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On Pachpatte, Pečarić and Love's inequalities

Chang-Jian Zhao · Winh-Sum Cheung

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Abstract In the article we establish new Hardy type inequalities similar to Pachpatte, Pečarić and Love's inequalities. These results in special cases yield Pachpatte, Pečarić and Love's results.

Keywords Hardy's inequality · Hölder's inequality · Pachpatte's inequality · Pečarić-Love's inequality

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1 Introduction

The classical inequality due to Hardy can be stated as follows (see [2]).

Theorem A. *If* $p > 1$, $f(x) \geq 0$ *for* $0 < x < \infty$, *and* $F(x) = \frac{1}{x}\int_0^x f(t)dt$, *then*

$$\int_0^\infty F(x)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx, \tag{1.1}$$

unless $f \equiv 0$.

A simple proof of Hardy's integral inequality was given in [3]. One of the best known and interesting generalizations of the inequality (1.1) given by Hardy [3] himself can be stated as follows.

Chang-Jian Zhao
Department of Mathematics
China Jiliang University
Hangzhou 310018, P.R. China
E-mail: chjzhao@163.com chjzhao@aliyun.com

Winh-Sum Cheung
Department of Mathematics
University of Hong Kong
Pokfulam Road, Hong Kong
E-mail: wscheung@hku.hk

Theorem B. *If $p > 1, m \neq 1$, $f(x) \geq 0$ for $0 < x < \infty$, and $F(x)$ is defined by*

$$F(x) = \int_0^x f(t)dt, m > 1; \quad F(x) = \int_x^\infty f(t)dt, m < 1,$$

then

$$\int_0^\infty x^{-m} F(x)^p dx < \left(\frac{p}{|m-1|}\right)^p \int_0^\infty x^{p-m} f(x)^p dx, \tag{1.2}$$

unless $f \equiv 0$. The constant is the best possible.

The inequalities given in (1.1) and (1.2) which later went by the name of Hardy′s inequalities, led to a great many papers dealing with alternative proofs, various generalizations, and numerous variants and applications in analysis (see [1], [5-6], [8-10] and [12-15]). In particular, Pachpatte [11] established some new inequalities (2.4) and (2.7) (see Section 2) similar to inequalities (1.1) and (1.2). Very recently, Leng and Feng [7] proved some new Hardy-type integral inequalities. In the present paper we establish new Hardy type inequalities similar to Pachpatte, Pečarić and Love's inequalities. These results in special cases yield Pachpatte, Pečarić and Love's results.

2 Main results

Theorem 2.1 *Let $0 \leq a < b < \infty, 0 \leq c < d < \infty, p > 0, q > 0$ and $\alpha > 0$ be constants. Let $w(x,y)$ and $r(x,y)$ be positive and locally absolutely continuous functions on $(a,b) \times (c,d)$. Let $f(x,y)$ be almost everywhere nonnegative and integrable function on $(a,b) \times (c,d)$, and let*

$$F(x,y) = \frac{1}{r(x,y)} \int_a^x \int_c^y \frac{r(s,t) f(s,t)}{st \log(bd/st)} dsdt,$$

for $(x,y) \in [a,b) \times [c,d)$, and $F(x,y) = o([(b-x)(d-y)]^{-q/p})$ as $x \to b^-, y \to d^-$. If $p > 1$ and

$$\begin{aligned} D(x,y) = 1 &- \frac{1}{q} \frac{x}{w(x,y)} \frac{\partial w(x,y)}{\partial x} \log\left(\frac{bd}{xy}\right) \\ &+ \frac{1}{q} \frac{x}{r(x,y)} \frac{\partial r(x,y)}{\partial x} \log\left(\frac{bd}{xy}\right) \geq \frac{1}{\alpha}, \end{aligned} \tag{2.1}$$

for almost all $(x,y) \in (a,b) \times (c,d)$, then

$$\begin{aligned} &\int_c^d \int_a^b w(x,y) \frac{1}{x} \left(\log\left(\frac{bd}{xy}\right)\right)^{q-1} F(x,y)^p dxdy \\ &\leq \left(\frac{\alpha p}{q}\right)^p \int_c^d \int_a^b \left[w(x,y) x^{p-1} G(x,y)^p \left(\log\left(\frac{bd}{xy}\right)\right)^{q+p-1} \right] dxdy, \end{aligned} \tag{2.2}$$

where

$$G(x,y) = \frac{1}{r(x,y)} \int_c^y \frac{r(x,t) f(x,t)}{xt \log(bd/xt)} dt.$$

If $0 < p < 1$ *and the reverse inequality in* (2.1) *holds, then the reverse inequality* (2.2) *also holds.*

