An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Tomul LXII, 2016, f. 2, vol. 1

Multiple solutions for a $p(x)$-Kirchhoff type problem with Neumann boundary condition

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Received: 15.XI.2013 / Revised: 19.I.2014 / Accepted: 21.I.2014

Abstract Using a recent result by Ricceri, we study the multiplicity of solutions to the following problem

$$\begin{cases} -M\left(L(u)\right)\left(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - |u|^{p(x)-2}u\right) = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a smooth bounded domain, ν is the outward unit normal to $\partial\Omega$, λ, μ are positive parameters, $p \in C(\overline{\Omega})$, $\inf_{x\in\overline{\Omega}} p(x) > N$, $L(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx$, $f, g : \Omega\times\mathbb{R} \to \mathbb{R}$ are Carathéodory functions, $f(x,t)$ is sublinear at infinity with respect to t, $M : \mathbb{R}_0^+ \to \mathbb{R}$ is continuous, nondecreasing and may be degenerate at zero.

Keywords $p(x)$-Kirchhoff type problems · Three critical points theorem · Sublinear term · Neumann boundary conditions

Mathematics Subject Classification (2010) 35D30 · 35J60

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1 Introduction

In this paper, we are interested in the existence of solutions for a class of $p(x)$-Kirchhoff type problems of the form

$$\begin{cases} -M\left(L(u)\right)\left(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - |u|^{p(x)-2}u\right) \\ \qquad\qquad = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is smooth bounded domain, ν is the outward unit normal to $\partial\Omega$, λ, μ are two positive parameters, $p \in C(\overline{\Omega})$, $\inf_{x\in\overline{\Omega}} p(x) > N$, $L(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions, $M : \mathbb{R}_0^+ := [0, +\infty) \to \mathbb{R}$ is a continuous, nondecreasing function and satisfies the following condition

(M_0) There exist $m_0 > 0$ and a constant $\alpha > 1$ such that $M(t) \geq m_0 t^{\alpha-1}$, $\forall t \in \mathbb{R}_0^+$.

Since the first equation in (1.1) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density (see [3]). Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

presented by Kirchhoff in 1883 (see [16]). This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In recent years, elliptic problems involving Kirchhoff type operators have been studied in many papers, we refer interested readers to [1,2,4–11,22]. In [10,11], the authors firstly studied the existence of solutions for a class of $p(x)$-Kirchhoff type equations with Ambrosetti-Rabinowitz type conditions. In [2,22], superlinear problems were studied by using a variant of the three critical points theorem by Ricceri [21]. In [1], the authors considered $p(x)$-Kirchhoff type problems in the asymptotically linear case and the probems with sublinear terms and Dirichlet boundary conditions were studied in [5,9]. Motivated by the results introduced in [4,6–8], we study the existence of multiple solutions for problem (1.1) with sublinear term and Neumann boundary condition. In particular, the Kirchhoff function $M(t)$ may be zero at zero. The result of this paper improves the previous ones in [18,23], in which the authors considered problem (1.1) in the special case $M(t) \equiv 1$ and $\mu = 0$. More precisely, using the three critical points theorem by Ricceri [20] we prove the existence of at least three solutions for problem (1.1). It is clear that our situation here is different from those introduced in [2,6,9]. In order to state the main result of this paper let us assume that the following conditions hold:

(F_0) There exist $C > 0$ and a function $q(x) \in C(\overline{\Omega})$, $q^+ < \alpha p^-$ such that $|f(x,t)| \leq C(1 + |t|^{q(x)-1}), \forall (x,t) \in \Omega \times \mathbb{R}$;

(F_1) There exist $t_0 > 1$ and $R > 0$ such that $f(x,t) < 0$ when $|t| \in (0,1)$ and $f(x,t) \geq R$ when $|t| \in (t_0, +\infty)$.

(G_0) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $G(.,0) \in L^1(\Omega)$ and $\sup_{|t|\leq k} |g(x,t)| \leq h_k(x)$, for all $k > 0$ and almost every $x \in \Omega$, where $h_k \in L^1(\Omega)$, $G(x,t) = \int_0^t g(x,s)\, ds$.

