Let $\mathbb{P}$ be a contraction resolvent then the associated *Resolvent* $\mathbb{V} := (V_p)_{p>0}$ is defined by $V_p = \int_0^\infty e^{-ps} P_s\, ds$. It is known that $\mathbb{V}$ satisfies the following properties

1. $p\|V_p u\|_2 \leq \|u\|_2$ for each $u \in L^2(m)$ and $p > 0$ (*contraction property*),
2. $\lim_{p\to\infty} \|pV_p u - u\|_2 = 0$ for each $u \in L^2(m)$ (*strong continuity*).

3. For all $0 < p < q$:

$$V_p = V_q + (q-p)V_qV_p, \quad \text{and} \quad V_qV_p = V_pV_q. \tag{2.1}$$

The generator A of $\mathbb{P}$ is given in terms of $\mathbb{V}$ by $Au := \lim_{p\to\infty} p(pV_pu - u)$.

A is also called the generator of $\mathbb{V}$. Following ([5], p. 91), $V_p(L^2(m)) \subset D(A)$ and

$$AV_pu = pV_pu - u, \qquad p > 0, u \in L^2(m). \tag{2.2}$$

We denote by $V := \sup_{p>0} V_p$ and $\mathcal{D}(V)$ the set of all $u \in \mathcal{F}$ such that $Vu \in \mathcal{F}$.

We say that V satisfies the "unicity property" (UP) if $\forall u, w \in \mathcal{D}(V) : Vu = Vw \Longrightarrow u = w$.

2.2 Excessive functions and exit laws

Let $\mathbb{P}$ be a contraction semigroup on E with resolvent $\mathbb{V}$ and let $p \geq 0$.

A nonnegative measurable function f is said to be p-excessive for $\mathbb{P}$ if

(i) f is *p-supermedian* for $\mathbb{P}$, i.e. $e^{-pt}P_tf \leq f$ for each $t > 0$.
(ii) $\lim_{t\to 0} e^{-pt}P_tf(x) = f(x)$, for all $x \in E$.

Following ([5], VII, 18), (i) and (ii) are equivalent respectively to

(i)' $qV_{q+p}f \leq f$ for each $q > 0$.
(ii)" $\lim_{q\to\infty} qV_{q+p}f(x) = f(x)$, for all $x \in E$.

We say that f is $\mathbb{P}$-*potential* if it is 0-excessive for $\mathbb{P}$ and $\lim_{t\to\infty} P_tf = 0$, *m.a.e.*

A $\mathbb{P}$-exit law is a family $\varphi := (\varphi_t)_{t>0}$ of elements of $L^2_+(m)$ satisfying the *exit equation* (1.1) (cf [5], p. 38).

A $\mathbb{V}$-exit law is a family $(f_p)_{p>0}$ of elements of $\mathcal{F}$ satisfying (1.2) (cf [5], p. 39-40). It is easy to show that the mapping $p \to f_p$ is decreasing and f_p is p-supermedian for each $p > 0$.

Example 2.1 Let $\mathbb{P}$ be a contraction semigroup and let $\mathbb{V} = (V_p)_{p>0}$ be the associated resolvent.

1. From the resolvent equation, the family $(V_pf)_{p>0}$ is an exit law for $\mathbb{V}$ for every $f \in \mathcal{F}$.
2. Let f is a supermedian function for $\mathbb{P}$ and let $f_p := f - pV_pf$ for $p > 0$. Then $(f_p)_{p>0}$ is a $\mathbb{V}$-exit law. Moreover f_p is p-excessive for $\mathbb{P}$.
3. $\mathbb{V}$ is said to be *m-basic* if there exists a measurable function $G_p : E \times E \to [0, \infty]$ such that $V_pu(x) = \int u(y)\, G_p(x, y)\, m(dy)$ for each $p > 0, u \in \mathcal{F}, x \in E$. In this case G is called the density of $\mathbb{V}$. Then, for fixed $y \in E$, $(G_p(., y))_{p>0}$ is a $\mathbb{V}$-exit law (cf. [5], XII, 72).

3 Characterisation of resolvent's exit laws

Proposition 3.1 *Let $\mathbb{P}$ be a contraction semigroup on E and let $\varphi = (\varphi_t)_{t>0}$ be a $\mathbb{P}$-exit law. Then the family $(f_p)_{p>0}$ defined by (1.3) is a $\mathbb{V}$-exit law, moreover f_p is p-excessive for $\mathbb{P}$ and $(P_tf_p)_{t>0} \subset L^2(m)$, for each $p > 0$.*

