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The graded primary radical of a graded submodule

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Abstract Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we define the graded primary radical of a graded submodule and give a number of its properties.

Keywords Graded primary submodule · Graded primary radical

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1 Introduction

The concept of the graded radical of a graded submodule of a graded module over graded commutative ring has been introduced and studied by various authors, (see, for example [2,4,6]). Here we introduce the concept of the graded primary radical of a graded submodule over graded commutative ring and give a number of its properties. We also introduce the concept of graded primary radical submodule and review some results about it.

Before we state some results, let us introduce some notations and terminologies. Let G be a group with identity e and R be a commutative ring. Then R is a G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$, for all $g, h \in G$. We denote this by (R, G). The elements of R_g are called homogeneous of degree g where R_g are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \bigcup_{g\in G} R_g$. Let I be an ideal of R. Then I is called graded ideal of (R, G) if $I = \bigoplus_{g\in G}(I \cap R_g)$. Thus, if $x \in I$, then $x = \sum_{g\in G} x_g$ with $x_g \in I$. An ideal of a G-graded ring need not be G-graded. For simplicity, we will denote the graded ring (R, G) by R. Let R be a G-graded ring and M an R-module. We say that M is a G-graded R-module (or graded R-module) if there exists a family of subgroups

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$\{M_g\}_{g\in G}$ of M such that $M = \bigoplus_{g\in G} M_g$ (as abelian groups) and $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. Here, R_gM_h denotes the additive subgroup of M consisting of all finite sums of elements r_gs_h with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g\in G} M_g$ and the elements of $h(M)$ are called homogeneous. Let $M = \bigoplus_{g\in G} M_g$ be a graded R-module and N a submodule of M. Then N is called a graded submodule of M if $N = \bigoplus_{g\in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the g-component of N. For more details, one can look in [3].

Let R be a G-graded ring and M a graded R-module. The graded radical of a graded ideal I, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$. A proper graded ideal P of R is said to be graded prime ideal if whenever $r, s \in h(R)$ with $rs \in P$, then either $r \in P$ or $s \in P$. A proper graded ideal P of R is said to be graded primary ideal if whenever $r, s \in h(R)$ with $rs \in P$, then either $r \in P$ or $s \in Gr(P)$ (see [5]). A proper graded submodule N of a graded R-module M is said to be graded prime submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $r \in (N :_R M) = \{r \in R : rM \subseteq N\}$ or $m \in N$ (see [1,2]). A proper graded submodule N of a graded R-module M is said to be graded primary submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $m \in N$ or $r \in Gr((N :_R M))$ (see [4]). The graded radical of a graded submodule N of a graded R-module M, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M, then $Gr_M(N) = M$ (see [2,4,6]). A graded R-module M is said to be graded finitely generated if there exist $x_{g_1}, x_{g_2}, ..., x_{g_n} \in h(M)$ such that $M = Rx_{g_1} + \cdots + Rx_{g_n}$.

2 The results

The following Lemma is known, but we write it here for the sake of references.

Lemma 2.1 *Let R be a G-graded ring and M a graded R-module. Then the following hold:*

(*i*) *If N is a graded submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded submodules of M.*
(*ii*) *If N and K are graded submodules of M, then $N + K$ and $N \cap K$ are also graded submodules of M and $(N :_R M)$ is a graded ideal of R.*
(*iii*) *Let $\{N_\lambda\}$ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodules of M.*
(*iv*) *A proper graded ideal P of R is a graded prime if and only if whenever J_1, J_2 are graded ideals of R with $J_1J_2 \subseteq P$, either $J_1 \subseteq P$ or $J_2 \subseteq P$.*

Definition 2.2 *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M.*

(*i*) *The graded primary radical of N in M denoted by $P\text{-}Gr_M(N)$ and is defined to be the intersection of all graded primary submodules of M containing N. Should there be no graded primary submodule of M containing N, then we put $P\text{-}Gr_M(N) = M$. By Lemma 2.1, it is easy to see that $P\text{-}Gr_M(N)$ is a graded submodule of M containing*

N. On the other hand if $M = R$ is a graded M-module and N is a graded ideal of R, it is clear that N is a graded submodule of M. Denote $P\text{-}Gr_R(N)$ the graded primary radical of N when $R = M$.

(ii) We say N is a graded primary radical submodule if $P\text{-}Gr_M(N) = N$.