Definition 1.1 *We say that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of problem (1.1) if*

$$M\left(L(u)\right) \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv \right) dx$$
$$- \lambda \int_\Omega f(x,u) v\, dx - \mu \int_\Omega g(x,u) v\, dx = 0,$$

for all $v \in W^{1,p(x)}(\Omega)$, where $L(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) dx$.

The result of this paper is presented in the following theorem.

Theorem 1.2 *Assume that $\inf_{x\in\overline{\Omega}} p(x) > N$ for any $x \in \overline{\Omega}$ and the conditions (M_0), (F_0)-(F_1). Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number $\delta > 0$ such that, for each $\lambda \in \Lambda$ and every Carathéodory function $g : \Omega \times \mathbb{R}$ satisfying the condition (G_0), there exists $\mu^* > 0$ such that for each $\mu \in [0, \mu^*]$, problem (1.1) has at least three solutions whose norms are less than δ.*

2 Preliminaries

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the book of Musielak [19] and the papers of Kováčik and Rákosník [17] and Fan et al. [14,15]. Set $C_+(\overline{\Omega}) := \{h : \ h \in C(\overline{\Omega}),\ h(x) > 1$, for all $x \in \overline{\Omega}\}$. For any $h \in C_+(\overline{\Omega})$ we define $h^+ = \sup_{x\in\Omega} h(x)$ and $h^- = \inf_{x\in\Omega} h(x)$. For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space $L^{p(x)}(\Omega) = \{u :$ a measurable real-valued function such that $\int_\Omega |u(x)|^{p(x)} dx < \infty\}$. We recall the following so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \quad \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$ then there exists a continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of

$L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

Proposition 2.1 (see [15]) *If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold*

$$|u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+} \tag{2.1}$$

provided $|u|_{p(x)} > 1$ while

$$|u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} \tag{2.2}$$

provided $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \;\Leftrightarrow\; \rho_{p(x)}(u_n - u) \to 0. \tag{2.3}$$

Next, we define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : \; |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm $\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}$, which is a separable and reflexive Banach space. It has the following equivalent norm

$$\|u\| = \inf \left\{ \mu > 0 : \; \int_\Omega \left(\left| \frac{u(x)}{\mu} \right|^{p(x)} + \left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

Let

$$I(u) = \int_\Omega \left(|u|^{p(x)} + |\nabla u|^{p(x)} \right) dx, \quad L(u) = \int_\Omega \frac{1}{p(x)} \left(|u|^{p(x)} + |\nabla u|^{p(x)} \right) dx,$$

then there are the following relations

$$\|u\| < 1 (= 1, > 1) \Leftrightarrow I(u) < 1 (= 1, > 1), \tag{2.4}$$

$$\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq I(u) \leq \|u\|^{p^+}, \tag{2.5}$$

$$\|u\| < 1 \Rightarrow \|u\|^{p^+} \leq I(u) \leq \|u\|^{p^-}. \tag{2.6}$$

Remark 2.1 If $N < p^- \le p(x)$ for any $x \in \overline{\Omega}$, by Theorem 2.2 in [14], we deduce that $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$. Since $N < p^-$ it follows that $W^{1,p^-}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Thus, we deduce that $W^{1,p(x)}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Defining $\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)|$, we find that there exists a positive constant $c > 0$ such that $\|u\|_\infty \le c\|u\|, \forall u \in W^{1,p(x)}(\Omega)$.

Finally, for proving our result in the next section, we introduce the following lemma.

Lemma 2.2 (see [20]) *Let $(X, \|.\|)$ be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous fucntional whose Gâteaux derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(i) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda J(u)) = \infty$ *for all* $\lambda > 0$;
(ii) *There are* $r \in \mathbb{R}$ *and* $u_0, u_1 \in X$ *such that* $\Phi(u_0) < r < \Phi(u_1)$;
(iii) $\inf_{u\in\Phi^{-1}((-\infty,r])} J(u) > \frac{(\Phi(u_1)-r)J(u_0)+(r-\Phi(u_0))J(u_1)}{\Phi(u_1)-\Phi(u_0)}$.

Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number δ such that each $\lambda \in \Lambda$, and every continuously Gâteaux differentiable functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\mu^* > 0$ such that for each $\mu \in [0, \mu^*]$, the equation $\Phi'(u) + \lambda J'(u) + \mu\Psi'(u) = 0$ has at least three solutions in X whose norms are less than δ.

3 Proof of the main result

In this section, we will prove the proof of Theorem 1.2 in details by using Lemma 2.2. We use the letter C_i to denote a general positive constant which value may change from line to line.

Let us define the functionals $\Phi, J : X := W^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$\Phi(u)=\widehat{M}\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx\right), J(u)=-\int_\Omega F(x,u)\,dx, \tag{3.1}$$

where $\widehat{M}(t) = \int_0^t M(s)\,ds$ and $F(x,t) = \int_0^t f(x,s)\,ds$. It is easy to see that $\Phi, J \in C^1(X,\mathbb{R})$ with the derivatives given by

$$\Phi'(u)(v) = M\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx\right)\cdot \\ \cdot\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v + |u|^{p(x)-2}uv\right)\,dx$$

and

$$J'(u)(v) = -\int_\Omega f(x,u)v\,dx,$$

for any $u, v \in X$. In order to prove Theorem 1.2 it is enough to verify that Φ and J satisfy the hypotheses of Lemma 2.2.

Now let us show that the operator $\Phi' : X \to X^*$ is invertible on X. On account of the well-known Minty-Browder theorem (Theorem 26.A(d) of [24]), it suffices to prove that Φ is strictly convex, hemicontinuous and coercive in the sense of monotone operators.

We have

$$L'(u)(v) = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv)\, dx,$$

for all $u, v \in W^{1,p(x)}(\Omega)$. The mapping $L' : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))^*$ is a homeomorphism; moreover it is bounded, strictly monotone and of type (S_+) (see [13, Proposition 3.1]). So, by Proposition 25.10 of [24], L is strictly convex.

Now let $u, v \in X$ with $u \neq v$, and $\lambda, \mu \in (0,1)$ with $\lambda + \mu = 1$. Since L' is strictly monotone, L is strictly convex. Moreover, since the Kirchhoff function M is nondecreasing, $\widehat{M}$ is convex in $[0, +\infty)$. Thus, we have $\widehat{M}\Big(L(\lambda u + \mu v)\Big) < \widehat{M}(\lambda L(u) + \mu L(v)) \leq \lambda \widehat{M}(L(u)) + \mu \widehat{M}(L(v))$. This shows that Φ is strictly convex.

For any $u \in X$ with $\|u\| > 1$, by (2.5), one has

$$\frac{\Phi'(u)(u)}{\|u\|} = \frac{M(L(u)) \int_\Omega \Big(|\nabla u|^{p(x)} + |u|^{p(x)}\Big)\, dx}{\|u\|} \geq \frac{m_0}{(p^+)^{\alpha-1}}\|u\|^{\alpha p^- - 1},$$

from which we have the coercivity of Φ'. Standard arguments ensure that Φ' is hemicontinuous. Thus, in view of Theorem 26.A(d) of [24] there exists $(\Phi')^{-1} : X^* \to X$ and it is bounded. Let us prove that $(\Phi')^{-1}$ is continuous by showing that it is sequentially continuous.

Let $\{f_m\}$ be a sequence converging strongly to $f \in X^*$ and let $u_m = (\Phi')^{-1}(f_m)$, $u = (\Phi')^{-1}(f)$. So, $\{u_m\}$ is bounded in X and without loss of generality, we can assume that it converges weakly to a certain $u_0 \in X$. Since $\{f_m\}$ converges to f, it is easy to prove that

$$\lim_{m\to\infty} \Phi'(u_m)(u_m - u_0) = \lim_{m\to\infty} f_m(u_m - u_0) = 0.$$

As the operator Φ' is of type (S_+), we get that $\{u_m\}$ converges strongly to u_0. The continuity and injectivity of Φ', then guarantee that $\{u_m\}$ converges strongly to u, so $(\Phi')^{-1}$ is continuous.