Proof. Let $p > 0$. From (1.1) we have $P_t f_p = \int_0^\infty e^{-ps}\, \varphi_{s+t}\, ds = V_p \varphi_t$ for each $t > 0$. Since $\varphi_t \in L^2(m)$ and $V_p(L^2(m)) \subset L^2(m)$, then $P_t f_p \in L^2(m)$. Moreover

$$e^{-pt}\, P_t f_p = \int_0^\infty e^{-p(s+t)}\, \varphi_{s+t}\, dt = \int_t^\infty e^{-ps}\, \varphi_s\, ds, \qquad t > 0. \tag{3.1}$$

Therefore f_p is p-excessive for $\mathbb{P}$. Using Fubini's Theorem and (1.1), we obtain for each $0 < p < q$

$$V_p f_q = \int_0^\infty e^{-pt} \int_0^\infty e^{-qs}\, P_t \varphi_s\, ds\, dt = \int_0^\infty e^{-qs} \int_0^\infty e^{-pt}\, P_s \varphi_t\, dt\, ds = V_q\, f_p.$$

Moreover, (2.2) yields $(q-p)V_p P_t f_q = qV_q P_t f_p - pV_p P_t f_q = AV_q P_t f_p + P_t f_p - AV_p P_t f_q - P_t f_q = P_t f_p - P_t f_q$. Hence

$$(q-p)e^{-(p+q)t} V_p P_t f_q = e^{-(p+q)t} P_t f_p - e^{-(p+q)t} P_t f_q, \qquad t > 0. \tag{3.2}$$

On the other hand, it follows from (2.1) that

$$e^{-qt} V_p P_t f_q = \int_t^\infty e^{-qs}\, V_p \varphi_s\, ds, \qquad t > 0. \tag{3.3}$$

Letting $t \to 0$ in (2.2) and (3.2) we deduce (1.2). □

Remark 3.1 The converse is not true in general. Indeed, let $\mathbb{P} := (P_t)_{t\geq 0}$ be the semigroup of right-translations on $\mathbb{R}$, endowed with its Lebesgue measure $m(dx) = dx$, i.e. $P_t u(x) := u(x-t)$ for $t \geq 0, x \in \mathbb{R}$ and $u \in \mathcal{F}$. Let f be a $\mathbb{P}$-potential function such that f don't belongs to the range of V and let $f_p = f - pV_p f$. It is known that each $\mathbb{P}$-exit law φ is closed, i.e $\varphi_t = P_t l$ for some $l \in \mathcal{F}_+$. But there exist no $l \in \mathcal{F}_+$ such that $f_p = V_p l$. Hence f_p can not be in the form (1.3).

3.1 Assumption (C)

Let $\mathbb{P}$ be a contraction semigroup on E. We suppose that $t \to P_t u$ is differentiable on $]0,\infty[$ for each $u \in L^2(m)$ and the mapping $u \to \frac{\partial P_t u}{\partial t}$ is continuous on $L^2(m)$. Equivalently $P_t(L^2(m)) \subset D(A)$ and there exists a function $K :]0,\infty[\to]0,\infty[$ such that

$$\|AP_t u\|_2 \leq K(t)\, \|u\|_2, \qquad t > 0, u \in L^2(m). \tag{3.4}$$

Example 3.1 1. We say that $\mathbb{P}$ satisfies the *sector condition* if there exists a constant $M > 0$ such that for all $u, v \in D(A)$, $\langle -Au, v\rangle \leq M\langle -Au, u\rangle^{1/2} \cdot \langle -Av, v\rangle^{1/2}$. In this case, the condition (C) is fulfilled (cf. [21], p. 254-255). In particular, if $\mathbb{P}$ is m-symmetric, i.e $\langle P_t v, w\rangle = \langle v, P_w\rangle$, for all $t > 0$ and $v, w \in L^2(m)$, then the sector condition is always satisfied for $M = 1$.

2. If the generator A of $\mathbb{P}$ is bounded, then $\mathbb{P}$ satisfies the condition (C).

Lemma 3.2 *Let $\mathbb{P}$ be a contraction semigroup satisfying the condition (C) and let $(f_p)_{p>0}$ be a $\mathbb{V}$-exit law such that $(P_t f_p)_{t,p>0} \subset L^2(m)$. Then the mappings $p \to P_t f_p$ and $p \to AP_t f_p$ are differentiable with values in $L^2(m)$ for each $t > 0$ and*

$$\frac{\partial}{\partial p} P_t f_p = -V_p P_t f_p \quad \textit{and} \quad \frac{\partial}{\partial p} AP_t f_p = -pV_p P_t f_p + P_t f_p. \tag{3.5}$$

Proof. First $P_t f_p = P_{t/2}(P_{t/2} f_p) \in D(A)$, for all $t > 0$. Let $q > p$, then from (1.2) and Fubini's theorem we have

$$\frac{P_t f_p - P_t f_q}{q - p} = V_q P_t f_p, \qquad t > 0. \tag{3.6}$$

Letting $q \to p$ in (3.6) and using the dominated convergence Theorem, we find the first equality of (3.5).

