An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Tomul LXII, 2016, f. 2, vol. 1

Lipschitz nonlinear operators and resolvents

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Received: 2.II.2014 / Accepted: 6.II.2014

Abstract We investigate the nonlinear Lipschitz operators, their resolvents and the complete maximum principle, in connection with the approach of Dellacherie and van Gool. We present relevant examples.

Keywords Lipschitz function · Complete maximum principle · Nonlinear resolvent

Mathematics Subject Classification (2010) 26A16 · 35B50 · 60J35

1 Introduction

In the linear potential theory the kernels which satisfy the complete maximum principle and the sub-Markovian resolvents are main investigation tools. In particular, it was a main goal to find conditions on a given kernel, such that it becomes the initial kernel of a sub-Markovian resolvent (see, e.g., [1] and [3]).

The aim of this paper is to investigate the nonlinear bounded (Lipschitz) operators and the nonlinear sub-Markovian resolvents. In this theory it is also a central result to show that a nonlinear operator satisfying a (nonlinear) complete maximum principle is the initial kernel of a nonlinear sub-Markovian resolvent. It was stated by Dellacherie in [2] and a complete proof is due to van Gool (cf. [4]).

We give a new proof for the fact that a nonlinear Lipschitz operator satisfying the complete maximum principle generates a regular sub-Markovian resolvent. Our approach is new and we use some techniques which allows a better analogy with the linear case. A basic ingredient is the couple of conjugated functions (cf. [5]).

The organization of the paper is as follows. In Section 2 we introduce our setup, following essentially [2], [4], and [6] (see also [7] and [8]). Then we recall some results concerning the conjugated functions (see [2] and [6]).

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The main results are presented in Section 3 (Theorem 3.2). The last section is an addendum and contains the proofs of several results from Section 2.

2 The general framework

In this work, $\mathcal{L}$ is a linear lattice of bounded real valued functions on a non-empty set X. The lattice $\mathcal{L}$ is supposed to be complete with respect to the uniform norm (denoted by $\|\cdot\|$). The symbols $T,\ V,\ N,\ V_p$ (where p is a positive number) denote functions from $\mathcal{L}$ into $\mathcal{L}$ (in what follows we shall say that they are functions on $\mathcal{L}$) and I is the identity map of $\mathcal{L}$. For every $f,\ g$ functions in $\mathcal{L}$, $\{f > g\} := \{x \in X : f(x) > g(x)\}$, and similar notation will be used for $\{x \in X : f(x) \geq g(x)\}$.

Definition 2.1 ([2]) *(i) The function T on $\mathcal{L}$ is called* nonlinear operator *on $\mathcal{L}$ if T is an increasing mapping on $\mathcal{L}$.*

(ii) A nonlinear operator T on $\mathcal{L}$ is called nonlinear bounded *operator on $\mathcal{L}$ if T is Lipschitz function on $\mathcal{L}$ and* lip$T$ *denotes the Lipschitz constant of T (i.e.* lip $T :=$ $\inf\{C \in (0,\infty) : \|Tf - Tg\| \leq C\|f - g\|,\ f,\ g \in \mathcal{L}\}$).

(iii) We say that the nonlinear bounded operator T on $\mathcal{L}$ is sub-Markovian *if* lip$T \leq 1$.

Lemma 2.2 ([4]; see [6] also for the sub-Markovian operator) *For every T a nonlinear bounded operator on $\mathcal{L}$ we have that*

$$\text{lip}T = \inf\{C \in (0,\infty) :\ \textit{for all } r \in (0,\infty),\quad \textit{for all } f,\ g \in \mathcal{L},$$
$$f \leq g + r \Rightarrow Tf \leq Tg + Cr\}.$$

Remark 2.1 ([6]) It is clear that for every function T on $\mathcal{L}$ the following assertions are equivalent.

(i) The function T is nonlinear sub-Markovian operator.

(ii) The mapping T has the following property:

$$\text{for all } f,\ g \in \mathcal{L},\quad \text{for all } \alpha \in (0,\infty) : f \leq g + \alpha \Rightarrow Tf \leq Tg + \alpha.$$

Definition 2.3 *(i) We shall say that T* satisfies the weak complete maximum principle *if for all $f,\ g \in \mathcal{L}$ and for all $\alpha \in (0,\infty)$, $f + Tf \leq g + Tg + \alpha$ on $\{f > g\} \Rightarrow Tf \leq Tg + \alpha$.*

(ii) We shall say that T satisfies the complete maximum principle *if for all $f,\ g \in \mathcal{L}$ and for all $\alpha \in (0,\infty)$, $Tf \leq Tg + \alpha$ on $\{f > g\} \Rightarrow Tf \leq Tg + \alpha$.*

Remark 2.2 (i) It is clear that every function T which satisfies the complete maximum principle satisfies the weak complete maximum principle too.