Remark 2.1 Let $f(x,y), w(x,y)$ and $r(x,y)$ reduce to $f(x)$, $w(x)$ and $r(x)$, respectively and in view of the following suitable modifications in Theorem 2.1,

$$F(x,y) \to F(x) = \frac{1}{r(x)} \int_a^x \frac{r(t)f(t)}{t\log(b/t)} dt,$$
$$D(x,y) \to D(x) = 1 - \frac{1}{q}\frac{xw'(x)}{w(x)} \log\left(\frac{b}{x}\right) + \frac{1}{q}\frac{xr'(x)}{r(x)} \log\left(\frac{b}{x}\right) \geq \frac{1}{\alpha}, \tag{2.3}$$

and $G(x,y) \to G(x) = \frac{f(x)}{x\log(b/x)}$, then (2.1) changes to the following result

$$\int_a^b w(x)\frac{1}{x}\left(\log\left(\frac{b}{x}\right)\right)^{q-1} F(x)^p dx$$
$$\leq \left(\frac{\alpha p}{q}\right)^p \int_a^b \left[w(x)x^{-1}f(x)^p \left(\log\left(\frac{b}{x}\right)\right)^{q-1}\right] dx. \tag{2.4}$$

If $0 < p < 1$ and the reverse inequality in (2.3) holds, then the reverse inequality (2.4) also holds. This is just a well-known result established by Pachpatte [11].

On the other hand, if $w(x) = 1$, the inequality (2.4) reduces to

$$\int_a^b \frac{1}{x}\left(\log\left(\frac{b}{x}\right)\right)^{q-1} F(x)^p dx \leq \left(\frac{\alpha p}{q}\right)^p \int_a^b \left[x^{-1}f(x)^p \left(\log\left(\frac{b}{x}\right)\right)^{q-1}\right] dx.$$

This is just an inequality recently established by Pečarić and Love in [14]. This in turn is the further generalization of the inequality given by Pachpatte in [13].

Theorem 2.2 *Let* $0 < a < b < \infty, 0 < c < d < \infty, p > 0, q > 0$ *and* $\beta > 0$ *be constants. Let* $w(x,y)$ *and* $r(x,y)$ *be positive and locally absolutely continuous functions on* $(a,b)\times(c,d)$. *Let* $f(x,y)$ *be almost everywhere nonnegative and integrable function on* $(a,b)\times(c,d)$, *and let*

$$F(x,y) = \frac{1}{r(x,y)} \int_x^b \int_y^d \frac{r(s,t)f(s,t)}{st\log(st/ac)} dsdt,$$

$(x,y) \in (a,b]\times(c,d]$, *and* $F(x,y) = o([(x-a)(y-c)]^{-q/p})$ *as* $x \to a^+, y \to c^+$. *If* $p > 1$ *and*

$$E(x,y) = 1 + \frac{1}{q}\frac{y}{w(x,y)}\frac{\partial w(x,y)}{\partial y} \log\left(\frac{xy}{ac}\right)$$
$$- \frac{1}{q}\frac{y}{r(x,y)}\frac{\partial r(x,y)}{\partial y} \log\left(\frac{xy}{ac}\right) \geq \frac{1}{\beta}, \tag{2.5}$$

for almost all $(x,y) \in (a,b)\times(c,d)$, *then*

$$\begin{aligned}&\int_c^d\int_a^b w(x,y)\frac{1}{y}\left(\log\left(\frac{xy}{ac}\right)\right)^{q-1}F(x,y)^p dxdy\\ &\le\left(\frac{\beta p}{q}\right)^p\int_c^d\int_a^b\left[w(x,y)y^{p-1}H(x,y)^p\left(\log\left(\frac{xy}{ac}\right)\right)^{q+p-1}\right]dxdy,\end{aligned} \tag{2.6}$$

where

$$H(x,y)=\frac{1}{r(x,y)}\int_x^a\frac{r(s,y)f(s,y)}{ys\log(ys/ac)}ds.$$

If $0<p<1$ *and the reverse inequality in* (2.5) *holds, then the reverse inequality* (2.6) *also holds.*

Remark 2.2 Let $f(x,y), w(x,y)$ and $r(x,y)$ reduce to $f(x)$, $w(x)$ and $r(x)$, respectively and with suitable modifications in Theorem 2.2, then (2.6) changes to the following inequality

$$\begin{aligned}&\int_a^b w(x)\frac{1}{x}\left(\log\left(\frac{x}{b}\right)\right)^{q-1}F(x)^p dx\\ &\le\left(\frac{\beta p}{q}\right)^p\int_a^b\left[w(x)x^{-1}f(x)^p\left(\log\left(\frac{x}{b}\right)\right)^{q-1}\right]dx.\end{aligned} \tag{2.7}$$

If $0<p<1$ and the reverse inequality in (2.5) holds, then the reverse inequality (2.7) also holds. This is just a well-known result established by Pachpatte [14].

