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Functions preserving slowly oscillating double sequences

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Abstract A double sequence $\boldsymbol{x} = \{x_{k,l}\}$ of points in $\boldsymbol{R}$ is slowly oscillating if for any given $\varepsilon > 0$, there exist $\alpha = \alpha(\varepsilon) > 0$, $\delta = \delta(\varepsilon) > 0$, and $N = N(\varepsilon)$ such that $|x_{k,l} - x_{s,t}| < \varepsilon$ whenever $k, l \geq N(\varepsilon)$ and $k \leq s \leq (1+\alpha)k$, $l \leq t \leq (1+\delta)l$. We study continuity type properties of factorable double functions defined on a double subset $A \times A$ of $\boldsymbol{R}^2$ into $\boldsymbol{R}$, and obtain interesting results related to uniform continuity, sequential continuity, and a newly introduced type of continuity of factorable double functions defined on a double subset $A \times A$ of $\boldsymbol{R}^2$ into $\boldsymbol{R}$.

Keywords Multiple sequences and series · Matrix methods · Continuity and related questions

Mathematics Subject Classification (2010) 40B05 · 40C05

1 Introduction

In 1900, Pringsheim [21] introduced the concept of convergence of real double sequences. Four years later, Hardy [11] introduced the notion of regular convergence for double sequences in the sense that double sequence has a limit in Pringsheim's sense and has one sided limits (see also [8, 22]). A considerable number of papers which appeared in recent years study double sequences from various points of view (see [1, 6, 7, 15–19]). Some results in the investigation are generalizations of known results concerning simple sequences to certain classes of double sequences, while other results reflect a specific nature of the Pringsheim convergence (e.g., the fact that a double sequence may converge without being bounded). First usage of the slowly oscillating concept of real single sequences goes back to beginning of twentieth century ([10, 1907,

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Hardy], and ([13, 1910, Landou]) while the slowly oscillating concept of real double sequences seems to be first studied in [12, 1939, Knopp] (see also [9], and [14]).

The aim of this paper is to investigate slowly oscillating double sequences and newly defined types of continuities for factorable double functions.

2 Preliminaries

Throughout this paper a factorable double sequence will mean a double sequence $\boldsymbol{x} = \{x_{m,n}\}$ of real numbers which can be written in the form that $x_{m,n} = x_m^{m,n}.x_n^{m,n}$ where $x_m^{m,n}$ and $x_n^{m,n}$ are real numbers for each $m, n \in \boldsymbol{N}$; a factorable double function f will mean a real valued function f defined on a double subset $E \times E$ of $\boldsymbol{R}^2$ such that there are $f^{x_1,x_2}(x_1)$, and $f^{x_1,x_2}(x_2)$ satisfying $f(x_1, x_2) = f^{x_1,x_2}(x_1).f^{x_1,x_2}(x_2)$ for all $(x_1, x_2) \in E \times E$; and a double sequence $(f_{m,n})$ of two dimensional factorable real-valued functions from a double interval $I \times I$ of $\boldsymbol{R}^2$ will be called uniformly P-convergent to a function f, if for each $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $m, n > N$ implies that $|f_{mn}(x) - f(x)| < \varepsilon$ for all $x \in I \times I$. A (single) sequence $\boldsymbol{x} = (x_k)$ is said to be λ-statistically convergent to a number L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0,$$

where (λ_n) is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, and $I_n = [n - \lambda_n + 1, n]$ for each $n \in \boldsymbol{N}$.

Definition 2.1 (Pringsheim, **1900 [21]**) A double sequence $\boldsymbol{x} = \{x_{k,l}\}$ is Cauchy provided that, given an $\varepsilon > 0$ there exists an $N \in \boldsymbol{N}$ such that $|x_{k,l} - x_{s,t}| < \varepsilon$ whenever $k, l, s, t > N$.

Definition 2.2 (Pringsheim, **1900 [21]**) A double sequence $\boldsymbol{x} = \{x_{k,l}\}$ has a *Pringsheim limit* L (denoted by P-lim $x = L$) provided that, given an $\varepsilon > 0$ there exists an $N \in \boldsymbol{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. Such an $\boldsymbol{x}$ is described more briefly as "P-convergent".