Since every proper graded submodule of a graded finitely generated module is contained in a graded prime, see [6, Corollary 2.11] and every graded prime submodule is graded primary, we can conclude the following Lemma.

Lemma 2.3 *Let R be a G-graded ring, M a graded finitely generated R-module. Then every proper graded submodule of M is contained in a graded primary submodule of M.*

Theorem 2.4 *Let R be a G-graded ring, M a graded R-module and N, K graded submodules of M. Then the following hold:*

(i) $N \subseteq P\text{-}Gr_M(N)$.
(ii) *If* $N \subseteq K$, *then* $P\text{-}Gr_M(N) \subseteq P\text{-}Gr_M(K)$.
(iii) $P\text{-}Gr_M(N \cap K) \subseteq P\text{-}Gr_M(N) \cap P\text{-}Gr_M(K)$.
(iv) $P\text{-}Gr_M(P\text{-}Gr_M(N)) = P\text{-}Gr_M(N)$.
(v) $P\text{-}Gr_M(N + K) = P\text{-}Gr_M(P\text{-}Gr_M(N) + P\text{-}Gr_M(K))$.
(vi) *If* $N = M$, *then* $P\text{-}Gr_M(N) = M$. *Moreover, if M is graded finitely generated, then* $P\text{-}Gr_M(N) = M$ *if and only if* $N = M$.

Proof. (i) It is clear.

(ii) Suppose that $N \subseteq K$ and let P be a graded primary submodule of M with $K \subseteq P$, it follows that $N \subseteq P$. Hence $P\text{-}Gr_M(N) \subseteq P\text{-}Gr_M(K)$.

(iii) By (ii), we have $P\text{-}Gr_M(N \cap K) \subseteq P\text{-}Gr_M(N)$ and P - $Gr_M(N \cap K) \subseteq P\text{-}Gr_M(K)$. Thus $P\text{-}Gr_M(N \cap K) \subseteq P\text{-}Gr_M(N) \cap P\text{-}Gr_M(K)$.

(iv) By (i) and (ii), we conclude that $P\text{-}Gr_M(N) \subseteq P\text{-}Gr_M(P\text{-}Gr_M(N))$. Now, let P be a graded primary submodule of M such that $N \subseteq P$. Then by definition of $P\text{-}Gr_M(N)$, $P\text{-}Gr_M(N) \subseteq P$. Hence $P\text{-}Gr_M(P\text{-}Gr_M(N)) \subseteq P\text{-}Gr_M(N)$. Thus $P\text{-}Gr_M(P\text{-}Gr_M(N)) = P\text{-}Gr_M(N)$.

(v) By (i), $N \subseteq P\text{-}Gr_M(N)$ and $K \subseteq P\text{-}Gr_M(K)$. So $N + K \subseteq P\text{-}Gr_M(N) + P\text{-}Gr_M(K)$. By (ii), we conclude that $P\text{-}Gr_M(N + K) \subseteq P\text{-}Gr_M(P\text{-}Gr_M(N) + P\text{-}Gr_M(K))$. Since N, $K \subseteq N + K$, we have $P\text{-}Gr_M(N)$, $P\text{-}Gr_M(K) \subseteq P\text{-}Gr_M(N + K)$. So $P\text{-}Gr_M(N)+ P\text{-}Gr_M(K) \subseteq P\text{-}Gr_M(N + K)$. By (ii) and (iv), we have $P\text{-}Gr_M(P\text{-}Gr_M(N)+P\text{-}Gr_M(K)) \subseteq P\text{-}Gr_M(N+K)$. Thus $P\text{-}Gr_M(N+K) = P\text{-}Gr_M(P\text{-}Gr_M(N) + P\text{-}Gr_M(K))$.

(vi) Suppose that $N = M$, So $P\text{-}Gr_M(N) = P\text{-}Gr_M(M) = M$. Now, let M be a graded finitely generated and $P\text{-}Gr_M(N) = M$. Suppose to the contrary that $N \neq M$. By Lemma 2.3, $N \subseteq P$ for some graded primary submodule P of M. Therefore $P\text{-}Gr_M(N) \neq M$, which is a contradiction. □

Theorem 2.5 *Let R be a G-graded ring, M a graded R-module and N, K graded submodules of M such that whenever $N \cap K \subseteq P$, we have either $N \subseteq P$ or $K \subseteq P$ for any graded primary submodule P of M. Then $P\text{-}Gr_M(N \cap K) = P\text{-}Gr_M(N) \cap P\text{-}Gr_M(K)$.*