(ii) If the function T satisfies the weak complete maximum principle then it is nonlinear operator on $\mathcal{L}$.

Lemma 2.4 ([6], see [4] also) *For a function T on $\mathcal{L}$ the following assertions are equivalent.*

(i) The application T satisfies the complete maximum principle.

(ii) For all $p \in (0,\infty)$, the mapping pT satisfies the weak complete maximum principle.

Definition 2.5 ([2]) *Assume that the following identities hold on $\mathcal{L}$:*

$$(I+T)(I-N)=I=(I-N)(I+T). \tag{2.1}$$

Then T is called the conjugate function *of N, respectively, N is called* the anticonjugate function *of T. Moreover, the pair (T,N) is called a pair of* conjugated functions *on $\mathcal{L}$.*

Remark 2.3 (i) It is easy to verify that for all functions $T,\ N$ the following assertions are equivalent: (a) (T,N) is a pair of conjugated functions; (b) the following identities hold: $T(I-N)=N$, $N(I+T)=T$; (c) $I+T$ is invertible and $(I+T)^{-1}=I-N$; (d) $I-N$ is invertible and $(I-N)^{-1}=I+T$.

(ii) If (T,N) is a pair of conjugated functions on $\mathcal{L}$ then we have that $N=T(I+T)^{-1}$ and $T=N(I-N)^{-1}$.

(iii) Moreover the function T has an anticonjugate function if and only if $I+T$ is invertible.

Theorem 2.6 ([6] and [4]) *Suppose that $I+T$ is invertible and $N=T(I+T)^{-1}$. The following assertions are equivalent.*

(i) The function N is nonlinear sub-Markovian operator and T is nonlinear operator.

(ii) The application T satisfies the weak complete maximum principle.

(iii) The mapping T is nonlinear operator and satisfies the following property: for all $f,\ g\in\mathcal{L}$, for all $\alpha\in(0,\infty)$, $f+Tf\leq g+Tg+\alpha\Rightarrow Tf\leq Tg+\alpha$.

Proposition 2.7 ([4]) *If the function T is Lipschitz and $t\in\mathbb{R}$ is such that* $\mathrm{lip}(tT)<1$, *then the application $I+tT$ is invertible.*

Proof. Let f be a function in $\mathcal{L}$ and T_f be the function on $\mathcal{L}$ defined by $T_fu:=f-tTu$, for all $u\in\mathcal{L}$. Since T_f is contraction on $\mathcal{L}$, there exists a unique element $u_f\in\mathcal{L}$ such that $T_fu_f=u_f\Leftrightarrow f=u_f+tTu_f$. Therefore $I+tT$ is surjective function and being injective too, it is invertible. □

Remark 2.4 Let T be a Lipschitz function on $\mathcal{L}$. If $\mathrm{lip}T=0$, then T is a constant function on $\mathcal{L}$, i.e. there exists $\omega\in\mathcal{L}$ such that $Tf=\omega$ for all $f\in\mathcal{L}$. Therefore $(I+tT)(I-tT)=I=(I-tT)(I+tT)$, for all $t\in\mathbb{R}$, hence $I+tT$ is invertible for all $t\in\mathbb{R}$.

From now on we suppose that $\mathrm{lip}T>0$.

(i) By the previous proposition, for all $t\in(-\frac{1}{\mathrm{lip}T},\frac{1}{\mathrm{lip}T})$, $I+tT$ is invertible. Let us denote $T(I+tT)^{-1}$ by N_t. In view of Remark 2.3.(i) we have that (tT,tN_t) is a pair of conjugated functions on $\mathcal{L}$.

(ii) Moreover $(-\frac{1}{\mathrm{lip}T},\frac{1}{\mathrm{lip}T})\subset\{t\in\mathbb{R}:I+tT\text{ is invertible }\}=:J$, and for all $s,t\in J$, $(I+(t-s)N_s)(I+(s-t)N_t)=I$, that is $((s-t)N_t,(s-t)N_s)$ and $((t-s)N_s,(t-s)N_t)$ are pairs of conjugated functions on $\mathcal{L}$.