On the other hand, if $w(x)=1$, the inequality (2.7) reduces to

$$\int_a^b\frac{1}{x}\left(\log\left(\frac{x}{b}\right)\right)^{q-1}F(x)^p dx\le\left(\frac{\beta p}{q}\right)^p\int_a^b\left[x^{-1}f(x)^p\left(\log\left(\frac{x}{b}\right)\right)^{q-1}\right]dx.$$

This is just an inequality recently established by Pečarić and Love in [14]. This in turn is the further generalization of the inequality given by Pachpatte in [13].

3 Proof of Theorems

Proof of Theorem 2.1 If $u(x,y)=w(x,y)F(x,y)^p$ and in view of

$$F(x,y)=\frac{1}{r(x,y)}\int_a^x\int_c^y\frac{r(s,t)f(s,t)}{st\log(bd/st)}dsdt,$$

for $(x,y)\in[a,b)\times[c,d)$, then

$$\begin{aligned}\frac{\partial u(x,y)}{\partial x}&=\frac{\partial w(x,y)}{\partial x}F(x,y)^p+w(x,y)pF(x,y)^{p-1}\\ &\times\left(\frac{1}{r(x,y)}\int_c^y\frac{r(x,t)h(x,t)}{xt\log(bd/xt)}dt\right.\\ &\left.-\frac{\partial r(x,y)/\partial x}{r^2(x,y)}\int_a^x\int_c^y\frac{r(s,t)f(s,t)}{st\log(bd/st)}dsdt\right).\end{aligned} \tag{3.1}$$

Let

$$\frac{\partial v(x,y)}{\partial x}=\frac{1}{x}\left(\log\left(\frac{bd}{xy}\right)\right)^{q-1},$$

then

$$v(x,y)=-\frac{1}{q}\left(\log\left(\frac{bd}{xy}\right)\right)^{q}. \tag{3.2}$$

From (3.1), (3.2) and integrating by parts with respect to x, we have

$$\begin{aligned}
&\int_c^d\int_a^b w(x,y)\frac{1}{x}\left(\log\left(\frac{bd}{xy}\right)\right)^{q-1}F(x,y)^p dxdy\\
&=\int_c^d\Bigg\{-w(x,y)F(x,y)^p\frac{1}{q}\left(\log\left(\frac{bd}{xy}\right)\right)^{q}\Bigg|_{x=a}^{x=b}\\
&+\int_a^b\frac{1}{q}\left(\log\left(\frac{bd}{xy}\right)\right)^{q}\Bigg[\frac{\partial w(x,y)}{\partial x}F(x,y)^p+w(x,y)pF(x,y)^{p-1}\\
&\times\left(G(x,y)-\frac{\partial r(x,y)/\partial x}{r^2(x,y)}\int_a^x\int_c^y\frac{r(s,t)f(s,t)}{st\log(bd/st)}dsdt\right)\Bigg]dx\Bigg\}dy,
\end{aligned} \tag{3.3}$$

where

$$G(x,y)=\frac{1}{r(x,y)}\int_c^y\frac{r(x,t)f(x,t)}{xt\log(bd/xt)}dt.$$

From (3.3), we observe

$$\begin{aligned}
&\int_c^d\int_a^b D(x,y)w(x,y)\left(\log\left(\frac{bd}{xy}\right)\right)^{q-1}F(x,y)^p dxdy\\
&\le\frac{p}{q}\int_c^d\int_a^b\left[w(x,y)\left(\log\left(\frac{bd}{xy}\right)\right)^{q}G(x,y)F(x,y)^{p-1}\right]dxdy\\
&=\frac{p}{q}\int_c^d\int_a^b\left[w(x,y)^{1/p}x^{(p-1)/p}G(x,y)\left(\log\left(\frac{bd}{xy}\right)\right)^{q-(q-1)(p-1)/p}\right]\\
&\times\left[w(x,y)^{(p-1)/p}x^{-(p-1)/p}\left(\log\left(\frac{bd}{xy}\right)\right)^{(q-1)(p-1)/p}F(x,y)^{p-1}\right]dxdy.
\end{aligned} \tag{3.4}$$

