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On closed semiregular rings

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Abstract A ring R is called *right closed semiregular* if for every closed principal right ideal aR of R, there exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$. The class of right closed semiregular rings contains all semiregular rings and all right principally extending rings. Some properties of these rings are studied and some results about semiregular rings and right principally extending rings are extended.

Keywords Semiregular rings · Closed semiregular rings · Principally extending rings

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1 Introduction

Semiregular rings were initiated by Nicholson in 1976. Following [2], a ring R is called a *semiregular* ring if for each $a \in R$, there exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$. Semiregular rings and their generalizations have been studied by many authors (see [4–7]). In this note, we define right closed semiregular rings, as a generalization of semiregular rings. A ring R is called *right closed semiregular* if for every closed principal right ideal aR of R, there exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$. Clearly, any semiregular ring is right closed semiregular but right closed semiregular rings need not be semiregular (see Example 3.1). In this paper our aim is to generalize some corresponding known results on semiregular rings. For example, we examine when direct sum of right closed semiregular rings is right closed semiregular. We investigate when image of a right closed semiregular ring R is also right closed semiregular. Moreover, we give some sufficient conditions under which a right closed semiregular ring is semiregular.

Throughout this paper R will denote an associative ring with identity, M a unitary right R-module. We will use the notation $N \leq^e M$ to indicate that N is an essential submodule of M (i.e. $\forall 0 \neq L \leq M, L \cap N \neq 0$); $N \ll M$ to indicate that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). The notation $N \leq^{\oplus} M$ denotes that N is a direct

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summand of M. For any non-empty subset X of R, $r(X)$ is reserved for the right annihilator of X in R. If $X = a$, we usually abbreviate it to $r(a)$.

2 Semiregular rings

Recall that an element $a \in R$ is *von Neumann regular* if $a \in aRa$. A ring R is called *von Neumann regular* if, for any $a \in R$, a is von Neumann regular.

Lemma 2.1 *The following conditions are equivalent for an element a of a ring R:*
(1) *There exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$.*
(2) *There exists $e^2 = e \in Ra$ such that $a(1-e) \in J(R)$.*
(3) *There exists a von Neumann regular element $b \in R$ with $a - b \in J(R)$.*
(4) *There exist two right ideals A and B of R such that $aR = A \oplus B$, where A is a direct summand of R and B is small in R.*

Proof. By [2, Lemma 2.1]. □

Lemma 2.2 *The following are equivalent for a right indecomposable ring R:*
(1) *R is semiregular;*
(2) *For every $1 \neq a \in R$, $aR \ll R$.*

Proof. It is clear. □

Let K and N be submodules of an R-module M. K is called a *supplement* of N in M if $M = K + N$ and K is minimal with respect to this property, or equivalently, $M = K + N$ and $K \cap N \ll K$ (see [3]).

Lemma 2.3 *A ring R is semiregular if and only if every principal right ideal I of R has a supplement which is a direct summand.*

Proof. Since R is a semiregular ring, for all $a \in R$, R has a decomposition $R = A \oplus B$ with $A \subseteq aR$ and $B \cap aR \ll R$. Thus $R = aR + B$ and so B is a supplement of aR which is a summand of R. Conversely, let $R = aR + B$, $aR \cap B \ll B$ and B is a direct summand of R. Hence there exists $A \subseteq aR$ with $R = A \oplus B$ and so R is semiregular. □

Proposition 2.4 *Let R be a semiregular ring. If $R = A + B$ such that $B \leq^{\oplus} R_R$ and $A \cap B$ is principal, then B contains a supplement of A in R_R.*

Proof. As R is semiregular, $A \cap B = I_1 \oplus I_2$ where $I_1 \leq^{\oplus} R_R$ and $I_2 \ll R_R$. Since $B \leq^{\oplus} R_R$, we have $I_2 \ll B$. Let $B = I_1 \oplus I_1'$. It implies that $A \cap B = I_1 \oplus (A \cap B \cap I_1') = I_1 \oplus (A \cap I_1')$. Let $\pi : I_1 \oplus I_1' \to I_1'$ be the natural projection. It follows that $A \cap I_1' = \pi(I_1 \oplus (A \cap I_1')) = \pi(A \cap B) = \pi(I_1 \oplus I_2) = \pi(I_2)$. Thus $A \cap I_1' \ll I_1'$ and $M = A + B = A + I_1 + I_1' = A + I_1'$. Therefore I_1' is a supplement of A in R_R that is contained in B. □