If for every $M > 0$ there are $n_1, n_2 \in \boldsymbol{N}$ such that $|x_{m,n}| > M$ whenever $m > n_1$, $n > n_2$, then $\boldsymbol{x} = \{x_{m,n}\}$ is said to be definitely divergent. This is denoted by P-$\lim |\boldsymbol{x}| = \infty$.

A double sequence $\boldsymbol{x} = \{x_{m,n}\}$ is bounded if there is an $M > 0$ such that $|x_{m,n}| < M$ for all $m, n \in \boldsymbol{N}$. Notice that a P-convergent double sequence need not be bounded.

Definition 2.3 (Patterson, **2000 [20]**) *A double sequence* $\boldsymbol{y}$ *is a double subsequence of* $\boldsymbol{x}$ *provided that there exist increasing index sequences* $\{n_j\}$ *and* $\{k_j\}$ *such that, if* $\{x_j\} = \{x_{n_j,k_j}\}$, *then* $\boldsymbol{y}$ *is formed by*

$$\begin{matrix} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & -. \end{matrix}$$

3 Results

Definition 3.1 ([2],[14]) *A double sequence* $\boldsymbol{x} = \{x_{k,l}\}$ *of points in* $\boldsymbol{R}$ *is called slowly oscillating if for any given* $\varepsilon > 0$*, there exist* $\alpha = \alpha(\varepsilon)$*,* $\delta = \delta(\varepsilon) > 0$*, and* $N = N(\varepsilon)$ *such that* $|x_{k,l} - x_{s,t}| < \varepsilon$*, if* $k, l \geq N(\varepsilon)$ *and* $k \leq s \leq (1+\alpha)k$*,* $l \leq t \leq (1+\delta)l$*.*

Any Cauchy double sequence is slowly oscillating, so any P-convergent double sequence is. The converse is easily seen to be false as in the single dimensional case as the following example shows.

Example 3.1 Write $s_n = \log n$ for each positive integer n. Then the double sequence defined by

$$\begin{matrix} s_1 & s_2 & s_3 & s_4 & \cdots \\ s_2 & s_2 & s_3 & s_4 & \cdots \\ s_3 & s_3 & s_3 & s_4 & \cdots \\ s_4 & s_4 & s_4 & s_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

is not P-convergent nor Cauchy, however it is a slowly oscillating double sequence.

Theorem 3.2 *If a factorable double function* f *defined on a double subset* $A \times A$ *of* $\boldsymbol{R}^2$ *is uniformly continuous, then it preserves factorable slowly oscillating double sequences from* $A \times A$*.*

Proof. Suppose that f is uniformly continuous, and let

$$\begin{matrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ x_{3,1} & x_{3,2} & x_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be any slowly oscillating factorable double sequence. To prove that $\{f(x_{n,m})\}$ is slowly oscillating, take any $\varepsilon > 0$. Uniform continuity of f implies that there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ for $x, y \in A \times A$ where the absolute value in the latter in $\boldsymbol{R}^2$. Since $\{x_{n,m}\}$ is slowly oscillating for this δ, there exist $\alpha_1 = \alpha_1(\delta) > 0$, $\delta_1 = \delta_1(\delta) > 0$ and $N = N(\delta)$ such that $|x_{k,l} - x_{s,t}| < \delta$, if $k, l \geq N(\delta)$ and $k \leq s \leq (1+\alpha_1)k$, $l \leq t \leq (1+\delta_1)l$. Hence $|x_{k,l} - x_{s,t}| < \delta$, if $k, l \geq N(\delta)$ and $k \leq s \leq (1+\alpha_1)k$, $l \leq t \leq (1+\delta_1)l$. It follows from this that $\{f(x_{n,m})\}$ is slowly oscillating. This completes the proof of the theorem. □

Theorem 3.3 *If a factorable double function* f *defined on a double subset* $A \times A$ *of* $\boldsymbol{R}^2$ *preserves factorable slowly oscillating double sequences from* $A \times A$*, then it preserves factorable* P*-convergent double sequences from* $A \times A$*.*

Proof. Suppose that f preserves factorable slowly oscillating double sequences from $A \times A$. Let