Proof. If $P\text{-}Gr_M(N \cap K) = M$, then clearly $P\text{-}Gr_M(N) = P\text{-}Gr_M(K) = M$ and hence $P\text{-}Gr_M(N \cap K) = P\text{-}Gr_M(N) \cap P\text{-}Gr_M(K)$. So we can assume that $P\text{-}Gr_M(N \cap K) \neq$

M. Then there exists a graded primary submodule P of M such that $N \cap K \subseteq P$. By hypothesis, either $N \subseteq P$ or $K \subseteq P$ and hence either $P\text{-}Gr_M(N) \subseteq P$ or $P\text{-}Gr_M(K) \subseteq P$. Since this is true for all graded primary submodule containing $N \cap K$, we conclude that $P\text{-}Gr_M(N) \cap P\text{-}Gr_M(K) \subseteq P\text{-}Gr_M(N \cap K)$. Now, from (iii) of Theorem 2.4, we have $P\text{-}Gr_M(N \cap K) \subseteq P\text{-}Gr_M(N) \cap P\text{-}Gr_M(K)$. Thus $P\text{-}Gr_M(N \cap K) = P\text{-}Gr_M(N) \cap P\text{-}Gr_M(K)$. □

We can generalize Theorem 2.5 as follows.

Theorem 2.6 *Let R be a G-graded ring, M a graded R-module and $N_1, \ldots, N_s$ a graded submodules of M such that whenever $\bigcap_{j=1}^{s} N_j \subseteq P$, we have $N_j \subseteq P$ for some $j = 1, 2, \ldots, s$, for any graded primary submodule P of M. Then $P\text{-}Gr_M(\bigcap_{j=1}^{s} N_j) = \bigcap_{j=1}^{s} P - Gr_M(N_j)$.*

By combining [2, Proposition 2.5(i)] and [5, Lemma 1.8], we have the following Lemma.

Lemma 2.7 *Let R be a G-graded ring, M a graded R-module and N a graded submodules of M. If N is a graded primary submodule of M, then $Gr((N :_R M))$ is a graded prime ideal of R.*

Theorem 2.8 *Let R be a G-graded ring, M a graded R-module and $N_1, \ldots, N_s$ graded submodules of M. If P is a graded primary submodule of M with $\bigcap_{j=1}^{s} N_j \subseteq P$, then either $N_j = P$ or $(N_j :_R M) \subseteq Gr((P :_R M))$ for some j.*

Proof. Assume the contrary. Then there exists an $n_1 \in N_1 \cap h(M) - P$ and an $t_j \in (N_j :_R M) \cap h(R) - Gr((P :_R M))$ for every $j \neq 1$. Hence $t_j n_1 \in N_j \cap N_1$ for every $j \neq 1$. So $t_2 t_3 \cdots t_s n_1 \in \bigcap_{j=1}^{s} N_j \subseteq P$. Since P is a graded primary submodule of M and $n_1 \notin P$, we have $t_2 t_3 \cdots t_s \in Gr((P :_R M))$. By Lemma 2.7, $Gr((P :_R M))$ is a graded prime ideal of R. So $t_j \in Gr((P :_R M))$ for some $2 \leq j \leq s$, which is a contradiction. □

By combining Theorem 2.6 and Theorem 2.8 we have the following Corollary.

Corollary 2.9 *Let R be a G-graded ring, M a graded R-module and $N_1, \ldots, N_s$ graded submodules of M such that whenever $\bigcap_{j=1}^{s} N_j \subseteq P$, we have $(N_j :_R M) \nsubseteq Gr((P :_R M))$ for every $j = 1, 2, \ldots, s$, for every graded primary submodule P of M. Then $P\text{-}Gr_M(\bigcap_{j=1}^{s} N_j) = \bigcap_{j=1}^{s} P - Gr_M(N_j)$.*

Theorem 2.10 *Let R be a G-graded ring, M a graded R-module and N, K graded submodules of M. If $Gr((N :_R M)) + Gr((K :_R M)) = R$, then $P\text{-}Gr_M(N \cap K) = P\text{-}Gr_M(N) \cap P\text{-}Gr_M(K)$.*