Proposition 2.8 ([4]) *We consider T a nonlinear bounded operator on $\mathcal{L}$ such that the function $I+T$ is invertible and the application $T(I+T)^{-1}$ is nonlinear operator. Then the nonlinear operator $T(I+T)^{-1}$ is bounded and $\frac{\mathrm{lip}T}{1+\mathrm{lip}T}\leq\mathrm{lip}T(I+T)^{-1}\leq\mathrm{lip}T$.*

Proof. For all $f,\ g \in \mathcal{L}$ and $r \in (0,\infty)$ with $f \leq g+r$ we have that $(I+T)^{-1}(\sup(f,g))-(I+T)^{-1}g \leq (I+T)(I+T)^{-1}(\sup(f,g)) - (I+T)(I+T)^{-1}g \leq r \Rightarrow T(I+T)^{-1}f - T(I+T)^{-1}g \leq T(I+T)^{-1}(\sup(f,g)) - T(I+T)^{-1}g \leq r\mathrm{lip}T$. According to Lemma 2.2 $T(I+T)^{-1}$ is nonlinear bounded operator, with $\mathrm{lip}T(I+T)^{-1} \leq \mathrm{lip}T$. Furthermore

$$\begin{aligned}
&(I+T)f-(I+T)g = f-g+Tf-Tg \leq (1+\mathrm{lip}T)\|f-g\| \\
&\Longrightarrow Tf-Tg = T(I+T)^{-1}(I+T)f - T(I+T)^{-1}(I+T)g \\
&\leq (\mathrm{lip}T(I+T)^{-1})\|(I+T)f-(I+T)g\| \\
&\leq (\mathrm{lip}T(I+T)^{-1})(1+\mathrm{lip}T)\|f-g\|,
\end{aligned}$$

i.e. $\mathrm{lip}T \leq (\mathrm{lip}T(I+T)^{-1})(1+\mathrm{lip}T)$. □

Remark 2.5 By Theorem 2.6, if the function T on $\mathcal{L}$ is such that T satisfies the weak complete maximum principle and $I+T$ is invertible, then $N = T(I+T)^{-1}$ is nonlinear sub-Markovian operator. Therefore if we assume that T is Lipschitz, then $\mathrm{lip}N = \mathrm{lip}T(I+T)^{-1} \leq \mathrm{lip}T$.

Proposition 2.9 *We suppose that (T,N) is a pair of conjugated functions on $\mathcal{L}$ and T satisfies the complete maximum principle. Then N satisfies the complete maximum principle.*

Proof. Let $f,\ g$ be functions in $\mathcal{L}$ and take $\alpha \in (0,\infty)$ such that $Nf \leq Ng+\alpha$ on $\{f > g\} \Leftrightarrow T(f-Nf) \leq T(g-Ng)+\alpha$ on $\{f > g\}$. Since $\{f-Nf > g-Ng\} = (\{f-Nf > g-Ng\} \cap \{f > g\}) \cup (\{f-Nf > g-Ng\} \cap \{f \leq g\})$, on the set $\{f-Nf > g-Ng\} \cap \{f > g\}$ we have that $T(f-Nf) \leq T(g-Ng)+\alpha \Leftrightarrow Nf \leq Ng+\alpha$. On the other hand for all $x \in \{f-Nf > g-Ng\} \cap \{f \leq g\}$ it follows that $0 \leq g(x)-f(x) < Ng(x)-Nf(x) \Rightarrow Nf(x) < Ng(x) < Ng(x)+\alpha$. Therefore $T(f-Nf) = Nf \leq Ng+\alpha = T(g-Ng)+\alpha$ on $\{f-Nf > g-Ng\}$, and by the complete maximum principle $Nf = T(f-Nf) \leq T(g-Ng)+\alpha = Ng+\alpha$ on X. □

3 Nonlinear resolvents

In this section we present the main results of our paper concerning the nonlinear sub-Markovian resolvents and their initial nonlinear operator.

Definition 3.1 ([2]) *Let $\mathcal{V} := (V_p)_{p>0}$ be a family of functions on $\mathcal{L}$.*
(i) If, for all $p,\ q \in (0,\infty)$, the following identity holds on $\mathcal{L}$

$$(I+(p-q)V_q)(I+(q-p)V_p) = I \tag{3.1}$$

then $\mathcal{V}$ is called nonlinear resolvent *on $\mathcal{L}$.*
(ii) Let $\mathcal{V}$ be a nonlinear resolvent on $\mathcal{L}$ and V be a function on $\mathcal{L}$.
(a) If, for every $p \in (0,\infty)$ we have

$$(I+pV)(I-pV_p) = I = (I-pV_p)(I+pV). \tag{3.2}$$

the function V is called the initial *function of the resolvent $\mathcal{V}$.*
(b) When pV_p is nonlinear sub-Markovian operator on $\mathcal{L}$, for all $p \in (0,\infty)$, the resolvent $\mathcal{V}$ is called sub-Markovian.

Remark 3.1 Let $\mathcal{V} := (V_p)_{p>0}$ be a nonlinear resolvent on $\mathcal{L}$ and V be a function on $\mathcal{L}$.

(i) Let us assume that V is the initial function of the resolvent $\mathcal{V}$.

(a) The conditions (3.2) are equivalent to the following equations $V_p = V(I - pV_p)$ and $V = V_p(I + pV)$, for all $p \in (0, \infty)$ and they mean that (pV, pV_p) is a pair of conjugated functions on $\mathcal{L}$.