In the same way, by the semigroup property we get

$$\frac{1}{q-p}(AP_t f_p - AP_t f_q) = \frac{1}{q-p}(AP_{t/2}(P_{t/2} f_p - P_{t/2} f_q)), \quad t > 0.$$

Therefore, (3.4) yields that

$$\|\frac{AP_t f_p - AP_t f_q}{q-p} - AP_t V_p f_p)\|_2 \leq 2K(t)\, \|\frac{P_{t/2} f_p - P_{t/2} f_q}{q-p} - P_{t/2} V_p f_p\|_2. \tag{3.7}$$

Letting $q \to p$ in (3.7), it follows from (2.2) that $\frac{\partial}{\partial p} AP_t f_p = -AP_t V_p f_p = -AV_p P_t f_p = -pV_p P_t f_p + P_t f_p$. □

Theorem 3.3 *Let $\mathbb{P}$ be a contraction semigroup satisfying the condition (C) and let $(f_p)_{p>0}$ be a $\mathbb{V}$-exit law such that f_p is p-excessive for $\mathbb{P}$ and $(P_t f_p)_{t>0} \subset L^2(m)$ for each $p > 0$. Then there exists a unique $\mathbb{P}$-exit law $\varphi := (\varphi_t)_{t>0}$ such that (1.3) holds.*

Proof. Let $p > 0$ fixed, the function $t \to \varphi_t^p$ given by $\varphi_t^p := pP_t f_p - AP_t f_p$ is well defined and lies in $L^2(m)$. Moreover

$$P_t \varphi_s^p = pP_t P_s f_p - AP_t(P_s f_p) = \varphi_{t+s}^p, \qquad t, s > 0. \tag{3.8}$$

Furthermore, (3.8) and (2.2) imply that

$$\begin{aligned}\int_0^\infty e^{-ps}\, \varphi_{t+s}^p\, ds &= pV_p P_t f_p - \int_0^\infty e^{-ps}\, AP_s(P_t f_p)\, ds \\ &= pV_p P_t f_p - A\int_0^\infty e^{-ps}\, P_s(P_t f_p)\, ds \\ &= pV_p P_t f_p - AV_p P_t f_p = P_t f_p.\end{aligned}$$

Hence $e^{-pt} P_t f_p = \int_t^\infty e^{-ps}\, \varphi_s^p\, ds$, for all $t > 0$. Letting $t \to 0$, we obtain

$$f_p = \int_0^\infty e^{-ps}\, \varphi_s^p\, ds. \tag{3.9}$$

Now, for each $t > 0$, it follows from (3.5) that $\frac{\partial}{\partial p}\, \varphi_t^p = P_t f_p + p\frac{\partial}{\partial p} P_t f_p - \frac{\partial}{\partial p} AP_t f_p = 0$. Hence the function $p \to \varphi_t^p$ is constant for each $t > 0$ and therefore $\varphi_t^p = \varphi_t^1 = P_t f_1 - AP_t f_1 := \varphi_t$. On the other hand, for all $p > 0$, we get from (2.2), $V_p \varphi_t = V_p \varphi_t^p = pP_t V_p f_p - AV_p P_t f_p = P_t f_p$. Then $V_p \varphi_t \geq 0$ for each $p > 0$ and consequently, by the strong continuity of $\mathbb{V}$, $\varphi_t = \lim_{p\to\infty} pV_p \varphi_t$ is nonnegative. So $\varphi = (\varphi_t)_{t>0}$ is a $\mathbb{P}$-exit law and from (3.9) the representation (1.3) holds. Let us now prove the uniqueness:

If $\phi = (\phi_t)_{t>0}$ is another $\mathbb{P}$-exit law such that $f_p = \int_0^\infty e^{-ps}\, \phi_s\, ds$, then by (1.1) we have $pP_t f_p = pV_p \phi_t = pV_p \varphi_t$ for $p, t > 0$. Letting $p \to \infty$ we deduce that $\phi_t = \varphi_t$ for each $t > 0$. □

3.2 Basic semigroups

Let $\mathbb{P}$ be a semigroup of kernels on E. We say that $\mathbb{P}$ is m-basic if there exists a measurable function $\delta_t : E \times E \to [0, \infty]$ such that $P_t u(x) = \int_E u(y)\, \delta_t(x, y)\, m(dy)$ for each $t > 0, u \in \mathcal{F}, x \in E$. In this case δ is called the density of $\mathbb{P}$ and it satisfies $\delta_{s+t}(x, y) = \int \delta_s(x, z)\, \delta_t(z, y)\, m(dz)$ for each $s, t > 0$ and $x, y \in E$.