Proof. Let P be a graded primary submodule of M containing $N \cap K$. Hence $Gr((N \cap K :_R M)) \subseteq Gr((P :_R M))$, it follows that $Gr((N :_R M)) \cap Gr((K :_R M)) \subseteq Gr((N \cap K :_R M)) \subseteq Gr((P :_R M))$. By Lemma 2.7, $Gr((P :_R M))$ is a graded prime ideal of R. Then either $Gr((N :_R M)) \subseteq Gr((P :_R M))$ or $Gr((K :_R M)) \subseteq Gr((P :_R M))$ by Lemma 2.1(iv). Assume that $Gr((N :_R M)) \subseteq Gr((P :_R M))$. Since $Gr((N :_R M)) + Gr((K :_R M)) = R$, we conclude that $Gr((K :_R M)) \nsubseteq Gr((P :_R M))$. We will show $N \subseteq P$. Suppose $t \in N \cap h(M) - P$ and let $r \in Gr((K :_R$

$M)) \cap h(R) - Gr((P :_R M))$. Then there exists $n \in \mathbb{Z}^+$ such that $r^n M \subseteq K$ and $r^n \notin Gr((P :_R M))$. Hence $r^n t \in N \cap K \subseteq P$. Since P is a graded primary submodule, either $t \in P$ or $r^n \in Gr((P :_R M))$, which is a contradiction. Thus $N \subseteq P$. Similarly, if $Gr((K :_R M)) \subseteq Gr((P :_R M))$ we conclude that $K \subseteq P$. By Theorem 2.5, we conclude that $P\text{-}Gr_M(N \cap K) = P\text{-}Gr_M(N) \cap P\text{-}Gr_M(K)$. □

Theorem 2.11 *Let R be a G-graded ring, M a graded finitely generated R-module and N, K graded submodules of M. Then $P\text{-}Gr_M(N) + P\text{-}Gr_M(K) = M$ if and only if $N + K = M$.*

Proof. ($\Rightarrow$) Suppose that $P\text{-}Gr_M(N) + P\text{-}Gr_M(K) = M$ and $N + K \neq M$. By Lemma 2.3, there exists a graded primary submodule P of M such that $N + K \subseteq P$. Since N, $K \subseteq N + K \subseteq P$, we have $P\text{-}Gr_M(N)$, $P\text{-}Gr_M(K) \subseteq P$. Hence $P\text{-}Gr_M(N) + P\text{-}Gr_M(K) \subseteq P$, which is a contradiction.

($\Leftarrow$) Since $N \subseteq P\text{-}Gr_M(N)$ and $K \subseteq P\text{-}Gr_M(K)$, it is clear that $N + K = M$ implies $P\text{-}Gr_M(N) + P\text{-}Gr_M(K) = M$. □

Let N be a proper graded submodule of a graded R-module M. Let Q be a graded primary ideal of R, we shall denote by $H(N, Q)$ the following subset of M, $H(N, Q) = \{m \in M : rm \in QM + N$, where $r \in h(R)$ and $r^n \notin Q$ for every $n \in \mathbb{Z}^+\}$. It is clear that $H(N, Q)$ is a graded submodule of M and $QM + N \subseteq H(N, Q)$.

Lemma 2.12 *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. If Q is a graded primary ideal of R, then $H(N, Q) = M$ or $H(N, Q)$ is a graded primary submodule of M.*

Proof. Let Q be a graded primary ideal of R. Suppose that $H(N, Q) \neq M$ and let $s \in h(R)$ and $m \in h(M)$ such that $sm \in H(N, Q)$ and $s \notin Gr(H(N, Q) :_R M)$. We show that $m \in H(N, Q)$. Since $sm \in H(N, Q)$, there exists $r \in h(R)$ such that $rsm \in QM + N$ and $r^n \notin Q$ for every $n \in \mathbb{Z}^+$. Since $s^n M \nsubseteq H(N, Q)$ for all $n \in \mathbb{Z}^+$, we have $s^n \notin Q$ for every $n \in \mathbb{Z}^+$. Since Q is a graded primary ideal of R, $(rs)^n \notin Q$ for every $n \in \mathbb{Z}^+$. Hence $m \in H(N, Q)$. Thus $H(N, Q)$ is a graded primary submodule of M. □

Theorem 2.13 *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Then $P\text{-}Gr_R(N) = \cap\{H(N, Q) : Q$ is a graded primary ideal of $R\}$.*

