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$(\theta, \phi)^*$-derivations on complex semi-prime Banach $*$-algebras

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Abstract In this paper, we prove that $(\theta, \phi)^*$-derivations on complex semi–prime $*$-algebras can be represented by double $(\theta, \phi)^*$-centralizers. As an application, we prove a result in automatic continuity of $(\theta, \phi)^*$-derivations and α-derivations.

Keywords $(\theta, \phi)^*$-derivation · Double (θ, ϕ)-centralizer · Semi-prime $*$-algebra

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1 Introduction

Let $\mathcal{A}$ be a complex algebra and θ, ϕ algebra endomorphisms of $\mathcal{A}$. An additive mapping $L : \mathcal{A} \to \mathcal{A}$ is called a left θ-centralizer if $L(xy) = L(x)\theta(y)$, for all $x, y \in \mathcal{A}$. Analogously, an additive mapping $R : \mathcal{A} \to \mathcal{A}$ is called a right ϕ-centralizer if $R(xy) = \phi(x)R(y)$, for all $x, y \in \mathcal{A}$.

Let L be a left θ-centralizer of $\mathcal{A}$ and R a right ϕ-centralizer of $\mathcal{A}$. Then the pair (L, R) is called a double (θ, ϕ)-centralizer of $\mathcal{A}$ if $\phi(x)L(y) = R(x)\theta(y)$, for all $x, y \in \mathcal{A}$.

An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a (θ, ϕ)-derivation if $d(xy) = d(x)\theta(y) + \phi(x)d(y)$, for all $x, y \in \mathcal{A}$, and d is said to be a Jordan (θ, ϕ)-derivation if $d(x^2) = d(x)\theta(x) + \phi(x)d(x)$, for all $x, y \in \mathcal{A}$. An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is called a $*$-derivation if $d(xy) = d(x)y^* + xd(y)$, for all $x, y \in \mathcal{A}$, and d is called a Jordan $*$-derivation if $d(x^2) = d(x)x^* + xd(x)$, for all $x \in \mathcal{A}$. For more details on $*$-derivations, we refer the reader to [4], [5] and [9]-[12]. Jordan $*$-derivations are closely connected with a question of representability of quadratic forms by bilinear forms (see [11]). Some

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algebraic properties of Jordan $*$-derivations were studied in [4]. A representation of Jordan $*$-derivations in terms of left and right centralizers on the algebra of compact operators on a Hilbert space was found by Bresar and Zalar in [5] (see also [10]). The authors in [5], considered a problem whether an additive mapping L satisfying a weaker condition $L(x^2) = L(x)x$ is automatically a left centralizer and obtained an affirmative answer on prime rings of characteristic not 2. Later, Zalar in [13] generalized this result on semi–prime rings of characteristic not 2. C. Lanski proved that every Jordan (θ, ϕ)-derivation is a (θ, ϕ)-derivation on 2-torsion free semi–prime rings (see [8]). As a consequence of this result Ali and Haetinger in [1] proved that every left Jordan θ-centralizer is a left θ-centralizer on 2-torsion free semi–prime rings.

α-derivations have been studied extensively in pure algebra and have not been treated systematically in Banach algebras. Note that if α is an automorphism of an algebra $\mathcal{A}$, then a linear mapping $\Delta : \mathcal{A} \to \mathcal{A}$ is called an α-derivation if it satisfies $\Delta(xy) = \Delta(x)\alpha(y) + x\Delta(y)$, for all $x, y \in \mathcal{A}$. We note that some authors use a slightly different definition for Δ as follows $\Delta(xy) = \Delta(x)y + \alpha(x)\Delta(y)$, for all $x, y \in \mathcal{A}$. Bresar and Villena [6] gave a proof for automatic continuity of α-derivations. Another proof was given by Hejazian and Janfada [7] by using the structure of separating ideals of an α-derivation on a Banach algebra.

In this paper, we verify the structure of $(\theta, \phi)^*$-derivations on complex semi–prime $*$-algebras. We prove that every $(\theta, \phi)^*$-derivation on a complex semi–simple Banach $*$-algebra is continuous. As a simple consequence of this result, we prove the automatic continuity of α-derivations.