We suppose that $\mathbb{V}$ is m-basic and m is $\mathbb{P}$-excessive, i.e. $\int_E P_t(x, A)\, m(dx) \leq m(A)$ and $\lim_{t\to 0} \int_E P_t(x, A)\, m(dx) = m(A)$, for all $t > 0$ and $A \in \mathcal{E}$. In this case m is a reference measure (cf [5], XII, 42). Following ([5], XII, 67 D), V satisfies (UP).

Corollary 3.4 *Let $\mathbb{P}$ be a contraction semigroup satisfying the condition (C). If $(P_t G_1(\cdot, y))_{t>0} \subset L^2(m)$ and $G(\cdot, y) := \sup_{p>0} G_p(\cdot, y) < \infty, m.a.e,$ for each $y \in E$, then there exists an m-basic semigroup of kernels $\mathbb{Q}$ with resolvent $\mathbb{V}$.*

Proof. Let $y \in E$ be fixed. According to ([5], XII, 72), $(G_p(\cdot, y))_{p>0}$ is a $\mathbb{V}$-exit law and $G_p(\cdot, y)$ is p-excessive for $\mathbb{P}$. From Theorem 3.3, there exists a unique $\mathbb{P}$-exit law $(\delta_t(\cdot, y))_{t>0}$ such that $G_p(\cdot, y) = \int_0^\infty e^{-pt}\, \delta_t(\cdot, y)\, dt$ for each $p > 0$. Since $\sup_{p>0} G_p = \lim_{p\to 0} G_p$, then we get

$$G(\cdot, y) = \int_0^\infty \delta_t(\cdot, y)\, dt \tag{3.10}$$

and by the monotone convergence Theorem

$$Vu = \lim_{p\to 0} V_p u = \int_E G(\cdot, y)\, u(y)\, m(dy) = \int_0^\infty P_s\, du, \qquad u \in \mathcal{F}.$$

Using (3.10), (1.1) and Fubini's Theorem we obtain, for all $t > 0$ and $u \in \mathcal{F}$

$$\begin{aligned} VP_t u = P_t V u &= \int_E P_t G(\cdot, y)\, u(y)\, m(dy) \\ &= \int_E P_t\Big(\int_0^\infty \delta_s(\cdot, y)\, ds\Big)\, u(y)\, m(dy) \\ &= \int_0^\infty P_s\Big(\int_E \delta_t(\cdot, y)\, u(y)\, m(dy)\Big)\, ds = V\Big[\int_E \delta_t(\cdot, y)\, u(y)\, m(dy)\Big]. \end{aligned}$$

In particular by (1.1) and (3.10) again we have

$$V\delta_{s+t}(\cdot, y) = V\Big(\int_E \delta_t(\cdot, z)\, \delta_s(z, y)\, m(dz) < G(\cdot, y), \qquad s, t > 0. \tag{3.11}$$

Since $G(\cdot, y) < \infty, m.a.e$ and $\mathbb{P}$ satisfies (UP), then (3.11) yields that δ is a density of an m-basic semigroup $\mathbb{Q}$. By Fubini's Theorem, we have for all $u \in \mathcal{F}$

$$\begin{aligned} \int_0^\infty e^{-pt} Q_t u\, dt &= \int_E \Big(\int_0^\infty e^{-pt}\, \delta_t(\cdot, y)\, dt\Big)\, u(y)\, m(dy) \\ &= \int_E G_p(\cdot, y)\, u(y)\, m(dy) = V_p u. \end{aligned}$$

Then $\mathbb{V}$ is the resolvent of $\mathbb{Q}$. $\square$

4 Subordination of resolvents' exit laws

For the following standard notions, we will refer to ([3], II-9), [4] and ([20], 4.3).

We consider $\mathbb{R}$ endowed with its Borel field $\mathcal{A}$, we denote by λ the Lebesgue measure on $[0,\infty[$ and by ε_t the Dirac measure at point t.

A *Bochner subordinator* is a family of probability measures on $\beta=(\beta_t)_{t>0}$ of sub-probability measures on $(\mathbb{R},\mathcal{A})$ such that

1. For each $t>0$, the measure $\beta_t\neq 0$ and $\beta_t([0,\infty[)=0$.
2. $\beta_t * \beta_s=\beta_{s+t}$, for all $s,t>0$.
3. $\lim_{t\to 0}\beta_t=\varepsilon_0$ vaguely.

For each $p>0$, let $\kappa:=\int_0^\infty e^{-ps}\beta_s\,ds$. It is known that κ_p is a bounded measure on $[0,\infty[$ and the associated potential $\kappa:=\kappa_0:=\sup_{p>0}\kappa_p=\int_0^\infty \beta_s\,ds$ is a Borel measure on $[0,\infty[$.