Next, we will verify that the condition (i) of Lemma 2.2 is fulfilled. In fact, by relation (2.5), we have

$$\begin{aligned}
\Phi(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\, dx\right) \\
&\geq \frac{m_0}{\alpha}\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\, dx\right)^\alpha \\
&\geq \frac{m_0}{\alpha(p^+)^\alpha} I(u) \\
&\geq \frac{m_0}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-}, \quad \forall u \in X, \quad \|u\| > 1.
\end{aligned} \tag{3.2}$$

On the other hand,

$$J(u) = -\int_\Omega F(x,u)\,dx = \int_\Omega -F(x,u)\,dx$$

and due to the assumption (F_1), there exists $C_1 > 0$ such that

$$|F(x,t)| \le C_1 \left(|t| + \frac{1}{q(x)} |t|^{q(x)} \right), \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Therefore,

$$\begin{aligned} J(u) &\ge -C_1 \int_\Omega |u|\,dx - C_1 \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\,dx \\ &\ge -C_2\|u\| - \frac{C_1}{q^-} \int_\Omega (|u|^{q^+} + u^{q^-})\,dx = -C_2\|u\| - \frac{C_1}{q^-}(|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}). \end{aligned} \tag{3.3}$$

Using Remark 2.1, we know that X is continuously embedded in $L^{q^\pm}(\Omega)$. Furthermore, we can find two positive constants $C_3 > 0$ such that

$$|u|_{q^+} \le C_3\|u\|, \quad |u|_{q^-} \le C_3\|u\|, \quad \forall u \in X. \tag{3.4}$$

From (3.3) and (3.4), we have

$$J(u) \ge -C_2\|u\| - C_4\|u\|^{q^+} - C_4\|u\|^{q^-}. \tag{3.5}$$

Combining (3.2) and (3.5), it follows that for all $u \in X$ with $\|u\| > 1$,

$$\begin{aligned} \Phi(u) + \lambda J(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx \right) - \lambda \int_\Omega F(x.u)\,dx \\ &\ge \frac{m_0}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^-} - \lambda C_2\|u\| - \lambda C_4(\|u\|^{q^+} + \|u\|^{q^-}). \end{aligned} \tag{3.6}$$

Since $1 < q^+ < \alpha p^-$, then $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda J(u)) = \infty$ and (i) is verified.

In the sequel, we will verify the conditions (ii) and (iii) in Lemma 2.2. Indeed, it follows from the assumptions (F_1) and (F_2) that $F(x,t)$ is increasing for $t \in (1,\infty)$ and decreasing for $t \in (0,1)$, uniformly with respect to $x \in \Omega$.

It is clear that $F(x,0) = 0$ and $F(x,t) \to \infty$ when $t \to \infty$, because of the definition of $F(x,t)$ and the assumption (F_2). Then there exists a real number $\delta > t_0$ such that $F(x,t) \ge 0 = F(x,0) \ge F(x,\tau)$, $\forall x \in X$, $t > \delta$, $\tau \in (0,1)$. Let a, b be two real numbers such that

$$0 < a < \min\left\{ 1, c, c\left(\frac{\alpha(p^+)^\alpha}{m_0} \right)^{\frac{1}{\alpha p^+}} \right\}, \tag{3.7}$$

where c is given in Remark 2.1 and $b > \delta$ satisfies

$$b^{\alpha p^-} |\Omega|^\alpha > 1. \tag{3.8}$$

From (F_2) we have $F(x,t) \le F(x,0)$ for all $t \in [0,a]$, which implies that

$$\int_\Omega \sup_{0\le t\le a} F(x,t)\,dx \le \int_\Omega F(x,0)\,dx = 0.$$

Furthermore, since $b > \delta$ we get $\int_\Omega F(x,b)\,dx > 0$ and thus,

$$\int_\Omega \sup_{0\le t\le a} F(x,t)\,dx \le 0 < r.\frac{\int_\Omega F(x,b)\,dx}{\widehat{M}\left(\int_\Omega \frac{1}{p(x)} b^{p(x)}\,dx\right)}. \tag{3.9}$$