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be any P-convergent factorable double sequence with P-limit L. Then the sequence

$$\begin{array}{cccccccc}
a_{1,1}\ L & a_{1,2}\ L & a_{1,3}\ L & ... & & & \\
L & L & L & L & L & L & ... \\
a_{2,1}\ L & a_{2,2}\ L & a_{2,3}\ L & \cdots & & & \\
L & L & L & L & L & L & ... \\
a_{3,1}\ L & a_{3,2}\ L & a_{3,3}\ L & \cdots & & & \\
L & L & L & L & L & L & ... \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

is also P-convergent with P-limit L. Since any P-convergent double sequence is slowly oscillating, this sequence is slowly oscillating. So the transformed sequence of the sequence is slowly oscillating. Thus it follows that

$$\begin{array}{ccccccc}
f(a_{1,1})\ f(L) & f(a_{1,2})\ f(L) & f(a_{1,3})\ f(L) & ... & & & \\
f(L) & f(L) & f(L) & f(L) & f(L) & f(L) & ... \\
f(a_{2,1})\ f(L) & f(a_{2,2})\ f(L) & f(a_{2,3})\ f(L) & \cdots & & & \\
f(L) & f(L) & f(L) & f(L) & f(L) & f(L) & ... \\
f(a_{3,1})\ f(L) & f(a_{3,2})\ f(L) & f(a_{3,3})\ f(L) & \cdots & & & \\
f(L) & f(L) & f(L) & L & f(L) & f(L) & ... \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

is a slowly oscillating double sequence. Hence

$$\begin{array}{cccc}
f(a_{1,1}) - f(L) & f(a_{1,2}) - f(L) & f(a_{1,3}) - f(L) & ... \\
f(a_{2,1}) - f(L) & f(a_{2,2}) - f(L) & f(a_{2,3}) - f(L) & \cdots \\
f(a_{3,1}) - f(L) & f(a_{3,2}) - f(L) & f(a_{3,3}) - f(L) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

a P-convergent factorable double sequence with P-limit 0. This implies that the transformed double sequence

$$\begin{array}{cccc}
f(a_{1,1}) & f(a_{1,2}) & f(a_{1,3}) & ... \\
f(a_{2,1}) & f(a_{2,2}) & f(a_{2,3}) & \cdots \\
f(a_{3,1}) & f(a_{3,2}) & f(a_{3,3}) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

is P-convergent with P-limit $f(L)$. This completes the proof of the theorem. □

Corollary 3.4 *If a factorable double function f defined on a double subset $A\times A$ of $\boldsymbol{R}^2$ preserves factorable slowly oscillating double sequences from $A \times A$, then it preserves λ-statistically convergent (single) sequences from $A \times A$.*

Proof. The proof follows from the regularity and subsequentiality of λ-statistically sequential method so is omitted (see [4]). □

Theorem 3.5 *Suppose that $A \times A$ is a bounded subset of $\boldsymbol{R}^2$. A two dimensional factorable real-valued function is uniformly continuous on $A\times A$ if and only if it preserves factorable slowly oscillating double sequences from $A\times A$.*

Proof. It immediately follows from Theorem 3.2 that two dimensional uniformly continuous functions preserve slowly oscillating sequences. Conversely, suppose that f defined on $A \times A$ is not uniformly continuous. Then there exists an $\varepsilon > 0$ such that for any $\delta > 0$ there exist $(a,b),(\bar{a},\bar{b}) \in A \times A$ with $\sqrt{(a-\bar{a})^2+(b-\bar{b})^2} < \delta$ but $|f(a,b)-f(\bar{a},b)| \geq \varepsilon$, $\left|f(a,b)-f(a,\bar{b})\right| \geq \varepsilon$, and $\left|f(a,b)-f(\bar{a},\bar{b})\right| \geq \varepsilon$, respectively. Thus for each positive integer n we can choose $(a_n,b_n),(\bar{a}_n,\bar{b}_n) \in A \times A$ with $\sqrt{(a_n-\bar{a}_n)^2+(b_n-\bar{b}_n)^2} < \frac{1}{n}$ but $|f(a_n,b_n)-f(\bar{a}_n,b)| \geq \varepsilon$, $|f(a_n,b_n)-f(a_n,\bar{b}_n)| \geq \varepsilon$, and $\left|f(a_n,b_n)-f(\bar{a}_n,\bar{b}_n)\right| \geq \varepsilon$. Then since $A \times A$ is bounded there exists a slowly oscillating double subsequence of the double sequence $\{a_n,b_n\}$ by a simple extension of Bolzano-Weierstrass theorem, $\{a_{n_k},b_{n_k}\}$ say. Thus the corresponding double sequence $\{\bar{a}_{n_k},\bar{b}_{n_k}\}$ has a slowly oscillating double subsequence, say $\{\bar{a}_{n_{k_m}},\bar{b}_{n_{k_m}}\}$. It is easy to see that $\{\bar{a}_{n_{k_m}},\bar{b}_{n_{k_m}}\}$ is a slowly oscillating sequence. Since f preserves slowly oscillating double sequences by the hypothesis, $\{f(a_{n_{k_m}},b_{n_{k_m}})\}$ and $\{f(\bar{a}_{n_{k_m}},\bar{b}_{n_{k_m}})\}$ are slowly oscillating. This is impossible. This contradiction completes the proof of the theorem. □