Corollary 2.5 *Let R be a semiregular principal ideal ring. If $R = A + aR$, then aR contains a supplement of A in R_R.*

Proof. By hypothesis, we have $aR = B_1 \oplus B_2$ where $B_1 \leq^{\oplus} R_R$ and $B_2 \ll R_R$. It shows that $R = A + B_1$. Since B_1 is a summand of R and $A \cap B_1$ is a principal right ideal of R, by Proposition 2.4, aR contains a supplement of A in R_R. □

It is well known that if $R = I_1 \oplus I_2$, where right ideal I_1 is simple as a submodule of the right module R_R and right ideal I_2 is uniserial as a submodule of the right module R_R with composition series $0 < A \subset I_2$, then R is semiperfect. Now we prove the following proposition:

Proposition 2.6 *Let I_1 and I_2 be right ideals of a ring R. If $R = I_1 \oplus I_2$, where I_1 is simple as a submodule of R_R and I_2 has composition series $0 < A \subset I_2$ as a submodule of R_R, then R is semiregular.*

Proof. Let B be a nonzero proper principal right ideal of R. Without loss of generality we may assume that $B = (a_1 + a_2)R$, where $0 \neq a_1 \in I_1$ and $0 \neq a_2 \in I_2$. It is clear that $B + I_1 = I_1 \oplus a_2R$ and that a_2R is either A or I_2. As I_1 is simple, we have that either $I_1 \leq B$ or $I_1 \cap B = 0$. If $I_1 \leq B$, then $a_2R = A$ and $B = I_1 \oplus A$, where $I_1 \leq^{\oplus} R_R$ and $A \ll R_R$. On the other hand, if $I_1 \cap B = 0$, then $B \oplus I_1 = I_1 \oplus a_2R$. If $a_2R = I_2$, then $B \leq^{\oplus} R_R$, and if $a_2R = A$, then $B \cong A$ is simple. But $R = B + I_2$ with $B \nleq I_2$ and hence $B \cap I_2 = 0$, which implies that $B \leq^{\oplus} R_R$. Therefore R is semiregular. □

3 Closed semiregular rings

A ring R is called *right closed semiregular* if for every closed principal right ideal aR of R, there exists $e^2 = e \in aR$ such that $(1 - e)a \in J(R)$.

Clearly, any semiregular ring is right closed semiregular but right closed semiregular rings need not be semiregular (see Example 3.1).

Recall that a ring R is called a *right principally extending* (or *right P-extending*) ring if every principal right ideal of R is essential in a direct summand of R. Obviously, any right P-extending ring is right closed semiregular.

Example 3.1 Let $\mathbb{Z}$ be the ring of all integers. Since $\mathbb{Z}$ is extending, it is right closed semiregular. But $\mathbb{Z}$ is not semiregular since $n\mathbb{Z}$ has no supplement in $\mathbb{Z}$, for any $n \geq 2$.

A *right distributive ring* is a ring whose lattice of right ideals is distributive.

Proposition 3.1 *Let R be a right distributive closed semiregular ring. Then, for every $e^2 = e \in R$, eR is right closed semiregular.*

Proof. Let aR be a closed principal right ideal of eR. Since eR is closed in R, aR is closed in R. Then there exists a right ideal I of R such that $R = aR + I$ and $aR \cap I \ll I$. Thus $eR = (eR \cap I) + aR$. Note that $I = (I \cap eR) \oplus (I \cap (1 - e)R)$ since R is a right distributive ring. Then $(eR \cap I) \cap aR = aR \cap I \ll (I \cap eR)$. Therefore eR is right closed semiregular. □

Proposition 3.2 *Let R be a semiprimitive ring. Then R is a right closed semiregular ring if and only if R is a right P-extending ring.*

Proof. It is clear. □

We know that any direct sum of semiregular rings is again semiregular [2, Theorem 1.10]. But the direct sum of closed semiregular rings need not be closed semiregular in general.

Example 3.2 Let $S = \mathbb{Z}[x]$, where $\mathbb{Z}$ is the ring of all integers. Set $R = S \oplus S$ then, by [1, Example 2.4], R is not P-extending. Since $J(R) = 0$, we see that R is not right closed semiregular.