Proof. Let $m_g \in V = \cap\{H(N, Q) : Q$ is a graded primary ideal of $R\}$ for $g \in G$. Let P be a graded primary submodule of M containing N. By [2, Proposition 2.5(i)], $(P :_R M)$ is a graded primary ideal of R. So $m_g \in H(N, (P :_R M))$. Then there exists $r \in h(R)$ such that $rm_g \in (P :_R M)M + N$ and $r^n \notin (P :_R M)$ for every $n \in \mathbb{Z}^+$. Hence $rm_g = sm' + n$, where $s \in (P :_R M) \cap h(R)$, $m' \in h(M)$ and $n \in N \cap h(M)$. Since $sM \subseteq P$ and $N \subseteq P$, we conclude that $rm_g = sm' + n \in P$. Since P is a graded primary submodule of M and $r^n \notin (P :_R M)$ for every $n \in \mathbb{Z}^+$, we have $m_g \in P$. Thus $V \subseteq P\text{-}Gr_M(N)$. Now we will prove the reverse inclusion. Let $t_g \in P\text{-}Gr_R(N)$ for $g \in G$ and let $H(N, Q) \in V$. By Lemma 2.12, $H(N, Q) = M$ or $H(N, Q)$ is a graded primary submodule of M. If $H(N, Q) = M$, then it is trivial that $t_g \in H(N, Q)$. So we can assume that $H(N, Q) \neq M$. Since $H(N, Q)$ is a graded primary submodule of M and $N \subseteq H(N, Q)$, we have $t_g \in H(N, Q)$. So $P\text{-}Gr_R(N) \subseteq V$. Therefore $P\text{-}Gr_R(N) = \cap\{H(N, Q) : Q$ is a graded primary ideal of $R\}$. □

Let M and M' be two graded R-modules. A homomorphism of graded R-modules $\varphi : M \to M'$ is a homomorphism of R-modules verifying $\varphi(M_g) \subseteq M'_g$ for every $g \in G$,.

Lemma 2.14 *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be an epimorphism of graded modules. Let N' be a graded submodule of M'. Then N' is a graded primary submodule of M' if and only if $\varphi^{-1}(N')$ is a graded primary submodule of M.*

Proof. ($\Rightarrow$) Suppose that N' is a graded primary submodule of M' and let $r \in h(R)$ and $m \in h(M)$ such that $rm \in \varphi^{-1}(N')$ and $m \notin \varphi^{-1}(N')$. Then $\varphi(rm) = r\varphi(m) \in N'$. Since N' is a graded primary submodule of M' and $\varphi(m) \notin N'$, we have $r^n \in (N' :_R M')$ for some $n \in \mathbb{Z}^+$, i.e., $r^n M' \subseteq N'$ and hence $r^n \varphi^{-1}(M') = r^n M \subseteq \varphi^{-1}(N')$, i.e., $r^n \in (\varphi^{-1}(N') :_R M)$. Thus $\varphi^{-1}(N')$ is a graded primary submodule of M.

($\Leftarrow$) Suppose that $\varphi^{-1}(N')$ is a graded primary submodule of M and let $s \in h(R)$ and $m' \in h(M')$ such that $sm' \in N'$ and $m' \notin N'$. Since φ is an epimorphism, there exists $m \in h(M)$ such that $\varphi(m) = m'$. Thus $s\varphi(m) = \varphi(sm) \in N'$. So $sm \in \varphi^{-1}(N')$. Since $\varphi^{-1}(N')$ is a graded primary submodule of M and $m \notin \varphi^{-1}(N')$, we have $s^n \in (\varphi^{-1}(N') :_R M)$ for some $n \in \mathbb{Z}^+$, i.e., $s^n M \subseteq \varphi^{-1}(N')$ and so $\varphi(s^n M) = s^n \varphi(M) = s^n M' \subseteq N'$, i.e., $s^n \in (N' :_R M')$. Therefore N' is a graded primary submodule of M'. □

Lemma 2.15 *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be an epimorphism of graded modules. Let N be a graded submodule of M such that $Ker\varphi \subseteq N$. If N is a graded primary submodule of M, then $\varphi(N)$ is a graded primary submodule of M'.*

