An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Tomul LXII, 2016, f. 2, vol. 1

β-topological hypergroups

H. Momenaee Kermani · S. Sh. Mousavi

Received: 17.IX.2013 / Revised: 15.I.2014 / Accepted: 29.I.2014

Abstract In this paper we introduce the notion of (quasi) β-topological hypergroup as a generalization of (quasi) topological group using the relation β on hypergroups. We will study some of their properties and present some examples. The relation between β-topological hypergroups and topological hypergroups [1,6] is also discussed.

Keywords Hypergroup · β-topological hypergroup · Topological group

Mathematics Subject Classification (2010) 20N20 · 22A30

1 Introduction

In recent decades hypergroups have been used in several branches of mathematics such as geometry, graph theory, probability, etc. Moreover, they have shown to be applicable to other sciences as well (see [4]). As a generalization of groups, it is reasonable to study hypergroups from geometric point of view. There are several important hypergroups, such as the hypergroup of joining points in the plane, whose hyperoperations are somehow "continuous". This fact makes the investigation on hypergroups with a topological structure more essential. Although the study of algebraic structures which are a topological space at the same time has opened very active research areas in mathematics, in case of hyperstructures this idea has not been seriously developed. The first attempt in this direction was made in [1] where the author introduced the notion of *topological hypergroup* as a hypergroup whose hyperoperation is continuous with respect to a topology τ on H and a topology τ_* on the power set of H. Later, this theory was developed in [6]. The presented definition in [1] requires that not only the hypergroup but also its power set be a topological space. Moreover, according to this definition, a topological hypergroup whose underlying hypergroup is in fact a group, is not necessarily a topological group. These observations persuade us to look for a stronger generalization of topological groups. In this paper, using the well-known

H. Momenaee Kermani, S. Sh. Mousavi
Mathematics Department
Shahid Bahonar University of Kerman
Kerman, Iran
E-mail: momenaee@uk.ac.ir; smousavi@uk.ac.ir

relation β on a hypergroup H, we generalize the concept of topological groups to hypergroups and introduce (quasi) β-topological hypergroups. To this end, we first define the generalized inverse function as a function from H to its power set $\mathscr{P}^*(H)$ and then propose two strategies to define the continuity of the generalized inverse function requiring no topology on $\mathscr{P}^*(H)$. This, together with the natural definition of the continuity of hyperoperation, provide us with two definitions. We finally show that one of the resulting definitions is stronger than the other and we called them β-topological hypergroup and quasi β-topological hypergroup, respectively. Some properties of these structures are studied in section 4 and in section 5 we provide some examples of β-topological hypergroups. In section 6, the relation between topological hypergroups and β-topological hypergroups is studied.

Let H be a nonempty set and $\mathscr{P}^*(H)$ be the set of all nonempty subsets of H and let $\circ$ be a hyperoperation or join operation on H, that is, $\circ$ is a function from $H \times H$ into $\mathscr{P}^*(H)$. If $(a, b) \in H \times H$, its image under $\circ$ in $\mathscr{P}^*(H)$ is denoted by $a \circ b$ or ab. The join operation is extended to subsets of H in a natural way, that is $A \circ B = \cup\{a \circ b \mid a \in A, b \in B\}$. The notation aA is used for $\{a\} \circ A$ and Aa for $A \circ \{a\}$. Generally, the singleton $\{a\}$ is identified with its member a. The structure $(H, \circ)$ is called a *hypergroup* if $aH = Ha = H$ for all $a \in H$ and $a(bc) = (ab)c$ for all $a, b, c \in H$.

Throughout, we assume that $(H, \circ)$ is a hypergroup. An element a in H is called a scalar, whenever for every $x \in H$, $x \circ a$ and $a \circ x$ are single-element sets. The set of *scalars* of H is denoted by $S(H)$ and if $x \in S(H)$, then we denote $x \circ y$ by $\{x \odot y\}$ and $y \circ x$ by $\{y \odot x\}$. If H has a scalar element, then $(S(H), \odot)$ is a group (see [3]).

Let K be a nonempty subset of H that is closed under the hyperoperation in H. If K is itself a hypergroup under the hyperoperation in H, then K is said to be a *subhypergroup* of H. Suppose $(H, \circ)$ and $(H', \circ')$ are two hypergroups. A function $f : H \to H'$ is called a *homomorphism* if $f(a \circ b) \subseteq f(a) \circ' f(b)$ for all a and b in S. We say that f is a *good homomorphism* if for all a and b in H, $f(a \circ b) = f(a) \circ' f(b)$.

Let R be an equivalence relation on H. For all pairs (A, B) of non-empty subsets of H, we set $A \overset{=}{R} B \Leftrightarrow a \; R \; b, \forall a \in A, \forall b \in B$. The relation R is called *strongly regular on the left* (*on the right*) if $x \; R \; y \Rightarrow a \circ x \overset{=}{R} a \circ y$ ($x \; R \; y \Rightarrow x \circ a \overset{=}{R} y \circ a$ respectively), for all $(x, y, a) \in H^3$. Moreover, R is called *strongly regular* if it is strongly regular on the right and on the left.

If R is a strongly regular relation, it is well known that the quotient $\frac{H}{R}$ is a group under the operation $R(x) \otimes R(y) = R(z), \quad \forall z \in x \circ y$. For all $n > 1$ define the relation β_n on H as follows: $a \; \beta_n \; b \Leftrightarrow \exists (x_1, ..., x_n) \in H^n \; : \; \{a, b\} \subseteq \prod_{i=1}^{n} x_i$ and $\beta = \bigcup_{i=1}^{n} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation. It is known [3] that β is a strongly regular relation. In fact, β is the least equivalence relation on H such that the quotient $\frac{H}{\beta}$ is a group. This group is called the *fundamental group* of H. The equivalence class containing x is denoted by $\frac{x}{\beta}$ and the identity element of the group $\frac{H}{\beta}$ is denoted by $\frac{e}{\beta}$.