Theorem 3.3 *Let $R = R_1 \oplus R_2$ be a right distributive ring. Then R is right closed semiregular if and only if each R_i is right closed semiregular.*

Proof. Let aR be a closed principal right ideal of R. Since R is right distributive, $aR = ((aR) \cap R_1) \oplus ((aR) \cap R_2)$. Let $a = a_1 + a_2$ where $a_1 \in R_1$ and $a_2 \in R_2$. Then $a_1R = aR \cap R_1$ and $a_2R = aR \cap R_2$. Thus there exists $A_i \leq R_i$ such that $R_i = a_iR + A_i$ and $(a_iR) \cap A_i \ll A_i$, for each $i = 1, 2$. Then $R = a_1R + a_2R + A_1 + A_2 = aR + A_1 + A_2$. Now we prove $aR \cap (A_1 + A_2) \ll A_1 + A_2$. Note that

$$\begin{aligned} aR \cap (A_1 + A_2) &= (aR \cap R_1 + aR \cap R_2) \cap (A_1 + A_2) \\ &\leq (A_1 \cap ((aR \cap R_1) + R_2)) + (A_2 \cap ((aR \cap R_2) + R_2)) \\ &\leq (aR \cap R_1) \cap (A_1 + R_2) + (aR \cap R_2) \cap (A_2 + R_1). \end{aligned}$$

On the other hand, $(aR \cap R_1) \cap (A_1 + R_2) = (a_1R) \cap (A_1 + R_2) \leq A_1 \cap (a_1R + R_2) \leq a_1R \cap (A_1 + R_2)$ implies that $a_1R \cap (A_1 + R_2) = A_1 \cap (a_1R + R_2) = (a_1R) \cap A_1$. Similarly, $a_2R \cap (A_2 + R_1) = A_2 \cap (a_2R + R_1) = (a_2R) \cap A_2$. Since $a_iR \cap A_i \ll A_i$, $a_1R \cap A_1 + a_2R \cap A_2 \ll A_1 + A_2$. Therefore, $aR \cap (A_1 + A_2) \ll A_1 + A_2$. The converse is clear by Proposition 3.1. $\square$

Proposition 3.4 *Let R be a ring. If any nonzero proper closed principal right ideal of R is maximal in R, then R is P-extending and hence R is right closed semiregular.*

Proof. Suppose that aR is a proper closed right ideal and maximal in R. Then aR is not essential in R. So, there exists $x \in R \backslash aR$ such that for all $0 \neq r \in R$ with $xr \neq 0$, then $xr \notin aR$. Set $K = aR + xR$, then K contains aR as a proper right ideal. Thus $K = R$ and, since $aR \cap xR = 0$, we have $R = xR \oplus aR$. $\square$

Proposition 3.5 *Let I be a small right ideal of a ring R and R/I a right closed semiregular ring. Suppose that every nonzero closed principal right ideal of R contains I. Then R is right closed semiregular.*

Proof. Let $f : R \to R/I$ denote the natural map. Let $0 \neq aR$ be closed in R and $f(aR) \leq^e K \leq R/I$, then $aR = aR + I = f^{-1}f(aR) \leq^e f^{-1}(K)$. Hence $aR = f^{-1}(K)$ and $f(aR) = K$ is a closed principal right ideal of R/I. Since R/I is right closed semiregular, there exists $H \leq R/I$ such that $f(aR) + H = R/I$ and $f(aR) \cap H \ll H$. Then $aR + f^{-1}(H) = R$ and $f^{-1}(f(aR) \cap H) = aR \cap f^{-1}(H) \ll f^{-1}(H)$. Thus R is right closed semiregular. $\square$

Proposition 3.6 *Let R be a right closed semiregular ring. Then:*

(1) *Every closed principal right ideal of $R/J(R)$ is a direct summand.*

(2) *If I is a right ideal of R such that $I \cap J(R) = 0$, then every closed principal right ideal $A \subseteq I$ is a direct summand of I.*

Proof. (1) Note that $R/J(R)$ contains no closed small principal right ideal. Thus every closed principal right ideal of $R/J(R)$ is a direct summand.

(2) It is clear by (1). $\square$

We denote $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal I of $R\}$. A module M is called a *singular module* provided $Z(M) = M$. M is called a *nonsingular module* provided $Z(M) = 0$. A ring R is called a *right singular* (resp. *right nonsingular*) ring if R_R is a singular (resp. nonsingular) module.