2 Main results

We start our work with the following known result:

Theorem 2.1 ([8],Theorem 2) *Let $\mathcal{R}$ be a 2-torsion-free semi-prime ring and d a Jordan (θ, ϕ)-derivation of $\mathcal{R}$ with θ or ϕ an automorphism of $\mathcal{R}$. Then d is a (θ, ϕ)-derivation of $\mathcal{R}$.*

Theorem 2.2 *Let $\mathcal{A}$ be a complex semi-prime algebra. We have the following assertions:*

(i) If L is a left Jordan θ-centralizer of $\mathcal{A}$ and θ an automorphism of $\mathcal{A}$, then L is a left θ-centralizer.
(ii) If R is a right Jordan ϕ-centralizer of $\mathcal{A}$ and ϕ an automorphism of $\mathcal{A}$, then R is a right ϕ-centralizer.

Proof. (i) We can prove the result by a same reasoning as Theorem 2.1, with condition $\phi = 0$.

(ii) It follows from Theorem 2.1 by putting $\theta = 0$. $\square$

Definition 2.3 *We say that an additive mapping $D : \mathcal{A} \to \mathcal{A}$ is a $(\theta, \phi)^*$-derivation if $D(xy) = D(x)\theta(y^*) + \phi(x)D(y)$, for all $x, y \in \mathcal{A}$, and D is a Jordan $(\theta, \phi)^*$-derivation if $D(x^2) = D(x)\theta(x^*) + \phi(x)D(x)$, for all $x \in \mathcal{A}$.*

In particular, an $(id_{\mathcal{A}}, id_{\mathcal{A}})^*$-derivation is a $*$-derivation.

Lemma 2.4 *Let $\mathcal{A}$ be a complex $*$-algebra. If $D:\mathcal{A}\to\mathcal{A}$ is a $(\theta,\phi)^*$-derivation, then*

$$D(x)\theta(y^*)+\phi(x)D(y)+i\phi(x)D(iy)=iD(ix)\theta(y^*),$$

for all $x,y\in\mathcal{A}$.

Proof. It is clear that $D(xy)=-D(((ix)(iy))$, for all $x,y\in\mathcal{A}$. We have

$$D(xy)=D(x)\theta(y^*)+\phi(x)D(y),$$

for all $x,y\in\mathcal{A}$. On the other hand, we have

$$D((ix)(iy))=-iD(ix)\theta(y^*)+i\phi(x)D(iy),$$

for all $x,y\in\mathcal{A}$. The desired result follows from the last two equations. □

We now prove the main result of the paper as follows.

Theorem 2.5 *Let $\mathcal{A}$ be a complex semi-prime $*$-algebra. If $D:\mathcal{A}\to\mathcal{A}$ is a $(\theta,\phi)^*$-derivation, then there exists a unique double (θ,ϕ)-centralizer (L,R) such that $D(x)=L(x^*)-R(x)$, for all $x\in\mathcal{A}$. Moreover, if L is injective, then $\mathcal{A}$ is commutative.*

Proof. Define $R_1:\mathcal{A}\longrightarrow\mathcal{A}$ by $R_1(x):=2D(ix)+2iD(x)$. Clearly, R_1 is additive. We have

$$\begin{aligned}
R_1(x^2)-\phi(x)R_1(x)&=2D(ix^2)+2iD(x^2)-2\phi(x)D(ix)-2i\phi(x)D(x)\\
&=D((x+ix)^2)+2iD(x)\theta(x^*)+2i\phi(x)D(x)\\
&\quad-2\phi(x)D(ix)-2i\phi(x)D(x)\\
&=D(x)\theta(x^*)-iD(x)\theta(x^*)+D(ix)\theta(x^*)-iD(ix)\theta(x^*)\\
&\quad+\phi(x)D(x)+\phi(x)D(ix)+i\phi(x)D(x)+i\phi(x)D(ix)\\
&\quad+2iD(x)\theta(x^*)-2\phi(x)D(ix)\\
&=\{D(x)\theta(x^*)-iD(ix)\theta(x^*)+\phi(x)D(x)+i\phi(x)D(ix)\}\\
&\quad+i\{D(x)\theta(x^*)-iD(ix)\theta(x^*)+\phi(x)D(x)+i\phi(x)D(ix)\}.
\end{aligned}$$