The nonempty subset A of H is called a *complete part* of H if for all $n \geqslant 2$ and for all $(x_1, x_2, ..., x_n) \in H^n$ the following implication holds:

$$\prod_{i=1}^{n} x_i \cap \; A \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i \subseteq A.$$

It is not hard to see that if A is a complete part of H, then A^c the complement of A in H is also a complete part. For a non-empty subset A of H, the intersection of all complete parts of H containing A is called the *complete closure* of A in H and will be denoted by $\mathscr{C}(A)$.

Let $\varphi_{_H} : H \to \frac{H}{\beta}$ be the canonical projection. $\varphi_{_H}$ is a good homomorphism. The kernel of $\varphi_{_H}$ is called the *heart* (or core) of H and is denoted by $\omega_{_H}$. Note that $\omega_{_H}$ is a complete part (see [3]).

Theorem 1.1 ([3], Theorems 66, 67) *If A is a non-empty subset of H, one has $\omega_{_H} \circ H = H \circ \omega_{_H} = \mathscr{C}(A) = \varphi_{_H}^{-1}(\varphi_{_H}(A))$.*

2 β-inverse

In this section we define the β-inverse of a subset of a hypergroup and state some of its properties which are immediate consequences of the definition.

Definition 2.1 *Suppose that $(H, \circ)$ is a hypergroup and A is a nonempty subset of H. The β-inverse of A denoted by A_β^{-1} is defined by $A_\beta^{-1} \overset{def}{:=} \{h \in H \mid h \circ a \subseteq \omega_{_H}$ for some $a \in A\}$. For $a \in A$, $\{a_\beta^{-1}\}$ is denoted by a_β^{-1}.*

It is not hard to see that if $h \circ a \subseteq \omega_{_H}$, then $a \circ h \subseteq \omega_{_H}$.

Proposition 2.2 *Suppose that $(H, \circ)$ is a hypergroup and A, B are nonempty subsets of H. Then,*

(1) *$\varphi_{_H}(A_\beta^{-1}) = (\varphi_{_H}(A))^{-1}$. Thus, $A_\beta^{-1} = \varphi_H^{-1}(\varphi_{_H}(A))^{-1})$.*

(2) *If $A \subseteq B$, then $A_\beta^{-1} \subseteq B_\beta^{-1}$.*

(3) *A_β^{-1} is a nonempty complete part of H.*

(4) *$A \subseteq (A_\beta^{-1})_\beta^{-1}$. In fact, $(A_\beta^{-1})_\beta^{-1} = \mathscr{C}(A)$.*

(5) *$A_\beta^{-1} = (\mathscr{C}(A))_\beta^{-1}$.*

(6) *If $A \circ B \subseteq \omega_{_H}$, then $B \circ A \subset \omega_{_H}$. Moreover $A \subseteq B_\beta^{-1}$ and $B \subseteq A_\beta^{-1}$.*

(7) *If $\omega_{_H} \subseteq A \circ B$, then $A \cap B_\beta^{-1} \neq \emptyset$ and $B \cap A_\beta^{-1} \neq \emptyset$.*

(8) *$(A \circ B)_\beta^{-1} = B_\beta^{-1} \circ A_\beta^{-1}$.*

Proof. (1) Let $\frac{s}{\beta} = (\frac{a}{\beta})^{-1} \in (\varphi_{_H}(A))^{-1}$ for some $a \in A$. For all $l \in s \circ a$ we have $\frac{l}{\beta} = \frac{s}{\beta}\frac{a}{\beta} = \frac{e}{\beta}$. So, $s \circ a \subseteq \omega_{_H}$ and hence $\frac{s}{\beta} = \varphi_{_H}(s) \in \varphi_{_H}(A_\beta^{-1})$. Therefore, $(\varphi_{_H}(A))^{-1} \subseteq \varphi_{_H}(A_\beta^{-1})$. Conversely, let $\frac{s}{\beta} \in \varphi_{_H}(A_\beta^{-1})$ be given. Thus there exists $t \in A_\beta^{-1}$ such that $\varphi_{_H}(s) = \varphi_{_H}(t)$. Since $t \in A_\beta^{-1}$, there exists $a \in A$ such that $t \circ a \subseteq \omega_{_H}$. Therefore, $\frac{s}{\beta} = (\frac{a}{\beta})^{-1} \in (\varphi_{_H}(A))^{-1}$ and hence $\varphi_{_H}(A_\beta^{-1}) \subseteq (\varphi_{_H}(A))^{-1}$ and the proof is complete.

(2) The proof is obvious.

(3) Follows from (1).

(4) By part (1) and Theorem 1.1, $(A_\beta^{-1})_\beta^{-1} = \varphi_H^{-1}(\varphi_{_H}(A_\beta^{-1}))^{-1}) = \varphi_{_H}^{-1}(\varphi_{_H}(A)) = \mathscr{C}(A)$.

(5) The proof follows from (3) and (4).

(6) The proof is obvious.

(7) Let $\omega_H \subseteq A \circ B$, so there exists $a \in A$ and $b \in B$ such that $a \circ b \cap \omega_H \neq \emptyset$. Since ω_H is a complete part, $a \circ b \subseteq \omega_H$. Thus $a \in B_\beta^{-1}$ and $b \in A_\beta^{-1}$ and so $A \cap B_\beta^{-1} \neq \emptyset$ and $B \cap A_\beta^{-1} \neq \emptyset$.