(b) From the previous point and Remark 2.3.(ii), for every $p \in (0, \infty)$, $V_p = V(I + pV)^{-1}$, hence there exists at most one nonlinear resolvent on $\mathcal{L}$ satisfying the conditions (3.2).

(c) Similarly, for every $p \in (0, \infty)$, $V = V_p(I - pV_p)^{-1}$ which means that there exists at most one initial function of $\mathcal{V}$.

(d) By Lemma 2.4 and Theorem 2.6 it follows that the resolvent $\mathcal{V}$ is sub-Markovian if and only if V satisfies the complete maximum principle.

(ii) (a) The condition (3.1) (named the resolvent equation) means that, for every $p,\ q \in (0, \infty),\ ((p - q)V_q, (p - q)V_p)$ is a pair of conjugated functions on $\mathcal{L}$.

(b) If we suppose that $\mathcal{V}$ is sub-Markovian, then for all $q \in (0, \infty)$ and $p \in (q, \infty)$, the nonlinear operator $(p - q)V_p$ is sub-Markovian hence $(p - q)V_q$ satisfies the weak complete maximum principle (Theorem 2.6). Since $p \in (q, \infty)$ is arbitrary it follows that V_q satisfies the complete maximum principle, for all $q \in (0, \infty)$.

Theorem 3.2 *Let V be nonlinear bounded operator on $\mathcal{L}$ which satisfies the complete maximum principle. There exists a unique nonlinear sub-Markovian resolvent $\mathcal{V} := (V_p)_{p>0}$ on $\mathcal{L}$ having the following properties.*

(1) *The function V is the initial operator of $\mathcal{V}$.*

(2) *There exists $r \in (0, \infty)$ such that $\mathrm{lip}V_p \leq \frac{1}{r}$, for all $p \in (0, \infty)$.*

(3) *For all $f \in \mathcal{L}$ we have that $\lim_{p \searrow 0} \|V_p f - Vf\| = 0$.*

Proof. (see [4] or [6] for another proofs) Choose $r \in (0, \infty)$ with $\mathrm{lip}(rV) < 1$. By Propositions 2.7 and 2.8 the nonlinear operator $I + rV$ is invertible and if $V_r := V(I + rV)^{-1}$ then (rV, rV_r) is a pair of conjugated operators, rV_r is nonlinear sub-Markovian operator (Theorem 2.6) and $\mathrm{lip}rV_r = \mathrm{lip}rV(I + rV)^{-1} \leq \mathrm{lip}rV < 1$.

We suppose that for $k \in \mathbb{N}^*$ there exists $(I + krV)^{-1}$ and if $V_{kr} := V(I + krV)^{-1}$ then rV_{kr} is nonlinear sub-Markovian operator and $\mathrm{lip}rV_{kr} \leq \mathrm{lip}rV < 1$.

Define $(V_{kr})_r := V_{kr}(I + rV_{kr})^{-1}$ and remark that $(rV_{kr}, r(V_{kr})_r)$ is a pair of conjugated functions on $\mathcal{L}$. Since V_{kr} satisfies the complete maximum principle (Proposition 2.9), $r(V_{kr})_r$ is nonlinear sub-Markovian operator (Theorem 2.6). Therefore $\mathrm{lip}r(V_{kr})_r \leq \mathrm{lip}rV_{kr} \leq \mathrm{lip}rV < 1$. Moreover, $I + (k + 1)rV = (I + rV_{kr})(I + krV)$ is invertible and $V_{(k+1)r} := V(I + (k + 1)rV)^{-1} = V(I + krV)^{-1}(I + rV_{kr})^{-1} = V_{kr}(I + rV_{kr})^{-1} = (V_{kr})_r$.

That is $I + (k + 1)rV$ is invertible, $(k + 1)rV_{(k+1)r} = (k + 1)rV(I + (k + 1)rV)^{-1}$ is nonlinear sub-Markovian operator and $\mathrm{lip}rV_{(k+1)r} = \mathrm{lip}r(V_{kr})_r \leq \mathrm{lip}rV < 1$. Hence, for every $k \in \mathbb{N}^*$, $I + krV$ is invertible and the nonlinear operator $rV_{kr} = rV(I + krV)^{-1}$ is sub-Markovian such that $\mathrm{lip}rV_{kr} \leq \mathrm{lip}rV < 1$.