Let $\mathbb{P}$ be a contraction semigroup on E and let β be a Bochner subordinator. Then $\mathbb{P}^\beta:=(P_t^\beta)_{t>0}$, defined by $P_t^\beta u=\int_0^\infty P_s u\,\beta_t(ds)$ for each $u\in L^2(m)$, is a contraction semigroup on $L^2(m)$ (cf. [20], 4.3). It is said to be *subordinated* to $\mathbb{P}$ in the sense of Bochner by means of β.

We denote by A^β and $\mathbb{V}^\beta$, the resolvent and the generator associated respectively to $\mathbb{P}^\beta$. According to ([20] p. 269), $D(A)\subset D(A^\beta)$.

A Bochner subordiantor is said to of class $\mathcal{K}$ if the associated potential measure κ is absolutely continuous with respect to the Lebesgue measure l on $[0,\infty[$. In this case we say that β is a $\mathcal{K}$-subordinator.

Example 4.1 For the following examples we will refer to ([3], p. 129).

1. The family $(\varepsilon_t)_{t>0}$ is called the trivial subordinator and the associated potential measure $\kappa=\lambda$.

2. One sided stable subordinator: For each $\alpha\in]0,1[$ and $t>0$, let η_t^α be the unique probability measure such that $\mathcal{L}(\eta_t^\alpha)(r)=\exp(-tr^\alpha)$ for $r>0$. Then $\eta^\alpha=(\eta_t^\alpha)_{t>0}$ is subordinator called the one sided stable subordinator of index α. It is well known that the associated potential measure is given by $\kappa(ds)=1_{]0,\infty[}(s)\,\frac{s^{\alpha-1}}{\Gamma(\alpha)}\,ds$. For $\alpha=1/2$,

3. Γ-subordinator: For $t>0$, let $g_t(s)=1_{]0,\infty[}(1/\Gamma(t))\,s^{t-1}\exp(-s)$ and $\gamma_t:=g_t\cdot\lambda$. Then $\gamma:=(\gamma_t)_{t>0}$ is a subordinator, called the Γ-subordinator. In this case $\kappa:=\int_0^\infty \gamma_t\,dt=h\cdot l$ where $h(t)=\exp(-t)\int_0^\infty \frac{1}{\Gamma(s)}\,t^{s-1}\,ds$.

Notice that the trivial subordinator, the one-sided stable subordinator and the Gamma subordinator are $\mathcal{K}$-subordinators

For the following notion, we will refer to [9] and [10]. Let β be a $\mathcal{K}$-subordinator, in this case we may write $\kappa(dt)=\xi(t)\cdot l$ where $\xi:]0,\infty[\to\mathbb{R}$ is completely monotone (i.e. ξ is a C^∞-function and $(-1)^n\xi^n\geq 0$, for all integers $n\in\mathbb{N}$). Moreover ξ is integrable at 0 and $\kappa_p(dt)=\xi_p(t)\cdot dt$ where ξ_p is also a completely monotone and integrable function on $]0,\infty[$, for each $p>0$. Therefore ξ_p is the Laplace transform of a nonnegative measure ρ_p on $]0,\infty[$ such that $\rho_p(\{0\})=0$ and

$$\int_0^\infty \frac{1}{s}\,\rho_p(ds)=\frac{1}{p},\qquad p>0. \tag{4.1}$$

It can be seen also that $\xi_p\uparrow\xi$ as $p\to 0$.

For a $\mathcal{K}$-subordinator β, We have immediately

$$V_p^\beta = \int_0^\infty V_s \, \rho_p(ds), \qquad p > 0. \tag{4.2}$$

Proposition 4.1 *Let $\mathbb{V}$ be a contraction resolvent, let β be a Bochner subordinator and let $g := (g_p)_{p>0}$ be a $\mathbb{V}^\beta$-exit law such that $V_1 g_1 \in D(A^\beta)$. Then the family $f := (f_p)_{p>0}$ defined by $f_p := -A^\beta V_p g_1 + V_p g_1, p > 0$ is a $\mathbb{V}$-exit law.*

Proof. First, using (2.1), we have $V_p g_1 = V_1 g_1 + V_p V_1 g_1$ for each $p > 0$. consequently $V_p g_1 \in D(A^\beta)$. The fact that g_1 is 1-supermedian for $\mathbb{V}^\beta$ and $V_p V_r^\beta = V_r^\beta V_p$ for each $p, r > 0$ implies $V_p g_1$ is 1-supermedian for $\mathbb{V}^\beta$. Let B be the generator of the resolvent $(V_{p+1}^\beta)_{p>0}$, then $(V_p g_1)_{p>0} \subset D(B)$ and $f_p = -BV_p g_1 = \lim_{q\to\infty} q(V_p g_1 - qV_{q+1}^\beta V_p g_1)$ which is nonnegative as limit of nonnegative functions. We put $h_p := -A^\beta V_p g_1$. Then, from the resolvent equation, we have for all $p, q > 0$, $(q-p)V_q h_p = -(q-p)V_q A^\beta V_p g_1 = -(q-p)A^\beta V_q V_p g_1 = -A^\beta(V_p g_1 - V_q g_1) = h_p - h_q$. □