By applying Lemma 2.4, we conclude that the last expression is zero and therefore $R_1(x^2)=\phi(x)R_1(x)$, for all $x\in\mathcal{A}$. Due to Theorem 2.2 (ii), R_1 is a right ϕ-centralizer. In a similar way one can show that $L_1:\mathcal{A}\longrightarrow\mathcal{A}$ defined by $L_1(x):=2iD(x^*)-2D(ix^*)$ is additive and satisfies $L_1(x^2)=L_1(x)\theta(x)$. So L_1 is a left θ-centralizer, by Theorem 2.2 (i). Let $L:=-\frac{i}{4}L_1$ and $R:=\frac{i}{4}R_1$, then $D(x)=L(x^*)-R(x)$, for all $x\in\mathcal{A}$. We claim that (L,R) is a double (θ,ϕ)-centralizer of $\mathcal{A}$. Clearly, L is a left θ-centralizer and R is a right ϕ-centralizer of $\mathcal{A}$. We first show that $\phi(x)L(x^*)=R(x)\theta(x^*)$, for all $x\in\mathcal{A}$. We have

$$\begin{aligned}
\phi(x)L(x^*)&=\phi(x)(D(x)+R(x))=\phi(x)D(x)+\phi(x)R(x)\\
&\quad-L(x^{*2})+(L(x^*)-R(x))\theta(x^*)+R(x)\theta(x^*)\\
&=\phi(x)D(x)+R(x^2)-L(x^{*2})+D(x)\theta(x^*)+R(x)\theta(x^*)\\
&=\phi(x)D(x)-D(x^2)+D(x)\theta(x^*)+R(x)\theta(x^*)
\end{aligned}$$

and so
$$\phi(x)L(x^*) = R(x)\theta(x^*), \tag{2.1}$$
for all $x \in \mathcal{A}$. Replacing x by $x+y$ in (2.1) we get
$$\phi(x+y)L(x^*+y^*) = R(x+y)\theta(x^*+y^*),$$
for all $x, y \in \mathcal{A}$. It follows that
$$\phi(x)L(y^*) + \phi(y)L(x^*) = R(x)\theta(y^*) + R(y)\theta(x^*), \tag{2.2}$$
for all $x, y \in \mathcal{A}$. Replacing y by iy in (2.2) we find that
$$\phi(x)L(-iy^*) + i\phi(y)L(x^*) = -iR(x)\theta(y^*) + R(iy)\theta(x^*),$$
for all $x, y \in \mathcal{A}$. From the definition of L and R we conclude that $L(ix) = iL(x)$ and $R(ix) = iR(x)$, for all $x \in \mathcal{A}$. Thus
$$-i\phi(x)L(y^*) + i\phi(y)L(x^*) = -iR(x)\theta(y^*) + iR(y)\theta(x^*), \tag{2.3}$$
for all $x, y \in \mathcal{A}$. Comparing (2.2) and (2.3) we deduce that
$$\phi(x)L(y^*) = R(x)\theta(y^*), \tag{2.4}$$
for all $x, y \in \mathcal{A}$. Replacing y by y^* in (2.4), we get
$$\phi(x)L(y) = R(x)\theta(y),$$
for all $x, y \in \mathcal{A}$. This means that (L, R) is a double (θ, ϕ)-centralizer of $\mathcal{A}$. Assume that (L', R') is another double (θ, ϕ)-centralizer of $\mathcal{A}$. Then $L(x^*) - R(x) = L'(x^*) - R'(x)$, for all $x \in \mathcal{A}$. It follows that
$$(L - L')(x^*) = (R - R')(x), \tag{2.5}$$
for all $x \in \mathcal{A}$. Hence
$$(R - R')((xy)^*) = (L - L')(xy) = (L - L')(x)\theta(y), \tag{2.6}$$
for all $x, y \in \mathcal{A}$. Replacing y by iy in (2.5) we see that
$$-i(R - R')((xy)^*) = i(L - L')(x)\theta(y), \tag{2.7}$$
for all $x, y \in \mathcal{A}$. It follows from (2.6) and (2.7) that $(L - L')(x)\theta(y) = 0$, for all $x, y \in \mathcal{A}$. Semi-primeness of $\mathcal{A}$ implies that $L = L'$ and by (2.5), we have $R = R'$. This proves the uniqueness of (L, R).

We have $D(xy) = L((xy)^*) - R(xy)$, for all $x, y \in \mathcal{A}$. Expanding this identity we get
$$D(x)\theta(y^*) + \phi(x)D(y) = L(y^*)\theta(x^*) - \phi(x)R(y),$$
for all $x, y \in \mathcal{A}$. Therefore
$$L(x^*)\theta(y^*) - R(x)\theta(y^*) + \phi(x)L(y^*) - \phi(x)R(y) = L(y^*)\theta(x^*) - \phi(x)R(y),$$
for all $x, y \in \mathcal{A}$. By applying (2.4) and using the injectivity of L, we conclude that $xy = yx$, for all $x, y \in \mathcal{A}$. $\square$