Let p now be a positive number and choose k a natural number with $kr < p < (k + 1)r$. We have that $I + pV = (I + (p - kr)V_{kr})(I + krV)$, and $\mathrm{lip}(p - kr)V_{kr} \leq \mathrm{lip}rV_{kr} < 1$;

then $I+(p-kr)V_{kr}$ is invertible, $I+pV$ is invertible too and if

$$\begin{aligned} V_p :&= V(I+pV)^{-1} = V(I+krV)^{-1}(I+(p-kr)V_{kr})^{-1} \\ &= V_{kr}(I+(p-kr)V_{kr})^{-1} \Rightarrow \mathrm{lip} r V_p \\ &= \frac{r}{p-kr}\mathrm{lip}(p-kr)V_p \leq \frac{r}{p-kr}\mathrm{lip}(p-kr)V_{kr} = r\mathrm{lip}V_{kr} < 1. \end{aligned}$$

The general statements follow now by Remarks 2.4.(ii) and 3.1.(i)(b) and the properties (1) and (2) of the theorem are now clear.

To prove assertion (3) remark that for all $f \in \mathcal{L}$ and $p \in (0,\infty)$, $\|V_pf - Vf\| = \|V_pf - V_p(f+pVf)\| \leq \frac{1}{p}\|pVf\| = \|Vf\| \Rightarrow \|V_pf\| \leq 2\|Vf\|$. Therefore the set of functions $\{V_pf : p \in (0,\infty)\}$ is bounded in $(\mathcal{L}, \|\cdot\|)$, hence $\lim_{p\searrow 0}\|pV_pf\| = 0$ and $\|V_pf - Vf\| = \|V(f-pV_pf) - Vf\| \leq \frac{1}{r}\|pVf\| \Rightarrow \lim_{p\searrow 0}\|V_pf - Vf\| = 0$. □

Remark 3.2 We consider V and $\mathcal{V} := (V_p)_{p>0}$ as in the previous theorem and we suppose that $V\mathbf{0} = \mathbf{0}$ (which by the resolvent equation means that $V_p\mathbf{0} = \mathbf{0}$, for every $p \in (0,\infty)$ what it is equivalent to, or there exists $p \in (0,\infty)$ such that $V_p\mathbf{0} = \mathbf{0}$) (where $\mathbf{0}$ is the null function on X).

(i) Let f be a non-negative function in $\mathcal{L}$. Then for every $p \in (0,\infty)$ and $q \in (p,\infty)$ we have that $Vf \geq 0,\ V_pf \geq 0,\ Vf = V_p(f+pVf) \geq V_pf$, and $V_pf = V_q(f+(q-p)V_pf) \geq V_qf$. Therefore (as in the linear case [1]), $(V_pf)_{p>0}$ is uniformly decreasing to Vf on X.

(ii) Similarly for every non-positive function f in $\mathcal{L}$ it follows that $(V_pf)_{p>0}$ is uniformly increasing to Vf on X.

Theorem 3.3 *Let us consider $\mathcal{V} = (V_p)_{p>0}$ a nonlinear sub-Markovian resolvent on $\mathcal{L}$ which satisfies the following conditions:*

(a) There exists $r \in (0,\infty)$ with $\mathrm{lip}V_p \leq \frac{1}{r}$, for every $p \in (0,\infty)$.

(b) The set of functions $\{V_p\mathbf{0} : p \in (0,\infty)\}$ is bounded in $(\mathcal{L}, \|\cdot\|)$.

Then there exists V nonlinear operator on $\mathcal{L}$ with the following properties.

(1) The operator V satisfies the complete maximum principle.

(2) The nonlinear operator V is the initial operator of $\mathcal{V}$.

(3) For all $f \in \mathcal{L}$, $\lim_{p\searrow 0}\|V_pf - Vf\| = 0$.

(4) The (nonlinar) operator V is bounded with $\mathrm{lip}V \leq \frac{1}{r}$.

Proof. For every $f \in \mathcal{L}$ and $p,\ q \in (0,\infty)$ we have the following statements $\|V_pf\| \leq \|V_pf - V_p\mathbf{0}\| + \|V_p\mathbf{0}\| \leq \frac{1}{r}\|f\| + \|V_p\mathbf{0}\|$. Since $\{V_p\mathbf{0} : p \in (0,\infty)\}$ is a bounded subset in $(\mathcal{L}, \|\cdot\|)$, the set $\{V_pf : p \in (0,\infty)\}$ is also bounded in $(\mathcal{L}, \|\cdot\|)$. Moreover,

$$\begin{aligned} \|V_pf - V_qf\| = \|V_pf - V_p(f+(p-q)V_qf)\| &\leq \frac{|p-q|}{r}\|V_qf\| \\ \Rightarrow (p \mapsto V_pf) : (0,\infty) &\to (\mathcal{L}, \|\cdot\|) \text{ is uniformly continuous.} \end{aligned}$$