Proof. Suppose that N is a graded primary submodule of M and let $r \in h(R)$ and $m' \in h(M')$ such that $rm' \in \varphi(N)$ and $m' \notin \varphi(N)$. Since $rm' \in \varphi(N)$, there exists $t \in N \cap h(M)$ such that $\varphi(t) = rm'$. Since $m' \in h(M')$ and φ is an epimorphism, there exists $m \in h(M)$ such that $\varphi(m) = m'$. Thus $\varphi(t) = r\varphi(m)$ and hence $\varphi(t - rm) = 0$. So $t - rm \in Ker\varphi \subseteq N$ and hence $rm \in N$. Since N is a graded primary submodule of M and $m \notin N$, $r^n \in (N :_R M)$ for some $n \in \mathbb{Z}^+$, i.e., $r^n M \subseteq N$ and so $r^n M' \subseteq \varphi(N)$. Thus $\varphi(N)$ is a graded primary submodule of M'. □

Theorem 2.16 *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be an epimorphism of graded modules. If N' is a graded submodule of M', then $\varphi^{-1}(P\text{-}Gr_{M'}(N')) = P\text{-}Gr_M(\varphi^{-1}(N'))$.*

Proof. Let $t \in P\text{-}Gr_M(\varphi^{-1}(N'))$ and P be a graded primary submodule of M' containing N'. By Lemma 2.14, $\varphi^{-1}(P)$ is a graded primary submodule of M containing $\varphi^{-1}(N')$. Hence $P\text{-}Gr_M(\varphi^{-1}(N')) \subseteq \varphi^{-1}(P)$ and so $t \in \varphi^{-1}(P)$. Thus $\varphi(t) \in P$ and so $\varphi(t) \in P\text{-}Gr_{M'}(N')$ it follows that $t \in \varphi^{-1}(P\text{-}Gr_{M'}(N'))$. Thus $P\text{-}Gr_M(\varphi^{-1}(N')) \subseteq \varphi^{-1}(P\text{-}Gr_{M'}(N'))$. Now suppose that $s \in \varphi^{-1}(P\text{-}Gr_{M'}(N'))$ and Q be a graded primary submodule of M containing $\varphi^{-1}(N')$. It is clear that $ker\varphi \subseteq \varphi^{-1}(N')$. By Lemma 2.15, $\varphi(Q)$ is a graded primary submodule of M' containing N'. Thus $P\text{-}Gr_{M'}(N') \subseteq \varphi(Q)$, then $\varphi(s) \in P\text{-}Gr_{M'}(N') \subseteq \varphi(Q)$. So there exists $q \in Q$ such that $\varphi(s) = \varphi(q)$, then $s - q \in ker\varphi \subseteq \varphi^{-1}(N') \subseteq Q$. Hence $s \in Q$, so $s \in P\text{-}Gr_M(\varphi^{-1}(N'))$. Thus $\varphi^{-1}(P\text{-}Gr_{M'}(N')) \subseteq P\text{-}Gr_M(\varphi^{-1}(N'))$. Therefore $\varphi^{-1}(P\text{-}Gr_{M'}(N')) = P\text{-}Gr_M(\varphi^{-1}(N'))$. □

Lemma 2.17 *Let R be a G-graded ring and M_1, M_2 be two graded R-modules. Let $M = M_1 \oplus M_2$.*

(i) $N = P_1 \oplus M_2$ is a graded primary submodule of M if and only if P_1 is a graded primary submodule of M_1.

(ii) $N = M_1 \oplus P_2$ is a graded primary submodule of M if and only if P_2 is a graded primary submodule of M_2.

Proof. (i) ($\Rightarrow$) Suppose that $N = P_1 \oplus M_2$ is a graded primary submodule of M and let $rm \in P_1$ where $r \in h(R)$ and $m \in h(M)\backslash P_1$. Then $(m, 0) \notin N = P_1 \oplus M_2$. Since $N = P_1 \oplus M_2$ is a graded primary submodule of M, $r(m, 0) \in N$ and $(m, 0) \notin N$, we conclude that $r^n \in (N :_R M)$ for some $n \in \mathbb{Z}^+$, i.e., $r^n(M_1 \oplus M_2) \subseteq P_1 \oplus M_2$ and hence $r^n M_1 \subseteq P_1$. Thus P_1 is a graded primary submodule of M_1.