Lemma 3.7 *Let R/I be a right nonsingular ring. If H/I is a closed right ideal of R/I, then H is closed in R.*

Proof. Suppose that $I \subseteq H \leq^e T \leq R$. Then $\frac{T}{H} \cong \frac{T/I}{H/I}$ is singular. Since R/I is right nonsingular, $\frac{H}{I} \leq^e \frac{T}{I}$. Hence $\frac{H}{I} = \frac{T}{I}$ and $H = I$. This means that H is closed in R. $\square$

Theorem 3.8 *Let R be a right closed semiregular ring. Then any right nonsingular image of R is also right closed semiregular.*

Proof. Let R be a right closed semiregular ring and, for right ideal $I \subseteq R$, R/I a right nonsingular ring. Assume that $\frac{aR+I}{I}$ be a closed principal right ideal of R/I. By Lemma 3.7, $aR + I$ is closed in R and so aR is closed in R. Since R is right closed semiregular, there exists a right ideal K of R such that $R = aR+K$ and $aR \cap K \ll K$. Hence $\frac{R}{I} = \frac{aR+I}{I} + \frac{K+I}{I}$. We have $\frac{aR+I}{I} \cap \frac{K+I}{I} = \frac{(aR\cap K)+I}{I} \ll \frac{K+I}{I}$ since $aR \cap K \ll K$. Thus R/I is right closed semiregular. $\square$

Note that in Theorem 3.8 is not necessary that image of R to be right nonsingular. For example, $\mathbb{Z}$ is a right closed semiregular ring and, for any prime p, $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is simple and is right closed semiregular but $\mathbb{Z}_p$ is singular.

Corollary 3.9 *Let R be a right closed semiregular ring such that $R/J(R)$ is right nonsingular. Then $R/J(R)$ is right extending.*

Corollary 3.10 *Let R be a right nonsingular ring. Then the following are equivalent:*
(1) *R is right closed semiregular;*
(2) *Every closed cyclic submodule N of a nonsingular cyclic R-module M has a supplement in M;*
(3) *Every principal right ideal of R is right closed semiregular.*

Lemma 3.11 *Let R be a ring and I a right ideal of R. Assume that H/I is closed in R/I. If $r(a) = r(a+I)$ for all $a \in R\backslash I$, then H is closed in R.*

Proof. Suppose that $I \subseteq H \leq^e K \subseteq R$. Then for any $k \in K\backslash I$, there exists $r_0 \in R$ such that $0 \neq kr_0 \in H$. Since $r(k) = r(k+I)$, $kr_0 + I = (k+I)r_0 \neq 0$, hence $0 \neq kr_0 + I \in H/I$. Thus $H/I \leq^e K/I$. As H/I is closed in R/I, we have that $H = K$ and so H is closed in R. $\square$

Corollary 3.12 *Let R be a right closed semiregular ring and I a right ideal of R. If $r(a) = r(a+I)$ for all $a \in R\backslash I$, then R/I is also a right closed semiregular ring.*

Proof. By Lemma 3.11, for any closed right ideal of R/I, there exists a closed right ideal of R. The rest is similar to the proof of Theorem 3.8. $\square$

Note that if every right ideal of R is closed in R (for example, R is semisimple), then R is right closed semiregular if and only if R is semiregular. For another case, we give the following result which is the relation between right closed semiregular rings and semiregular rings.

Theorem 3.13 *Let R be a ring. Suppose that for any $a \in R$, there exists a closed principal right ideal I of R such that either $aR = I + T$ or $I = aR + T'$ for some right ideals $T, T' \ll R$. Then R is semiregular if and only if R is right closed semiregular.*

Proof. Suppose that R is a right closed semiregular ring and $a \in R$.

Case (1): Assume that there exists a closed principal right ideal I such that $aR = I + T$ for some small right ideal T of R. Then, by [3, 41.1(4)], R is semiregular.

Case (2): Assume that there exists a closed principal right ideal I of R such that $I = aR + T'$ for some small right ideal T' of R. Since R is a right closed semiregular ring, there exists a right ideal H of R such that $R = H + I$ and $H \cap I \ll H$. Hence $R = H + aR + T'$, and so $R = H + aR$ as $T' \ll R$. Note that $H \cap aR \leq H \cap I \ll H$. Thus R is semiregular. The converse is clear. $\square$

Note that rings that every principal right ideal of them is closed, satisfy the condition of Theorem 3.13 (for example, $\mathbb{Z}_p$ satisfies the condition of Theorem 3.13 for any prime p).

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