Therefore there exists $\lim_{p\searrow 0} V_pf$ (in $(\mathcal{L}, \|\cdot\|)$ and so that define $V : \mathcal{L} \to \mathcal{L},\ Vf := \lim_{p\searrow 0} V_pf$ and the assertion (3) follows. Furthermore, for all f and g elements in $\mathcal{L}$ $\|Vf - Vg\| = \lim_{p\searrow 0}\|V_pf - V_pg\| \leq \frac{1}{r}\|f-g\|$, i.e. V is Lipschitz with $\mathrm{lip}V \leq \frac{1}{r}$ (a part of the assertion (4)). On the other hand $V_pf = V_q(f+(q-p)V_pf)$, for all $f \in \mathcal{L}$, for all $p,q \in (0,\infty)$ and $\lim_{p\searrow 0}\|f+(q-p)V_pf - (f+qVf)\| = 0$, for all

$q \in (0,\infty)$, for all $f \in \mathcal{L}$. In view of the continuity of V_q, for all $f \in \mathcal{L}$, it follows that $Vf = \lim_{p\searrow 0} V_p f = \lim_{p\searrow 0} V_q(f+(q-p)V_p f) = V_q(f+qVf)$, for all $f \in \mathcal{L}$. Moreover

$$\|V_q(f+(q-p)V_pf) - V_q(f-pV_pf)\| \leq \frac{q}{r}\|V_pf\|, \quad \text{for all } f \in \mathcal{L}, p, q \in (0,\infty)$$
$$\Rightarrow V_pf = \lim_{q\searrow 0} V_q(f+(q-p)V_pf) = \lim_{q\searrow 0} V_q(f-pV_pf)$$
$$= V(f-pV_pf), \quad \text{for all } f \in \mathcal{L}, \quad \text{for all } p \in (0,\infty).$$

So that (pV, pV_p) is a pair of conjugated functions on $\mathcal{L}$ and the other assertions of the theorem follow now by Lemma 2.4 and Theorem 2.6. □

Corollary 3.4 *Let $\mathcal{V} := (V_p)_{p>0}$ be a nonlinear sub-Markovian resolvent on $\mathcal{L}$ such that $V_p\mathbf{0} = \mathbf{0}$ for every $p \in (0,\infty)$. Then the following assertions are equivalent.*

(i) There exists a nonlinear bounded operator V on $\mathcal{L}$ which is the initial operator of $\mathcal{V}$.

(ii) The set $\{\mathrm{lip}V_p : p \in (0,\infty)\}$ is bounded.

Proof. (i)⇒(ii) It is a consequence of Theorem 3.2.

(ii)⇒(i) Since $\{V_p\mathbf{0} : p \in (0,\infty)\} = \{\mathbf{0}\}$ we apply Theorem 3.3. □

Example 3.1 We present some examples (see [9]) and we study their essential properties concerning our aim. From now on $(X,\mathcal{X})$ is a measurable space and the notation $f \in \mathcal{X}$ means that the numerical function f on X is $\mathcal{X}$-measurable. Moreover $b\mathcal{X} := \{f \in \mathcal{X} : f \text{ is bounded}\}$.

(i) Let N (respectively $\mathcal{S}_N$) be a linear sub-Markovian kernel on $(X,\mathcal{X})$ (respectively the convex cone of N-supermedian functions ([1])). We define, for every $f \in \mathcal{X}$, their reduit function $Rf := R^N f := \inf\{s \in \mathcal{S}_N : s \geq f\} = R(f^+)$.

(a) Since, for all $s \in \mathcal{S}_N$, $Rs = s$ (in particular $R\mathbf{1} = \mathbf{1}$, where the symbol $\mathbf{1}$ denotes the function from X into $\mathbb{R}$ identically equal to 1), for all f, g functions in $b\mathcal{X}$ we have that $\|Rf - Rg\| \leq \|f-g\|$, i.e. R is a sub-Markovian nonlinear operator on $b\mathcal{X}$.

(b) If $f,\ g \in b\mathcal{X}$ and $\alpha \in (0,\infty)$ are such that $Rf \leq Rg + \alpha$ on $\{f > g\}$, then $f \leq Rf \leq Rg+\alpha$ on $\{f > g\}$, and $f \leq g \leq Rg \leq Rg+\alpha$ on $\{f \leq g\}$. Therefore (according to Mokobodzki theorem ([1])), $f \leq Rg + \alpha$, and $Rg + \alpha \in \mathcal{S}_N \Rightarrow Rf \leq Rg + \alpha$ i.e. R satisfies the complete maximum principle.

(c) In view of Theorem 3.2 there exists $\mathcal{V} := (V_p)_{p>0}$ a nonlinear sub-Markovian resolvent on $b\mathcal{X}$ such that R is the initial nonlinear operator of $\mathcal{V}$. Thus for all $p \in (0,\infty), V_p = R(I - pV_p)$, and $R = V_p(I + pR)$.