Corollary 2.6 *Let $\mathcal{A}$ be a complex semi-simple Banach $*$-algebra and D a $(\theta, \phi)^*$-derivation of $\mathcal{A}$ with θ and ϕ automorphisms of $\mathcal{A}$. Then D is continuous.*

Proof. By Theorem 2.5, it suffices to show that every one-side centralizer of $\mathcal{A}$ is continuous. Then Theorem 2.5 gives a unique double (θ, ϕ)-centralizer (L, R) such that $D(x) = L(x^*) - R(x)$, for all $x \in \mathcal{A}$. Suppose that $x_n, y \in \mathcal{A}$ satisfying $x_n \to 0$ and $L(x_n) \to y$ as $n \to \infty$. By the closed graph theorem, we have to show that $y = 0$. To this end, let $a \in \mathcal{A}$. Then there exists $b \in \mathcal{A}$ such that $a = \phi(b)$. Then

$$yay = y\phi(b)y = \lim_{n\to\infty} y\phi(b)L(x_n) = \lim_{n\to\infty} yR(b)\theta(x_n).$$

Since $\mathcal{A}$ is semi-simple, a classical theorem of Johnson (see [2]) yields $\theta(x_n) \to 0$ as $n \to \infty$. It follows that $yay = 0$ and so $y = 0$. In a similar way, one can show that R is continuous. Assume that $x_n \in \mathcal{A}$ satisfying $x_n \to 0$ as $n \to \infty$. On the other hand, we know that the involution is continuous on a semi-simple Banach $*$-algebra (see [3]). Then

$$D(x_n) = L(x_n^*) - R(x_n) \to 0$$

as $n \to \infty$. This completes the proof. □

As a simple consequence of our result, we prove the automatic continuity of α-derivations. Other elegant proofs are given by Bresar and Villena [6, Corollary 4.3] and Hejazian and Janfada [7, Corollary 3.2].

Corollary 2.7 *Let $\mathcal{A}$ be a semi-simple Banach algebra and Δ an α-derivation of $\mathcal{A}$ with α an automorphism of $\mathcal{A}$. Then Δ is continuous.*

Proof. We define $\theta(x) := \alpha(x^*)$ and $\phi(x) := x$. Then Δ is a $(\theta, id)^*$-derivation and the result follows from Theorem 2.5. □

Remark 2.1 Some authors use the following definition for α-derivations:

$$\Delta(xy) = \Delta(x)y + \alpha(x)\Delta(y),$$

for all $x, y \in \mathcal{A}$. In this case it suffices to define $\theta(x) := x^*$. Then Δ is a $(\theta, \alpha)^*$-derivation and the automatic continuity of Δ follows from Theorem 2.5.

Corollary 2.8 *Let $\mathcal{A}$ be a complex semi-prime $*$-algebra. Then we have the following assertions:*

(i) Every double (θ, ϕ)-centralizer of $\mathcal{A}$ is $\mathbb{C}$-linear.
(ii) Every $(\theta, \phi)^*$-derivation of $\mathcal{A}$ is $\mathbb{R}$-linear, but not $\mathbb{C}$-linear.

Proof. (i) Let (L, R) be a double (θ, ϕ)-centralizer of $\mathcal{A}$ and define $a := L(\lambda x) - \lambda L(x)$ for $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Let $z \in \mathcal{A}$. Then there exists $w \in \mathcal{A}$ such that $z = \phi(w)$. It follows that

$$\begin{aligned} aza &= a\phi(w)(L(\lambda x) - \lambda L(x)) = a(\phi(w)L(\lambda x) - \lambda\phi(w)L(x)) \\ &= a(R(w)\theta(\lambda x) - \lambda\phi(w)L(x)) = \lambda a(R(w)\theta(x) - \phi(w)L(x)) = 0. \end{aligned}$$

Therefore, $a = 0$. In a similar way, we can show that R is $\mathbb{C}$-linear.

(ii) Let D be a $(\theta, \phi)^*$-derivation of $\mathcal{A}$. Then by Theorem 2.5, there exists a unique double (θ, ϕ)-centralizer (L, R) such that $D(x) = L(x^*) - R(x)$, for all $x \in \mathcal{A}$. It is clear that D is $\mathbb{R}$-linear. On the other hand

$$D(ix) = L(-ix^*) - R(ix) = -i(L(x^*) + R(x)),$$

for all $x \in \mathcal{A}$. This means that D is not $\mathbb{C}$-linear. □

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