(8) Using part (1) we have

$$\begin{aligned}(A \circ B)_\beta^{-1} &= \varphi_H^{-1}((\varphi_H(A \circ B))^{-1}) = \varphi_H^{-1}((\varphi_H(A)\varphi_H(B))^{-1}) \\ &= \varphi_H^{-1}(\varphi_H(B)^{-1}\varphi_H(A)^{-1}) = \varphi_H^{-1}((\varphi_H(B))^{-1}(\varphi_H(A))^{-1}) \\ &= B_\beta^{-1} \circ A_\beta^{-1.}\end{aligned}$$

□

Proposition 2.3 *If H is a hypergroup and $a \in A$, then we have $a \circ a_\beta^{-1} = a_\beta^{-1} \circ a = \omega_H$.*

Proof. The proof is obvious. □

Proposition 2.4 *Let K be a subhypergroup of the hypergroup $(H, \circ)$. Then K is a complete part if and only if for each $k \in K$, we have $k_\beta^{-1} \subseteq K$.*

Proof. Let K be a complete part subhypergroup of H and $k \in K$ and $t \in k_\beta^{-1}$ be given. Thus $t \circ k \subseteq \omega_H \subseteq K$ and since K is close as hypergroup (see [3]), we have $t \in K$. Conversely, for an element $k \in K$ we have $\omega_H = k \circ k_\beta^{-1} \subseteq K$. Therefore $K \circ \omega_H = K$ and hence K is complete part. □

3 β-topological hypergroup

In this section, we pursue two approaches in order to generalize the notion of topological group. In the first approach, we use the basic definition of topological group and call the resulting structure a β-topological hypergroup of type I and in the second one we use Proposition 3.6 as an equivalent definition of topological group and call the resulting structure a β-topological hypergroup of type II. We will show that in some cases, the two definitions are equivalent and in the general case, the second definition is stronger than the first one. We also study some properties of both types of β-topological hypergroups.

As is well known [2], a *quasitopological group* consist of a group G and a topology τ on G such that for all a in G, the right action ρ_a and the left action λ_a on G are continuous mappings of the space G into itself and the inverse mapping is continuous. A *topological group* G consist of a group G and a topology τ on G such that the multiplication mapping $G \times G \to G$ and the inverse mapping $G \to G$ are continuous, when $G \times G$ is given the product topology. Any topological group is a quasitopological group but there several examples of quasitopological groups which are not topological groups.

Definition 3.1 *The hypergroup $(H, \circ)$ together with the topology τ_H on H is called a β-topological hypergroup of type I, if the following conditions hold.*

($(\mathcal{TH})_\mathcal{I} - 1$) If $x \circ y \subseteq U$ for $U \in \tau_H$ and $x, y \in H$, then there exist open sets V_x and V_y containing x and y respectively such that $V_x \circ V_y \subseteq U$;

$((\mathcal{TH})_{\mathcal{I}} - 2)$ For each $x \in H$ and $U \in \tau_{_H}$ if $x_\beta^{-1} \subseteq U$, then there exists an open complete part V such that $x \in V$ and $V_\beta^{-1} \subseteq U$.

Remark 3.1 The first condition is the definition of pseudo continuity of hyperoperation appeared in [1,6].

Lemma 3.2 *Let $(H, \circ, \tau_{_H})$ be a β-topological hypergroup of type I and let U be an open complete part. Then,*

(i) for each $x \in H$, $x \circ U$ and $U \circ x$ are open complete parts.
(ii) U_β^{-1} is an open complete part.

Proof. (i) We show that $x \circ U$ is open. To this end, let $t \in x \circ U$ be given. Since U is a complete part, we have $x_\beta^{-1} \circ t \subseteq U$. Suppose that $x_\beta^{-1} = \omega_{_H} \circ z$ for some $z \in H$. Therefore $z \circ t \subseteq U$ and hence there exist open sets V_z and V_t containing z and t respectively such that $V_z \circ V_t \subseteq U$. For each element $s \in V_t$ we have $x \circ z \circ s \subseteq x \circ U$. Since $x \circ z \subseteq \omega_{_H}$, $s \in \omega_{_H} \circ s \subseteq x \circ U$. The proof for $U \circ x$ is similar.

(ii) The proof follows immediately from the definition. □

Corollary 3.3 *Let $(H, \circ, \tau_{_H})$ be a β-topological hypergroup of type I. Then, for each open set U containing $\omega_{_H}$ there exists an open complete part V such that $\omega_{_H} \subseteq V \subseteq U$.*

Proof. We have $\omega_{_H} = w_\beta^{-1}$ for some $w \in \omega_{_H}$, so there exists an open complete part V such that $w \in V$ and $V_\beta^{-1} \subseteq U$. By Lemma 3.2, V_β^{-1} is open and contains $\omega_{_H}$. □

For the topology $\tau_{_H}$ on H we denote the quotient topology on $\frac{H}{\beta}$ by $\tau_{_H}^\beta$. It is straightforward to see that if $U \in \tau_{_H}$ is a complete part, then $\varphi_{_H}(U) \in \tau_{_H}^\beta$.

Theorem 3.4 *Suppose that $(H, \circ, \tau_{_H})$ is a β-topological hypergroup of type I. Then $(\frac{H}{\beta}, \otimes, \tau_{_H}^\beta)$ is a quasitopological group.*

Proof. The proof follows immediately from Lemma 3.2. □

A well known result of Ellis [5] says that any locally compact Haussdorf quasitopological group is in fact a topological group. So, we have the following corollary.

Corollary 3.5 *If $(H, \circ, \tau_{_H})$ is a β-topological hypergroup of type I such that the quotient space $(\frac{H}{\beta}, \otimes, \tau_{_H}^\beta)$ is a locally compact Hausdorff space, then $(\frac{H}{\beta}, \otimes, \tau_{_H}^\beta)$ is a topological group.*

Now we use the following equivalent definition of topological group in order to define β-topological hypergroups of type II.