(d) Let f be an element in $b\mathcal{X}$. We have that

$$f - \frac{p}{p+1}Rf \leq \frac{1}{p+1}Rf,\ \frac{1}{p+1}Rf \in \mathcal{S}_N \Rightarrow R\left(f - \frac{p}{p+1}Rf\right) \leq \frac{1}{p+1}Rf.$$

On the other hand for every $s \in \mathcal{S}_N$ with $s \geq f - \frac{p}{p+1}Rf$ it follows that

$$f \leq s + \frac{p}{p+1}Rf \Rightarrow Rf \leq s + \frac{p}{p+1}Rf,$$

i.e. $\frac{1}{p+1}Rf \leq s$.

Since $s \in \mathcal{S}_N$ is arbitrary with $s \geq f - \frac{p}{p+1}Rf$, we have that

$$\frac{1}{p+1}Rf \leq R\left(f - \frac{p}{p+1}Rf\right), \text{ and } \frac{1}{p+1}Rf = R\left(f - \frac{p}{p+1}Rf\right).$$

Therefore, for all f in $b\mathcal{X}$ and p in $(0,\infty)$

$$\frac{1}{p+1}Rf = R\left(f - \frac{p}{p+1}Rf\right) \Leftrightarrow (I+pR)\left(f - \frac{p}{p+1}Rf\right) = f.$$

Hence $(I+pR)^{-1} = I - \frac{p}{p+1}R$ and $V_p = R(I+pR)^{-1} = R(I - \frac{p}{p+1}R) = \frac{1}{p+1}R$.

(ii) For every $f \in b\mathcal{X}$ and $p \in (0,\infty)$ let us define $V_p f = \frac{1}{p+1}f + \frac{1}{p+1}$. It is easy to show that $(V_p)_{p>0}$ satisfy the resolvent equation (the relation (3.1)). For all $f,\ g \in b\mathcal{X}$ and $p \in (0,\infty)$ it follows that

$$\|pV_p f - pV_p g\| = \frac{p}{p+1}\|f-g\| \leq \|f-g\|,$$
$$\|V_p f - V_p g\| = \frac{1}{p+1}\|f-g\| \leq \|f-g\|.$$

Therefore $(V_p)_{p>0}$ is nonlinear sub-Markovian resolvent on $b\mathcal{X}$, $\mathrm{lip}V_p \leq 1$, for all $p \in (0,\infty)$ and $\{V_p\mathbf{0} : p \in (0,\infty)\} = \{\frac{1}{p+1} : p \in (0,\infty)\}$ is a bounded set. According to Theorem 3.3 the function $V : b\mathcal{X} \to b\mathcal{X}, Vf = \lim_{p\to 0} V_p f = f + \mathbf{1}$, for all $f \in b\mathcal{X}$ is the initial nonlinear operator of $(V_p)_{p>0}$.

(iii) If $V_p f = \frac{1}{p}f + \frac{1}{p}$, for every $f \in b\mathcal{X}$ and $p \in (0,\infty)$, then it is clear that $(V_p)_{p>0}$ is a nonlinear sub-Markovian resolvent on $b\mathcal{X}$.

Let us suppose that there exists the initial function of $(V_p)_{p>0}$ denoted by V. In view of the conditions (3.2) it follows that for all $p \in (0,\infty), V\mathbf{1} = V_p(\mathbf{1} + pV\mathbf{1}) \Leftrightarrow$ for all $p \in (0,\infty), V\mathbf{1} = V\mathbf{1} + \frac{2}{p}$, which is a contradictory relation. Hence the nonlinear sub-Markovian resolvent $(V_p)_{p>0}$ has no initial function.

4 Addendum: Proof of several results from Section 2

Proof of Lemma 2.2. We consider a nonlinear operator T on $\mathcal{L}$ and the following sets of positive numbers $\mathcal{A}_T := \{c \in (0,\infty) : \text{for all } f,g \in \mathcal{L}, \|Tf - Tg\| \leq c\|f-g\|\}$, $\mathcal{A}'_T := \{c \in (0,\infty) : \text{for all } r \in (0,\infty), f,g \in \mathcal{L}, \|f-g\| \leq r \Rightarrow \|Tf - Tg\| \leq cr\}$, and $\mathcal{A}''_T := \{c \in (0,\infty) : \text{for all } r \in (0,\infty), f,g \in \mathcal{L}, f \leq g + r \Rightarrow Tf \leq Tg + cr\}$, and we shall prove that $\mathcal{A}_T \subset \mathcal{A}'_T \subset \mathcal{A}''_T \subset \mathcal{A}_T$.

If $c \in \mathcal{A}_T$ and $r \in (0,\infty)$, $f,\ g \in \mathcal{L}$ are such that $\|f-g\| \leq r$, then $\|Tf - Tg\| \leq c\|f-g\| \leq cr \Rightarrow c \in \mathcal{A}'_T$.