Consider $u_0, u_1 \in X$, $u_0(x) = 0$ and $u_1(x) = b$ for any $x \in \Omega$, we define $r = \frac{m_0}{\alpha(p^+)^\alpha}(\frac{a}{c})^{\alpha p^+}$. From (3.7), we have $r \in (0,1)$. A simple computation implies $\Phi(u_0) = J(u_0) = 0$ and

$$\begin{aligned}\Phi(u_1) &\ge \frac{m_0}{\alpha}\left(\int_\Omega \frac{1}{p(x)} b^{p(x)}\,dx\right)^\alpha \\ &\ge \frac{m_0}{\alpha(p^+)^\alpha} b^{\alpha p^-}|\Omega|^\alpha > \frac{m_0}{\alpha(p^+)^\alpha}.1 > \frac{m_0}{\alpha(p^+)^\alpha}\left(\frac{a}{c}\right)^{\alpha p^+},\end{aligned} \tag{3.10}$$

$$J(u_1) = -\int_\Omega F(x,u_1(x))\,dx = -\int_\Omega F(x,b)\,dx. \tag{3.11}$$

Thus, we obtain $\Phi(u_0) < r < \Phi(u_1)$ and the condition (ii) in Lemma 2.1 is verified.

On the other hand, we have

$$\begin{aligned}-\frac{(\Phi(u_1)-r)J(u_0) + (r-\Phi(u_0))J(u_1)}{\Phi(u_1)-\Phi(u_0)} &= -r.\frac{J(u_1)}{\Phi(u_1)} \\ &= r.\frac{\int_\Omega F(x,b)\,dx}{\widehat{M}\left(\int_\Omega \frac{1}{p(x)} b^{p(x)}\,dx\right)} > 0.\end{aligned} \tag{3.12}$$

Next, we consider the case $u \in X$ with $\Phi(u) \le r < 1$. Since $r \ge \Phi(u) \ge \frac{m_0}{\alpha(p^+)^\alpha}(I(u))^\alpha$ we obtain $(I(u))^\alpha \le r.\frac{\alpha(p^+)^\alpha}{m_0} = (\frac{a}{c})^{\alpha p^+} < 1$, which shows that $\|u\| < 1$ by (2.4). Furthermore, by (2.6), it is clear that $\frac{m_0}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} \le \frac{m_0}{\alpha(p^+)^\alpha}(I(u))^\alpha \le \Phi(u) \le r$. Thus, using Remark 2.1, for all $u \in X$ with $\Phi(u) \le r$, we have

$$|u(x)| \le c\|u\| \le c\left(r.\frac{\alpha(p^+)^\alpha}{m_0}\right)^{\frac{1}{\alpha p^+}} = a, \quad \forall x \in \Omega. \tag{3.13}$$

The above inequality shows that

$$-\inf_{u\in\Phi^{-1}(-\infty,r]} J(u) = \sup_{u\in\Phi^{-1}(-\infty,r]} -J(u) \le \int_\Omega \sup_{0\le t\le a} F(x,t)\,dx \le 0. \tag{3.14}$$

It follows from (3.12) and (3.14) that

$$-\inf_{u\in\Phi^{-1}(-\infty,r]} J(u) < r.\frac{\int_\Omega F(x,b)\,dx}{\widehat{M}\left(\int_\Omega \frac{1}{p(x)} b^{p(x)}\,dx\right)}.$$

That is,

$$\inf_{u\in\Phi^{-1}(-\infty,r]} J(u) > \frac{(\Phi(u_1)-r)J(u_0) + (r-\Phi(u_0))J(u_1)}{\Phi(u_1)-\Phi(u_0)},$$

which means that the condition (iii) in Lemma 2.2 is verified.

Since the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the condition (G_0), the functional

$$\Psi(u) = -\int_\Omega G(x,u)\,dx$$

is well defined and continuously Gâteaux differentiable on X, with compact derivative, and one has $\Psi'(u)(v) = -\int_\Omega g(x,u)v\,dx$ for all $u, v \in X$. So, according to Lemma 2.2, there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number δ such that for each $\lambda \in \Lambda$, and every continuously Gâteaux differentiable functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\mu^* > 0$ such that for each $\mu \in [0,\mu^*]$, the equation $\Phi'(u) + \lambda J'(u) + \mu\Psi'(u) = 0$ has at least three solutions in X whose norms are less than δ. It follows that Theorem 1.2 holds.

Acknowledgements The authors would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript. This work was supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED), Grant N.101.02.2014.03.

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