Proposition 3.6 *Suppose that $(G, \cdot)$ is a group with identity e and $\tau_{_G}$ is a topology on G. $(G, \tau_{_G})$ is a topological group if and only if the following assertions hold.*

(1) *If $x \cdot y \in U$ for $U \in \tau_G$ and $x, y \in G$, then there exist V_x and V_y in $\tau_{_G}$ such that $x \in V_x$, $y \in V_y$ and $V_x \cdot V_y \subseteq U$;*

(2) *If $e \in U$ for $U \in \tau_{_G}$, then there exists an open set V containing e such that $V \cdot V^{-1} \subseteq U$.*

Definition 3.7 *The hypergroup $(H,\circ)$ together with the topology τ_H is called a β-topological hypergroup of type II, if the following conditions hold:*

($(\mathcal{TH})_{\mathcal{II}}-1$) If $x\circ y\subseteq U$ for $U\in\tau_H$ and $x,y\in H$, then there exist open sets V_x and V_y containing x and y respectively such that $V_x\circ V_y\subseteq U$;

($(\mathcal{TH})_{\mathcal{II}}-2$) For each $U\in\tau_H$ if $\omega_H\subseteq U$, then there exists $V\in\tau_H$ such that V is a complete part and $V\circ V_\beta^{-1}\subseteq U$.

Lemma 3.8 *Let $(H,\circ,\tau_H)$ be a β-topological hypergroup of type II and let U be an open complete part. Then:*

(i) for each $x\in H$, $x\circ U$ and $U\circ x$ are open complete parts;

(ii) U_β^{-1} is an open complete part.

Proof. (i) Similar to the proof of Lemma 3.2(i).

(ii) Let $x\in U_\beta^{-1}$ be given. By Proposition 2.2, we have $\omega_H\subseteq U\circ x$. By (i), $U\circ x$ is open and hence there exists an open complete part V containing ω_H such that $V\circ V_\beta^{-1}\subseteq U\circ x$. Thus, by Proposition 2.3 we have $V\circ(x\circ V)_\beta^{-1}\subseteq U$ and so $x\in x\circ V\subseteq U_\beta^{-1}$. □

Theorem 3.9 *Let $(H,\circ,\tau_H)$ be a β-topological hypergroup of type II. Then $(\frac{H}{\beta},\otimes,\tau_H^\beta)$ is a topological group.*

Proof. Let $\mathcal{U}\in\tau_H^\beta$ and $\varphi_H(x)\otimes\varphi_H(y)\in\mathcal{U}$ be given.

By Lemma 3.8, $(\varphi_H(x))^{-1}\otimes\mathcal{U}\otimes(\varphi_H(y))^{-1}$ is an open set in $\frac{H}{\beta}$ containing $\frac{e}{\beta}$. Thus there exists an open complete part V containing ω_H such that $V\circ V_\beta^{-1}\subseteq\varphi_H^{-1}((\varphi_H(x))^{-1}\otimes\mathcal{U}\otimes(\varphi_H(y))^{-1})$. Put $\mathcal{V}\overset{\text{def}}{:=}\varphi_H(V)$ and so $\mathcal{V}^{-1}=\varphi_H(V_\beta^{-1})$. Therefore, $\varphi_H(x)\in\varphi_H(x)\otimes\mathcal{V}\in\tau_H^\beta$ and $\varphi_H(y)\in\mathcal{V}^{-1}\otimes\varphi_H(y)\in\tau_H^\beta$, since $\mathcal{V}$ is open. Hence $(\varphi_H(x)\otimes\mathcal{V})\otimes(\mathcal{V}^{-1}\otimes\varphi_H(y))\subseteq\mathcal{U}$. The continuity of the inverse opration in the group $\frac{H}{\beta}$ follows from Lemma 3.8. □

Corollary 3.10 *Let $(H,\circ,\tau_H)$ be a β-topological hypergroup of type II. If $V\in\tau_H$ is a complete part and $x\circ y\subseteq V$ for some $x,y\in H$, then there exists open complete parts V_1, V_2 containing x, y respectively such that $V_1\circ V_2\subseteq V$.*

Theorem 3.11 *If $(H,\circ,\tau_H)$ is a β-topological hypergroup of type II, then it is a β-topological hypergroup of type I.*

Proof. Let $U\in\tau_H$ and $x_\beta^{-1}\subseteq U$ be given. There exists $z\in H$ such that $x_\beta^{-1}=\omega_H\circ z$. So by $((\mathcal{TH})_{\mathcal{II}}-1)$ there exists an open U' containing ω_H such that $U'\circ z\subseteq U$. Now, there exists an open complete part W containing ω_H such that $W\circ W_\beta^{-1}\subseteq U'$. Let $V=z_\beta^{-1}\circ W\circ W_\beta^{-1}$, then V is an open complete part containing x such that $V_\beta^{-1}\subseteq U$. □

So far we have observed that the fundamental group of a β-topological hypergroup of type I is a quasitopological group and the fundamental group of a β-topological hypergroup of type II is a topological group. This fact together with previous theorem justify the following definition.

Definition 3.12 *Let $(H,\circ)$ be a hypergroup and let $\tau_{_H}$ be a topology on H.*

(i) H is called a quasi β-topological hypergroup, whenever it is a β-topological hypergroup of type I;

(ii) H is called a β-topological hypergroup, whenever it is a β-topological hypergroup of type II.