Proposition 4.2 *Let $\mathbb{P}$ be a contraction semigroup let β be a $\mathcal{K}$-subordinator and let $(f_p)_{p>0}$ be a $\mathbb{V}$-exit law such that $f_0 := \sup_{p>0} f_p < \infty$, m.a.e. Suppose that V satisfies (UP), then the family $(f_p^\beta)_{p>0}$, given by (1.7), is a $\mathbb{V}^\beta$-exit law which is said to be subordinated to f by means of β.*

Proof. From (1.2) and Fubini's Theorem we get

$$V_s f_p^\beta = \int_0^\infty V_s f_r \, \rho_p(dr) = \int_0^\infty V_r f_s \, \rho_p(dr) = V_p^\beta f_s, \qquad s, p > 0.$$

It follows that

$$V_p^\beta f_q^\beta = \int_0^\infty V_s f_q^\beta \, \rho_p(ds) = \int_0^\infty V_q^\beta f_s \, \rho_p(ds) = V_q^\beta f_p^\beta, \qquad p, q > 0.$$

Moreover, from (2.1), we obtain for each $r > 0$ and $0 < p < q$ $V_r(f_p^\beta - f_q^\beta) = V_p^\beta f_r - V_q^\beta f_r = (q-p)V_q^\beta V_p^\beta f_r = (q-p)V_q^\beta V_r f_p^\beta = (q-p)V_r V_q^\beta f_p^\beta$. But f_0 is $\mathbb{V}$-supermedian, then we have by (4.2) and (4.1) $V_r V_q^\beta f_p^\beta = V_p^\beta V_q^\beta f_r \le V_p^\beta V_q^\beta f_0 \le \frac{1}{pq} f_0$ Letting $r \to 0$, we get

$$0 \le V(f_p^\beta - f_q^\beta) = (q-p)VV_q^\beta f_p^\beta \le \frac{1}{pq} f_0 < \infty \; m.a.e \tag{4.3}$$

Thus, from (4.3), both of $f_p^\beta - f_q^\beta$ and $(q-p)V_q^\beta f_p^\beta$ belong to $\mathcal{D}(V)$. The proof is achived by using (4.3) and (UP). □

Theorem 4.3 *Let $\mathbb{V}$ be a contraction resolvent and let β be a $\mathcal{K}$-subordinator such that V^β satisfies (UP). Let $g = (g_p)_{p>0}$ be a $\mathbb{V}^\beta$-exit law verifying $g_0 := \sup_{p>0} g_p < \infty$, m.a.e and $V_1 g_1 \in D(A^\beta)$. Then g is subordinated to a $\mathbb{V}$-exit law f.*

Proof. Let $h_p = -A^\beta V_p g_1$ for each $p > 0$. By (4.1) we have

$$\int_0^\infty \|V_s A^\beta V_q g_1\| \, \rho_p(ds) \leq \frac{1}{p} \|A^\beta V_q g_1\| < \infty. \tag{4.4}$$

We remind that $V_p^\beta = \int_0^\infty V_s \rho_p(ds)$ and we denote $h_p^\beta := \int_0^\infty h_s \, \rho_p(ds)$ for each $p > 0$. Using (4.4) and (2.2) we find

$$\begin{aligned} V_q h_p^\beta &= \int_0^\infty V_q h_s \, \rho_p(ds) = -\int_0^\infty A^\beta V_s V_q g_1 \, \rho_p(ds) = -\int_0^\infty V_s A^\beta V_q g_1 \, \rho_p(ds) \\ &= -A^\beta V_p^\beta V_q g_1 = V_q g_1 - p V_p^\beta V_q g_1 = V_q(g_1 - p V_p^\beta g_1). \end{aligned}$$

By an integration with respect to ρ_q we obtain $V_q^\beta h_p^\beta = V_q^\beta(g_1 - p V_p^\beta g_1)$ for each $p, q > 0$. From (1.2), it follows that

$$V_q^\beta(h_p^\beta + V_p^\beta g_1) = V_q^\beta(g_1 + (1-p)V_p^\beta g_1) = V_q^\beta g_p, \qquad p, q > 0. \tag{4.5}$$