($\Leftarrow$) Suppose that P_1 is a graded primary submodule of M_1 and let $r \in h(R)$ and $(m_1, m_2) \in h(M)$ such that $r(m_1, m_2) \in N = P_1 \oplus M_2$ and $(m_1, m_2) \notin N$. Since P_1 is a graded primary submodule of M_1, $rm_1 \in P_1$ and $m_1 \notin P_1$, we have $r^n M_1 \subseteq P_1$ for some $n \in \mathbb{Z}^+$. Hence $r^n M \subseteq P_1 \oplus M_2$ for some $n \in \mathbb{Z}^+$. Therefore $N = P_1 \oplus M_2$ is a graded primary submodule of M.

(ii) This proof is similar to that in case (i) and we omit it. □

Theorem 2.18 *Let R be a G-graded ring and M_1, M_2 be two graded R-modules and let $M = M_1 \oplus M_2$.*

(i) If N_1 is a proper graded submodule of M_1, $t \in P\text{-}Gr_{M_1}(N_1)$ if and only if $(t, 0) \in P\text{-}Gr_M(N_1 \oplus (0))$.

(ii) If N_2 is a proper graded submodule of M_2, $t \in P\text{-}Gr_{M_2}(N_2)$ if and only if $(0, t) \in P\text{-}Gr_M((0) \oplus N_2)$.

Proof. (i) ($\Rightarrow$) Suppose that $t \in P\text{-}Gr_{M_1}(N_1)$ and let P be a graded primary submodule of M containing $N_1 \oplus (0)$. Let $P_1 = \{m \in M_1 : (m, 0) \in P\}$. We want to show that either $P_1 = M_1$ or P_1 is a graded primary submodule of M_1. Assume that $P_1 \neq M_1$ and let $r \in h(R)$ and $m_1 \in h(M_1)$ such that $rm_1 \in P_1$ and $m_1 \notin P_1$. Then $r(m_1, 0) = (rm_1, 0) \in P$. Since P is a graded primary submodule of M and $(m_1, 0) \notin P$, we have $r^n \in (P :_R M)$ for some $n \in \mathbb{Z}$, i.e., $r^n M \subseteq P$. Hence $(r^n x, 0) \in P$ for every $x \in M_1$. So $r^n M_1 \subseteq P_1$. Therefore P_1 is a graded primary submodule of M_1. Since $N_1 \oplus (0) \subseteq P$, $N_1 \subseteq P_1$. Hence $t \in P_1$, it follows that $(t, 0) \in P$. So $(t, 0) \in P\text{-}Gr_M(N_1 \oplus (0))$.

($\Leftarrow$) Suppose that $(t, 0) \in P\text{-}Gr_M(N_1 \oplus (0))$. Let P_1 be a graded primary submodule of M_1 containing N_1. From (i) of Lemma 2.17, we conclude that $P_1 \oplus M_2$ is a graded primary submodule of M containing $N_1 \oplus (0)$. So $(t, 0) \in P_1 \oplus M_2$, it follows that $t \in P_1$. Thus $t \in P\text{-}Gr_{M_1}(N_1)$.

(ii) This proof is similar to that in case (i) and we omit it. □

Corollary 2.19 *Let R be a G-graded ring and M_1, M_2 be two graded R-modules and let $M = M_1 \oplus M_2$. If $N = N_1 \oplus N_2$, is a proper graded submodule of M, then $P\text{-}Gr_{M_1}(N_1) \oplus P\text{-}Gr_{M_2}(N_2) \subseteq P\text{-}Gr_M(N)$.*

Proof. Let $(m_1, m_2) \in P\text{-}Gr_{M_1}(N_1) \oplus P\text{-}Gr_{M_2}(N_2)$. Then $m_1 \in P\text{-}Gr_{M_1}(N_1)$ and $m_2 \in P\text{-}Gr_{M_2}(N_2)$. By Theorem 2.18, we have. $(m_1, 0) \in P\text{-}Gr_M(N_1 \oplus (0))$ and $(0, m_2) \in P\text{-}Gr_M((0) \oplus N_2)$. From (ii) of Theorem 2.4 (ii), we conclude that $P\text{-}Gr_M(N_1 \oplus (0)) \subseteq P\text{-}Gr_M(N)$ and $P\text{-}Gr_M((0) \oplus N_2) \subseteq P\text{-}Gr_M(N)$, it follows that $(m_1, 0), (0, m_2) \in P\text{-}Gr_M(N)$. Hence $(m_1, m_2) \in P\text{-}Gr_M(N)$. Therefore $P\text{-}Gr_{M_1}(N_1) \oplus P\text{-}Gr_{M_2}(N_2) \subseteq P\text{-}Gr_M(N)$. □

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