Theorem 3.13 *Let $(H,\circ,\tau_{_H})$ be a quasi β-toplogical hypergroup, then $(H,\circ,\tau_{_H})$ is a β-toplogical hypergroup if and only if for each open set U containing $\omega_{_H}$ there exists open sets V and W containing $\omega_{_H}$ such that $V\circ W\subseteq U$.*

Proof. For the if part suppose that U is open and contain $\omega_{_H}$. By the assumption and Corollary 3.3, there exist open complete parts V and W containing $\omega_{_H}$ such that $V\circ W\subseteq U$. Put $O\overset{\text{def}}{:=}V\cap W\cap V_\beta^{-1}\cap W_\beta^{-1}$. Thus O is an open complete part and $O\circ O_\beta^{-1}\subseteq U$. The converse is obvious. □

Theorem 3.14 *If $(H,\circ,\tau_{_H})$ is a quasi β-toplogical hypergroup and $\omega_{_H}$ is compact, then it is a β-toplogical hypergroup.*

Proof. Suppose that $\omega_{_H}\subseteq U$ and U is an open. Let $z\in\omega_{_H}$ be an arbitrary element, thus for each $t\in\omega_{_H}$ there exist open sets V_t and W_t containing z and t respectively, such that $V_t\circ W_t\subseteq U$. Since $\omega_{_H}\subseteq\bigcup_{t\in\omega_{_H}}W_t$ and $\omega_{_H}$ is a compact, we have $\omega_{_H}\subseteq\bigcup_{i=1}^{n}W_{t_i}$ for some $t_1,\cdots,t_n\in\omega_{_H}$. Put $U_z\overset{\text{def}}{:=}\bigcap_{i=1}^{n}V_{t_i}$ and $W_z\overset{\text{def}}{:=}\bigcup_{i=1}^{n}W_{t_i}$. So $z\in U_z$, $\omega_{_H}\subseteq W_z$ and $U_z\circ W_z\subseteq U$. Similarly since $\omega_{_H}\subseteq\bigcup_{z\in\omega_{_H}}U_z$, we have $\omega_{_H}\subseteq\bigcup_{j=1}^{m}U_{z_j}$ for some $z_1,\cdots,z_m\in\omega_{_H}$. Put $V\overset{\text{def}}{:=}\bigcup_{j=1}^{m}U_{z_j}$ and $W\overset{\text{def}}{:=}\bigcap_{j=1}^{m}W_{z_j}$. Therefore, V and W contain $\omega_{_H}$ and $V\circ W\subseteq U$. Hence by Theorem 3.13, $(H,\circ,\tau_{_H})$ is a β-topological hypergroup. □

The authors conjecture that there exists a quasi β-topological hypergroup which is not a β-topological hypergroup.

Notation 3.1 Let $(H,\circ)$ be a hypergroup. We define the collection τ_c^H as follows $\tau_c^H=\{\emptyset\}\cup\{U\subseteq H\mid U \text{ is complete part}\}$.

It is easy to see that (H,τ_c^H) is a topological space. We have the following proposition.

Proposition 3.15 *Let $(H,\circ,\tau_{_H})$ be a β-topological hypergroup. Then $(S(H),\odot)$ is a topological group by the subspace topology induced by the topology $\tau_{_H}\cap\tau_c^H$ on H.*

Proof. The continuity of the operation $\odot$ is obvious. Let $U\cap S(H)$ where $U\in\tau_{_H}\cap\tau_c^H$ be an open set in the topology of $S(H)$ containing $e_{_{S(H)}}$, the identity element of the group $S(H)$. Since $\omega_{_H}\subseteq U$, there exists an open complete part V containing $\omega_{_H}$ such that $V\circ V_\beta^{-1}\subseteq U$. Since $e_{_{S(H)}}\in\omega_{_H}$ and $(V\cap S(H))^{-1}\subseteq V_\beta^{-1}$, where $(V\cap S(H))^{-1}$ is the inverse of the set $V\cap S(H)$ in the group $S(H)$, we have $e_{_{S(H)}}\in V\cap S(H)$ and $(V\cap S(H))\odot(V\cap S(H))^{-1}\subseteq V\circ V_\beta^{-1}\cap S(H)\subseteq U\cap S(H)$. □

Corollary 3.16 *Suppose that $(H,\circ,\tau_{_H})$ is a β-topological hypergroup. If H is a group, then $(H,\odot,\tau_{_H})$ is a topological group.*

Proposition 3.17 *Suppose that $(H, \circ, \tau_H)$ is a β-topological hypergroup. Then $(\frac{H}{\beta}, \tau_H^\beta)$ is Hausdorff if and only if for all $x \notin \omega_H$ there exists an open complete part U containing x such that $U \cap \omega_H = \emptyset$.*

Proof. First suppose that $\frac{H}{\beta}$ is Hausdorff and $x \notin \omega_H$. Let $\varphi_H(y) = (\varphi_H(x))^{-1}$. Since $\varphi_H(x) \neq e$, there exist open sets $\mathcal{W}, \mathcal{U}$ and $\mathcal{V}$ containing $e_{\frac{H}{\beta}}, \varphi_H(x)$ and $\varphi_H(y)$ respectively, such that $\mathcal{U}\mathcal{V} \subseteq \mathcal{W}$ and $\mathcal{U} \cap \mathcal{W} = \emptyset$. Therefore $x \in \varphi_H^{-1}(\mathcal{U})$, $\omega_H \subseteq \varphi_H^{-1}(\mathcal{W})$ and $\varphi_H^{-1}(\mathcal{U}) \cap \varphi_H^{-1}(\mathcal{W}) = \emptyset$. For the converse, suppose that $\varphi_H(x) \neq \varphi_H(y)$. Thus $\varphi_H(x)\varphi_H(y') \neq e_{\frac{H}{\beta}}$, where $\varphi_H(y') = [\varphi_H(y)]^{-1}$ and hence $x \circ y' \cap \omega_H = \emptyset$. So for all $t \in x \circ y'$ there exists an open complete part U_t containing t such that $U_t \cap \omega_H = \emptyset$. Put $U = \bigcup_{t \in x \circ y'} U_t$ and hence $x \circ y' \subseteq U$ and $U \cap \omega_H = \emptyset$. Since U is an open complete part, by Corollary 3.10 there exist open complete parts U_x and $U_{y'}$ containing x and y' respectively, such that $U_x \circ U_{y'} \subseteq U$. Therefore $[\varphi_H(U_{y'})]^{-1}$ is open containing $\varphi_H(y)$ and $\varphi_H(U_x) \cap [\varphi_H(U_{y'})]^{-1} = \emptyset$. $\square$