Moreover $V_q^\beta g_p \leq V_q^\beta g_0 \leq \frac{1}{q} g_0$ because g_0 is supermedian for $\mathbb{V}^\beta$. Letting $q \to 0$ in (4.5), we get by the monotone class Theorem

$$0 \leq V^\beta(h_p^\beta + V_p^\beta g_1) = V^\beta g_p \leq \frac{1}{p} g_0. \tag{4.6}$$

Since $g_0 < \infty$, *m.a.e*, then (4.6) and (UP) imply that

$$g_p = h_p^\beta + V_p^\beta g_1, \qquad p > 0. \tag{4.7}$$

We put $f_p := h_p + V_p g_1$ for each $p > 0$. According to Proposition 4.2, $f := (f_p)_{p>0}$ is a $\mathbb{V}$-exit law and $g_p = f_p^\beta$ from (4.7). □

Remark 4.1 Let β be a $\mathcal{K}$-subordinator. We cite some situations when the unicity property is fulfilled for V and V^β

1. If $\mathbb{P}$ is m-basic and m is $\mathbb{P}$-excessive then $\mathbb{P}^\beta$ is also m-basic and m is $\mathbb{P}^\beta$-excessive. In this case both V and V^β satisfy (UP).

2. Let $\mathbb{P}$ be the semigroup of right process (cf. [5], XVI for example). From ([5], Proposition 1.1 and Theorem 4.2), (UP) is fulfilled for V. According to ([13], Theorem 3), (UP) is verified for V^β.

3. Let $E = \mathbb{R}^d$ and $\mathcal{E}$ be the Borel σ-field. Let μ be a convolution semigroup on $\mathbb{R}^d$ endowed with its Lebesgue measure $m = \lambda^d$, i.e. $\mu := (\mu_t)_{t>0}$ is a family of sub-probability measures on $\mathbb{R}^d$ satisfying $\mu_s * \mu_t = \mu_{s+t}$, for all $s, t > 0$ and $\lim_{t\to 0} \mu_t = \varepsilon_0$ vaguely. Let $P_t f = \mu_t * f$, for each $f \in \mathcal{F}, t > 0$. Following ([3], Sections 16.8 and 14.21), if $\int_0^\infty \mu_s \, ds := \lim_{t\to\infty} \int_0^t \mu_s \, ds$ is a Borel measure then V and V^β satisfy (UP).

5 Application

We want to represent $\mathbb{P}^\beta$-potentials in terms of the "intitial entities" namely in terms of the $\mathbb{P}$-exit law and the subordintor β.

Lemma 5.1 *Let $\varphi = (\varphi_t)_{t>0}$ be a $\mathbb{P}$-exit law. Then the function $\int_0^\infty \varphi_s\, \kappa(ds)$ is a $\mathbb{P}^\beta$-potential.*

In the sequel, $\mathbb{P}$ is a contraction semigroup satisfying the condition (C) and $\mathbb{V}$ is the associated resolvent.

Theorem 5.2 *Let β be a $\mathcal{K}$-subordinator such that $\mathbb{V}^\beta$ satisfies (UP) and let h be a $\mathbb{P}^\beta$-potential verifying $V_1(h - V_1^\beta h) \in D(A^\beta)$. Then there exists a $\mathbb{P}$-exit law $\varphi = (\varphi_t)_{t>0}$ such that*

$$h = \int_0^\infty \varphi_s \kappa(ds). \tag{5.1}$$

Proof. Let $g_p : h - pV_p^\beta h$ for each $p > 0$. Then $(g_p)_{p>0}$ is a $\mathbb{V}^\beta$-exit law, $V_1 g_1 \in D(A^\beta)$ and $g_0 := \sup_{p>0} g_p = h < \infty,\ m.a.e.$ It follows, from Theorem 4.4, that g is subordinated to a $\mathbb{V}$-exit law $f = (f_p)_{p>0}$. Let $r > 0$ be fixed, by Fubini's Theorem and (1.2) we get

$$V_r g_p = \int_0^\infty V_r f_s\, \rho_p(ds) = \int_0^\infty V_s f_r\, \rho_p(ds). \tag{5.2}$$

It is obvious that $(V_s f_r)_{s>0}$ is a $\mathbb{V}$-exit law and $V_s f_r$ is s-excessive for $\mathbb{P}$ for each $s > 0$. By applying Theorem 3.3, there exists a unique $\mathbb{P}$-exit law $(\varphi_u^r)_{u>0}$ such that

$$V_s f_r = \int_0^\infty e^{-su} \varphi_u^r\, du, \qquad s > 0. \tag{5.3}$$

Combining (5.2) and (5.3), we obtain

$$V_r g_p = \int_0^\infty \int_0^\infty e^{-su}\, \varphi_u^r\, du\, \rho_p(ds) = \int_0^\infty \varphi_u^r\, \mathcal{L}\rho_p(u)\, du = \int_0^\infty \varphi_u^r\, \xi_p(u)\, du.$$