It is well known that uniform limit of a sequence of continuous functions is continuous. This is also true for two dimensional factorable real-valued functions that preserve slowly oscillating double sequences, i.e. uniform limit of a sequence of two dimensional factorable real-valued functions preserving slowly oscillating double sequences from $A \times A$ of $\boldsymbol{R}^2$ also preserves slowly oscillating double sequences from $A \times A$.

Theorem 3.6 *If (f_n) is a sequence of two dimensional factorable real-valued functions preserving slowly oscillating double sequences from a double interval $I \times I$ of $\boldsymbol{R}^2$ and (f_n) is uniformly convergent to a function f, then f preserves slowly oscillating double sequences from $I \times I$.*

Proof. Let (x_{nk}) be a slowly oscillating double sequence and $\varepsilon > 0$. Then there exists a positive integer N such that $|f_n(a,b)-f(\bar{a},\bar{b})| < \frac{\varepsilon}{3}$ for all $(a,b),(\bar{a},\bar{b}) \in I \times I$ whenever $n \geq N$. As f_N preserves slowly oscillating double sequences from $I \times I$, there exist a $\delta > 0$ and a positive integer $N_1 = N_1(\varepsilon)$, greater than N, such that

$$|f_N(x_{k,l})-f_N(x_{s,t})| < \frac{\varepsilon}{3},$$

for $n \geq N_1$ and $k \leq s \leq (1+\delta)k$, $l \leq t \leq (1+\delta)l$. Now for $n \geq N_1$ and $k \leq s \leq (1+\delta)k$, $l \leq t \leq (1+\delta)l$. Thus for $n \geq N_1$ and $k \leq s \leq (1+\delta)k$, $l \leq t \leq (1+\delta)l$ we have

$$\begin{aligned} |f(x_{k,l})-f(x_{s,t})| &\leq |f(x_{k,l})-f_N(x_{k,l})| + |f_N(x_{k,l})-f_N(x_{s,t})| \\ &+ |f_N(x_{s,t})-f(x_{s,t})| \leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This completes the proof of the theorem. □

Theorem 3.7 *If $(f_{m,n})$ is a double sequence of two dimensional factorable real-valued functions preserving slowly oscillating double sequences from a double interval $I \times I$ of $\boldsymbol{R}^2$ and $(f_{m,n})$ is uniformly P-convergent to a function f, then f preserves slowly oscillating double sequences from $I \times I$.*

The proof is similar to the last theorem and as of such it is omitted.

4 Conclusion

It is easy to see that Cauchy double sequences are slowly oscillating double. The converse is easily seen to be false as in the single dimensional case ([3], [5], [23]). One should also note that there are nice connections between double slowly oscillating sequences and uniform continuity of two-dimensional real-valued functions. This is illustrated through the following theorem. Suppose that $I \times I$ is any two dimensional bounded interval. Then a two dimensional factorable real-valued function is uniformly continuous on $I \times I$ if and only if it is defined on $I \times I$ and preserves factorable double slowly oscillating sequences from $I \times I$. Extensions and variations of the above theorem was also presented.

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