Corollary 3.18 *Suppose that $(H, \circ, \tau_H)$ is a β-topological hypergroup. If $\frac{H}{\beta}$ is Hausdorff, then ω_H is close.*

Proposition 3.19 *For the β-topological hypergroup $(H, \circ, \tau_H)$ the following assertions are equivalent:*
(a) *φ_H is an open map;*
(b) *For each open set U, we have (U) is open;*
(c) *The inverse of each open set is open;*
(d) *For each open set U and $x \in H$ if $x_\beta^{-1} \cap U \neq \emptyset$, then there exists an open complete part V containing x such that $V_\beta^{-1} \subseteq (U)$.*

Proof. The proof is straightforward. $\square$

Proposition 3.20 *Suppose that $(H, \circ, \tau_H)$ is a β-topological hypergroup, φ_H is an open map and K is a subhypergroup of H. If K is a complete part, then $\overline{K}$, the closure of K, is a subhypergroup of H.*

Proof. First we show that $(\overline{K}, \circ)$ is closed under the hyperoperation. Suppose that $x, y \in \overline{K}$, $t \in x \circ y$ and $U \in \tau_H$ such that $t \in U$ are given. By Proposition 3.19, (U) is open and $x \circ y \subseteq (U)$. Now, there exist open complete parts V_x and V_y containing x and y respectively, such that $V_x \circ V_y \subseteq (U)$. Therefore there exist $k_1 \in V_x \cap K$ and $k_2 \in V_y \cap K$ and hence $k_1 \circ k_2 \subseteq (U)$. Let $k \in k_1 \circ k_2$ be given. So there exists $u \in U$ such that $\varphi_H(k) = \varphi_H(u)$. Therefore, $u \in (K) = K$ and hence $U \cap K \neq \emptyset$. Thus, $(\overline{K}, \circ)$ is closed under the hyperoperation.

Now we show that for each $a \in \overline{K}$, we have $a \circ \overline{K} = \overline{K}$. Let $x \in \overline{K}$ be given. There exists $h \in H$ such that $x \in a \circ h$. Let $O \in \tau_H$ and $h \in O$. Thus, $x \in a \circ (O)$. By Lemma 3.8, there exists $k \in K \cap (a \circ (O))$. So $a_\beta^{-1} \circ k \subseteq (O)$. Hence there exists an open complete part V containing a such that $V_\beta^{-1} \circ k \subseteq (O)$. Therefore, $K \cap V \neq \emptyset$ and by Proposition 2.4 we have $V_\beta^{-1} \cap K \neq \emptyset$. Thus, $(V_\beta^{-1} \cap K) \circ k \subseteq (O)$ which means that $K \cap O \neq \emptyset$. $\square$

4 Examples

Example 4.1 Define the hyperoperation $\circ$ on $H = \mathbb{R}^2$ by $(x_1, y_1) \circ (x_2, y_2) = [x_1, x_2] \times [y_1, y_2]$. Therefore $\omega_{_H} = H$ and $\varphi_{_H}$ is an open map. It is easy to see that $(H, \circ, \tau_{_H})$ is a β-topological hypergroup, where τ_H is the standard topology on H.

Example 4.2 Suppose that $(G, \cdot)$ is an abelian group, H is a set containing G and $f : H \twoheadrightarrow G$ is a surjection. Define the hyperoperation $\circ$ on H by $h_1 \circ h_2 \overset{\text{def}}{:=} f^{-1}(f(h_1) \cdot f(h_2))$. It is easy to see that $(H, \circ)$ is a hypergroup and for all $h \in H$ we have $\frac{h}{\beta} = \{h' \mid f(h) = f(h')\} = f^{-1}(f(h))$. Also, $\forall a \in \omega_{_H}, \forall h \in H, \frac{a}{\beta} \otimes \frac{h}{\beta} = \frac{h}{\beta} \iff \forall a \in \omega_{_H}, \forall h \in H, \forall t \in a \circ h, \frac{t}{\beta} = \frac{h}{\beta} \iff \forall a \in \omega_{_H}, f(a) = e_G$. Therefore $\omega_{_H} = f^{-1}(e_G)$, where e_G is the identity element of the group G and hence for all $h \in H$, $h_\beta^{-1} = f^{-1}((f(a))^{-1})$, where $(f(a))^{-1}$ is the inverse of $f(a)$ in the group G. Now let G be the group $(\mathbb{Q}, +)$, $H = \mathbb{R}$ and $f : \mathbb{R} \twoheadrightarrow \mathbb{Q}$ be the function defined by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \in C, \end{cases}$$

where $C = \mathbb{R} - \mathbb{Q}$ is the irrational numbers. Consider the hypergroup $(H, \circ)$ as above. Thus $\omega_{_H} = \{0\} \cup C$ and for all $a \in H$,

$$a_\beta^{-1} = \begin{cases} \{-a\}, & \text{if } a \in \mathbb{Q} - \{0\}; \\ \{0\} \cup C, & \text{if } a \in \{0\} \cup C. \end{cases}$$

Define $\tau_{_H} \overset{\text{def}}{:=} \{\emptyset, H\} \cup \{U \mid U \subseteq C\}$, so $(H, \circ, \tau_{_H}$ is a β-topological hypergroup and $\varphi_{_H}$ is not an open map, since for each $U \subseteq C$ we have $\varphi_{_H}^{-1}(\varphi_{_H}(U)) = \{0\} \cup C \notin \tau_{_H}$.