Since $g_p \uparrow h, m.a.e$ and $\xi_p \uparrow \xi$, then by letting $p \to 0$ and using the monotone class Theorem we have $V_r h = \int_0^\infty \varphi_u^r\ \xi(u)\, du = \int_0^\infty \varphi_u^r\ \kappa(du)$. On the other hand, (1.2) yields that $\varphi_u^r = P_u V_1 f_r - A P_u V_1 f r = P_u V_r f_1 - A P_u V_r f_1 = V_r(P_u f_1 - A P_u f_1) = V_r \varphi_u$, where $\varphi_u = P_u f_1 - A P_u f_1$. Hence

$$V_r h = V_r \int_0^\infty \varphi_u\, \kappa(du), \qquad r > 0. \tag{5.4}$$

By integration of (4.4) with respect to each measure ρ_p, it follows that

$$p V_p^\beta h = p V_p^\beta \int_0^\infty \varphi_u\ \kappa(du), \qquad p > 0. \tag{5.5}$$

Letting $p \to \infty$ in (5.5), then (5.1) holds. □

Corollary 5.3 *Suppose that V^γ satisfies (UP). Then for each $\mathbb{P}^\gamma$-potential h verifying $(P_t^\gamma h)_{t>0} \subset L^2(m)$, there exists a unique $\mathbb{P}$-exit law φ such that*

$$h = \int_0^\infty \int_0^\infty e^{-t}\varphi_t \, \frac{1}{\Gamma(s)} \, t^{s-1} \, ds \, dt.$$

Proof. Following ([4], p. 874), $P_t^\gamma(L^2(m)) \subset D(A^\gamma)$. By the semigoup property, $P_t^\gamma h = P_{t/2}^\gamma(P_{t/2}^\gamma h) \in D(A^\gamma)$. Moreover we have $V_1 = P_1^\gamma$, it follows that $V_1(h - V_1^\gamma h) = P_1^\gamma h - V_1^\gamma(P_1^\gamma h) \in D(A^\gamma)$. By applying Theorem 5.2, the assertion holds. □

Corollary 5.4 *Suppose that m is $\mathbb{P}$-excessive and $\mathbb{P}$ is m-basic. Let h be a $\mathbb{P}^\beta$-potential satisfying $(P_t h)_{t>0} \subset L^2(m)$. Then there exists a unique $\mathbb{P}$-exit law $\varphi = (\varphi_t)_{t>0}$ such that* (5.1) *holds.*

Proof. Let $t>0$ be fixed, then $P_t h$ is a $\mathbb{P}^\beta$-potential and $V_1(P_t h - V_1^\beta P_t h) \in V_1(L^2(m)) \subset D(A) \subset D(A^\beta)$. It follows, from Theorem 5.2, that there exists a unique $\mathbb{P}$-exit law $(\varphi_s^t)_{s>0}$ such that

$$P_t h = \int_0^\infty \varphi_s^t \, \kappa(ds), \tag{5.6}$$

where $\varphi_s^t = -AP_s f_1^t + P_s f_1^t$ and $f_1^t = -A^\beta V_1(P_t h - V^\beta P_t h) + V_1(P_t h - V_1^\beta P_t h)$. Using the semigroup property we obtain $\varphi_s^t = P_t \varphi_s$ where $\varphi_s = AP_{s/2}A^\beta V_1(P_{s/2}h - V_1^\beta P_{s/2}h) - A^\beta V_1(P_s h - V_1^\beta P_s h) + P_s h - V_1^\beta P_s h$. Since $\varphi_s \in L^2(m)$, for all $s > 0$, then the strong continuity of $\mathbb{P}$ yields that $\varphi_s = \lim_{t\to 0} P_t \varphi_s = \lim_{t\to 0} \varphi_s^t \geq 0$. By the semigroup property again, it is obvious that $(\varphi_s)_{s>0}$ satisfies (1.1). Consequently $(\varphi_s)_{s>0}$ is a $\mathbb{P}$-exit law and $P_t h = P_t \int_0^\infty \varphi_s \kappa(ds)$. By integration with each measure β_t we get

$$P_t^\beta h = P_t^\beta \int_0^\infty \varphi_s \, \kappa(ds), \qquad t > 0. \tag{5.7}$$

We conclude by letting $t \to 0$ in (5.7) and using Lemma 5.1. Finally, we shall prove the unicity. So assume that there exists a $\mathbb{P}$-exit law $(\psi_t)_{t>0}$ such that (5.1) holds. The fact that $V^\beta = \int_0^\infty P_s \, \kappa(ds)$ gives us $P_t h = V^\beta \varphi_t = V^\beta \psi_t, t > 0$. Since $P_t h < \infty, m.a.e$, then $\varphi_t = \psi_t$ by (UP). □

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