Suppose now that $c \in \mathcal{A}'_T$ and let us take $r \in (0,\infty)$ and $f,\ g \in \mathcal{L}$ such that $f \leq g + r$. Then we have $f \leq \sup(f,g) \leq g + r \Rightarrow \|\sup(f,g) - g\| \leq r$. Therefore $\|T(\sup(f,g)) - Tg\| \leq cr \Rightarrow Tf \leq T(\sup(f,g)) \leq Tg + cr$, i.e. $c \in \mathcal{A}''_T$. For every $c \in \mathcal{A}''_T$ and $f,\ g \in \mathcal{L}$ it follows $f \leq g + \|f-g\|, g \leq f + \|f-g\| \Rightarrow Tf \leq Tg + c\|f-g\|, Tg \leq Tf + c\|f-g\|$, hence $\|Tf - Tg\| \leq c\|f-g\|$, for all $f,g \in \mathcal{L} \Rightarrow c \in \mathcal{A}_T$. □

Proof of the assertion of Remark 2.1. Property (i) (respectively (ii)) is equivalent to the conditions $\mathbf{1} \in \mathcal{A}_T$ (respectively $\mathbf{1} \in \mathcal{A}_T''$). Now the assertion follows by Lemma 2.2. □

Proof of Lemma 2.4. The implication (i)⇒(ii) is clear.

(ii)⇒(i) Let $f,\ g$ be functions in $\mathcal{L}$ and $\alpha,\ \epsilon,\ p$ be positive numbers such that $Tf \leq Tg + \alpha$ on $\{f > g\} (\Rightarrow \sup\{(Tf - Tg)(x) : x \in \{f > g\}\} \leq \alpha < \alpha + \epsilon)$ and $\frac{1}{p}\sup\{(f-g)(x) : x \in X\} < \alpha + \epsilon + \inf\{(Tg - Tf)(x) : x \in \{f > g\}\}$. Then, for all $x \in \{f > g\}$ we have that $\frac{1}{p}(f(x) - g(x)) < \alpha + \epsilon + Tg(x) - Tf(x) \Leftrightarrow f(x) + pTf(x) < g(x) + pTg(x) + p(\alpha + \epsilon)$.

Since the nonlinear operator pT satisfies the weak complete maximum principle it follows that $pTf \leq pTg + p(\alpha + \epsilon)$ on $X \Leftrightarrow Tf \leq Tg + \alpha + \epsilon$ on X, and ϵ being arbitrary positive number $Tf \leq Tg + \alpha$. □

Proof of Theorem 2.6. (i)⇒(ii) We consider $f,\ g \in \mathcal{L}$ and $\alpha \in (0, \infty)$ such that $f + Tf \leq g + Tg + \alpha$ on $\{f > g\}$, and we define $u := \inf(f + Tf, g + Tg + \alpha)$. Since (T, N) is a pair of conjugated functions on $\mathcal{L}$ and N is sub-Markovian we remark the following relations:

(1) $Nu \leq Tf$; (2) $Nu \leq Tg + \alpha$; (3) $u = f + Tf$ on $\{f > g\}$.

If $j := u - Nu$, then $u = (I + T)j$ and we shall prove that $f \leq j$ on X.

If $x \in \{f > g\}$, then $f(x) \leq j(x)$ is equivalent to $f(x) \leq u(x) - Nu(x) = f(x) + Tf(x) - Nu(x)$, and in view of condition (1) the last inequality is true.

When $x \in \{f \leq g\}$ we distinguish the following cases: $u(x) := f(x) + Tf(x)$ and the inequality is proved similarly to the previous case, respectively $u(x) = g(x) + Tg(x) + \alpha$ and the inequality $f(x) \leq j(x)$, means that $Nu(x) \leq g(x) - f(x) + Tg(x) + \alpha$ and according to the hypothesis and the condition (2) it is also true.

By the inequality $f \leq j$ and the monotony of T, it follows that $f + Tf \leq (I + T)j = (I + T)(I - N)u = u \leq g + Tg + \alpha \Rightarrow Tf = N(f + Tf) \leq N(g + Tg + \alpha) \leq N(I + T)g + \alpha = Tg + \alpha$.

(ii)⇒(iii) It is clear.

(iii)⇒(i) Let $f,\ g$ be functions in $\mathcal{L}$ and $\alpha \in (0, \infty)$ be such that $f \leq g + \alpha$.

Then

$$\begin{aligned} f = (I + T)(I - N)f &\leq (I + T)(I - N)g + \alpha \\ \Leftrightarrow f - Nf + T(f - Nf) &\leq g - Ng + T(g - Ng) + \alpha. \end{aligned}$$

By the hypothesis (iii) $Nf = T(f - Nf) \leq T(g - Ng) + \alpha = Ng + \alpha$, i.e. N is sub-Markovian (Remark 2.1). □

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