Example 4.3 Put $H = \mathbb{R}^2 - \{(0, y) \mid y \in \mathbb{R}\}$. Define the hyperoperation $\circ$ on H by $\forall (x_1, y_1), (x_2, y_2) \in H,\ \ (x_1, y_1) \circ (x_2, y_2) \overset{\text{def}}{:=} \{(x_1 x_2, z) \mid z \in \mathbb{R}\}$. $(H, \circ)$ is a hypergroup, $\omega_{_H} = \{(1, y) \mid y \in \mathbb{R}\}$, $\underset{\beta}{(}(x, y)) = \{(x, z) \mid z \in \mathbb{R}\}$ and $(x, y)_\beta^{-1} = \{(1/x, z) \mid z \in \mathbb{R}\}$. Consider $\tau_{_H}$ as the subspace topology of the standard topology of $\mathbb{R}^2$, so it is easy to see that $(H, \circ, \tau_{_H})$ is a β-topological hypergroup. Note that $\frac{H}{\beta} \cong (\mathbb{R}^*, \cdot)$ the multiplicative group of the non-zero elements of $\mathbb{R}$ and $\varphi_{_H}$ is an open mapping.

Example 4.4 Let $(G, +)$ be an abelian group and suppose that $a \in G$ have order 2. Define the hyperoperation $\circ$ on G by $x \circ y = \{x + y, x + y + a\}$. Thus, $(G, \circ)$ is a (complete) hypergroup and $\omega_{_H} = \{g \in G \mid \forall x \in G, x \in x \circ g\} = \{0, a\}$, where 0 is the identity element of the group G. Therefore for each $g \in G$, $\underset{\beta}{(}g) = \{g, g + a\}$ and $g_\beta^{-1} = \{-g, a - g\}$. Now let G be the group $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ and $a = \bar{2}$. So the table of hyperoperation $\circ$ on $\mathbb{Z}_4$ defined above is as follows:

$\circ$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}, \bar{2}$	$\bar{1}, \bar{3}$	$\bar{0}, \bar{2}$	$\bar{1}, \bar{3}$
$\bar{1}$	$\bar{1}, \bar{3}$	$\bar{0}, \bar{2}$	$\bar{1}, \bar{3}$	$\bar{0}, \bar{2}$
$\bar{2}$	$\bar{0}, \bar{2}$	$\bar{1}, \bar{3}$	$\bar{0}, \bar{2}$	$\bar{1}, \bar{3}$
$\bar{3}$	$\bar{1}, \bar{3}$	$\bar{0}, \bar{2}$	$\bar{1}, \bar{3}$	$\bar{0}, \bar{2}$

Therefore, $\omega_H = \{\bar{0}, \bar{2}\} = \bar{0}_\beta^{-1} = \bar{2}_\beta^{-1}$, $\bar{1}_\beta^{-1} = \{\bar{1}, \bar{3}\}$ and $\bar{3}_\beta^{-1} = \{\bar{1}, \bar{3}\}$. Let $\tau_H \overset{\text{def}}{:=} \{\emptyset, \{\bar{3}\}, H\}$. Thus, $(H, \circ, \tau_H)$ is a β-topological group.

Example 4.5 Suppose that $(G, \cdot, \tau_G)$ is a topological group and $(H, \circ)$ is a hypergroup. For all $g \in G$, put $H_g = H \times \{g\}$ and define the hypergroup $(K, \circ)$ where $K = \bigcup_{g \in G} H_g$ and $k_1 \circ k_2 = \{(h, g_1 \cdot g_2) \mid h \in h_1 \circ h_2\}$ for all $k_1 = (h_1, g_1)$ and $k_2 = (h_2, g_2)$ in K. It is easy to see that $\omega_K = H_e$ where $e \in G$ is the identity element of G, for each $(h, g) \in K$ we have $(h, g)_\beta^{-1} = H \times \{g^{-1}\}$, where g^{-1} is the inverse element of g in group G and $\frac{K}{\beta} \cong G$.

Define the topology τ_K on K as follows $\mathcal{U} \in \tau_K$ if and only if $\mathcal{U} = \bigcup_{g \in U} H_g$ for some $U \in \tau_G$. Let $(h, g) \in K$ and $\mathcal{U} \in \tau_K$ such that $(h, g)_\beta^{-1} \subseteq \mathcal{U}$ be given. Thus $\mathcal{U} = \bigcup_{g \in U} H_g$, where $U \in \tau_G$. So $H \times \{g^{-1}\} \subseteq \mathcal{U}$. Therefore, $g^{-1} \in U$ and since G is a topological group hence there exists $V \in \tau_G$ containing g such that $V^{-1} \subseteq U$, where V^{-1} is the inverse of V in the group G. Put $\mathcal{V} \overset{\text{def}}{:=} \bigcup_{g \in V} H_g$. Thus, $\mathcal{V}$ is a complete and open set containing (h, g) and $\mathcal{V}_\beta^{-1} = \bigcup_{g \in V^{-1}} H_g \subseteq \bigcup_{g \in U} H_g = \mathcal{U}$. Therefore, $(K, \circ, \tau_K)$ is a β-topological hypergroup and φ_K is an open map.

As a special case of example 4.5 we introduce two β-topological hypergroup as follows.

Example 4.6 Let a topological group $(G, \cdot, \tau)$ together with a normal closed subgroup H be given. Define the set $\mathcal{X}$ as follows $\mathcal{X} \overset{\text{def}}{:=} \{U \cdot H \mid U \in \tau\}$. It is easy to see that $\cup\mathcal{X} = G$ and for all $U_1 \cdot H$ and $U_2 \cdot H$ in $\mathcal{X}$ we have $(U_1 \cap U_2) \cdot H \subseteq U_1 \cdot H \cap U_2 \cdot H$. Let τ_G be the topology on G generated by $\mathcal{X}$. The mapping $* : G \times G \to \mathscr{P}^*(H)$ defined by $x \circ y \overset{\text{def}}{:=} xyH$ is a hyperoperation on G and $(G, \circ)$ is a hypergroup. In this case $\omega_G = H$ and hence $\frac{G}{\beta} = \frac{G}{H}$. The topology τ_G^β on $\frac{G}{\beta}$ is exactly the quotient topology on $\frac{G}{H}$ induced by canonical projection $\pi_G : G \to \frac{G}{H}$. Note that for all $x \in G$, $\varphi_G(x) = \pi_G(x)$. We can easily see that $(G, \circ, \tau_G)$ is a β-topological hypergroup.

Example 4.7 Suppose that $\{A_g\}_{g \in G}$ is a family of mutually disjoin sets where G is a topological group with topology τ_G and $H = \bigcup_{g \in G} A_g$. Define the hyperoperation $\circ$ on H by $x \circ y \overset{\text{def}}{:=} A_{st}$, where $x \in A_s$ and $y \in A_t$. It is easy to see that $(H, \circ)$ is a hypergroup with $\omega_H = A_e$, where e is the identity element of G and $\frac{H}{\beta} \cong G$. Define the topology τ_H on H by $U \in \tau_H \Leftrightarrow U = A_V$ where $V \in \tau_G$, $A_V = \bigcup_{v \in V} A_v$. It is easy to see that H is a β-topological hypergroup.

Example 4.8 Suppose that $H = \{e, a, b\}$. Consider the hypergroup $(H, \circ)$, where $\circ$ is defined on H as follows:

$\circ$	e	a	b
e	e	a	b
a	a	$\{e, b\}$	a
b	b	a	$\{e, b\}$

In fact $(H, \circ)$ is a polygroup. Define $\tau_{_H} \overset{\text{def}}{:=} \{\emptyset, \{e\}, H\}$. So, $\omega_{_H} = \{e, b\} = e_\beta^{-1} = b_\beta^{-1}$ and $a_\beta^{-1} = \{a\}$. It is easy to see that $(H, \circ, \tau_{_H})$ is a β-topological hypergroup. Note that $\varphi_{_H}$ is not an open map.

5 Relation between quasi β-topological hypergroup and topological hypergroup

In this section we show that the condition $(\mathcal{TH})_{\mathcal{I}} - 2$ is in association with the continuity of the map $i : H \to \mathscr{P}^*(H)$ which is defined by $i(x) = x_\beta^{-1}$ for each $x \in H$.

Let $(H, \tau_{_H})$ be a topological space. In [6] it was mentioned that the family $\mathscr{U}$ consisting of all sets $S_U = \{V \in \mathscr{P}^*(H) \mid V \subseteq U\}, U \in \tau_{_H}$, is a base of topology $\tau_{\mathscr{U}}$ on $\mathscr{P}^*(H)$. The topology $\tau_{\mathscr{U}}$ on $\mathscr{P}^*(H)$ is called the *upper topology* on $\mathscr{P}^*(H)$ induced by the topology $\tau_{_H}$ on H. Recall that the hypergroup H is called a topological hypergroup with respect to a topology $\tau_{_H}$ on H and a topology τ_* on $\mathscr{P}^*(H)$, whenever the hyperoperation is continuous with respect to $\tau_{_H} \times \tau_{_H}$ and τ_*. Now, we have the following theorem.

Theorem 5.1 *Let $(H, \circ)$ be a hypergroup and $(H, \tau_{_H})$ be a topological space. Then H is a quasi β-topological hypergroup if and only if H is a topological hypergroup and i is continuous with respect to $\tau_{_H}$ and $\tau_{\mathscr{U}}$.*

Proof. Let $U \in \tau_{_H}$ and $x \in H$ such that $x_\beta^{-1} \subseteq U$ be given. It is easy to see that $i^{-1}(S_U) = \{x \in H \mid x_\beta^{-1} \subseteq U\}$ is complete part. Since i is continuous, $i^{-1}(S_U)$ is open and we have $(i^{-1}(S_U))_\beta^{-1} \subseteq U$. The converse is obvious. $\square$

References

1. Ameri, R. – *Topological (transposition) hypergroups*, Ital. J. Pure Appl. Math., 13 (2003), 171–176.
2. Arhangel'skii, A.; Tkachenko, M. – *Topological Groups and Related Structures*, Atlantis Studies in Mathematics, 1, Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.
3. Corsini, P. – *Prolegomena of hypergroup theory*, Supplement to Riv. Mat. Pura Appl. Aviani Editore, Tricesimo, 1993.
4. Corsini, P.; Leoreanu, V. – *Applications of Hyperstructure Theory*, Advances in Mathematics (Dordrecht), 5, Kluwer Academic Publishers, Dordrecht, 2003.
5. Ellis, R. – *Locally compact transformation groups*, Duke Math. J., 24 (1957), 119–125.
6. Hošková, Š. – *Topological hypergroupoids*, Comput. Math. Appl., 64 (2012), 2845–2849.
7. Marty, F. – *Sur uni generalization de la notion de group*, 8th Congress Math. Scandenaves, Stockholm, 1934, 45–49.
8. Montgomery, D.; Zippin, L. – *Topological transformation groups*, Interscience Publishers, New York–London, 1955