Using (2.1) and applying Hölder's inequality with indices p, $p/(p-1)$ on the right side of (3.4), we obtain

$$\begin{aligned}
&\int_c^d\int_a^b w(x,y)\frac{1}{x}\left(\log\left(\frac{bd}{xy}\right)\right)^{q-1}F(x,y)^p dxdy\\
&\le\frac{\alpha p}{q}\left\{\int_c^d\int_a^b\left[w(x,y)x^{p-1}G(x,y)^p\left(\log\left(\frac{bd}{xy}\right)\right)^{q+p-1}\right]dxdy\right\}^{1/p}\\
&\times\left\{\int_c^d\int_a^b\left[w(x,y)x^{-1}\left(\log\left(\frac{bd}{xy}\right)\right)^{q-1}F(x,y)^p\right]dxdy\right\}^{(p-1)/p}.
\end{aligned} \tag{3.5}$$

Dividing both sides of (3.5) by the second integral factor on the right side of (3.5) and raising both sides to the pth power, we obtain

$$\begin{aligned}&\int_c^d\int_a^b w(x,y)\frac{1}{x}\left(\log\left(\frac{bd}{xy}\right)\right)^{q-1}F(x,y)^p dxdy\\&\le\left(\frac{\alpha p}{q}\right)^p\int_c^d\int_a^b\left[w(x,y)x^{p-1}G(x,y)^p\left(\log\left(\frac{bd}{xy}\right)\right)^{q+p-1}\right]dxdy.\end{aligned}$$

Similarly, if $0<p<1$ and (2.1) is reversed, the inequality (2.2) is also reversed.

Proof of Theorem 2.2. From the hypotheses of Theorem 2.2, we obtain

$$\begin{aligned}\frac{\partial}{\partial y}\Big(w(x,y)F(x,y)^p\Big)&=\frac{\partial w(x,y)}{\partial y}F(x,y)^p+w(x,y)pF(x,y)^{p-1}\\&\times\left(-\frac{1}{r(x,y)}\int_x^b\frac{r(s,y)h(s,y)}{sy\log(sy/ac)}ds\right.\\&\left.-\frac{\partial r(x,y)/\partial x}{r^2(x,y)}\int_x^b\int_y^d\frac{r(s,t)f(s,t)}{st\log(st/ac)}dsdt\right),\end{aligned}\tag{3.6}$$

and

$$\frac{\partial}{\partial y}\left(-\frac{1}{q}\left(\log\left(\frac{xy}{ac}\right)\right)^q\right)=\frac{1}{y}\left(\log\left(\frac{xy}{ac}\right)\right)^{q-1}.\tag{3.7}$$

From (3.6)-(3.7) and integrating by parts with respect to y, we have

$$\begin{aligned}&\int_c^d\int_a^b w(x,y)\frac{1}{y}\left(\log\left(\frac{xy}{ac}\right)\right)^{q-1}F(x,y)^p dxdy\\&=\int_a^b\left\{-w(x,y)F(x,y)^p\frac{1}{q}\left(\log\left(\frac{xy}{ac}\right)\right)^q\Bigg|_{y=c}^{y=d}\right.\\&+\int_c^d\frac{1}{q}\left(\log\left(\frac{xy}{ac}\right)\right)^q\left[\frac{\partial w(x,y)}{\partial x}F(x,y)^p+w(x,y)pF(x,y)^{p-1}\right.\\&\left.\left.\times\left(H(x,y)-\frac{\partial r(x,y)/\partial x}{r^2(x,y)}\int_a^x\int_c^y\frac{r(s,t)f(s,t)}{st\log(bd/st)}dsdt\right)\right]dy\right\}dx,\end{aligned}\tag{3.8}$$

where

$$H(x,y)=-\frac{1}{r(x,y)}\int_x^b\frac{r(s,y)f(s,y)}{sy\log(sy/ac)}ds.$$

From (2.5) and (3.8), we observe

$$\begin{aligned}
&\int_c^d\int_a^b E(x,y)w(x,y)\left(\log\left(\frac{xy}{ac}\right)\right)^{q-1}F(x,y)^p dxdy\\
&\leq \frac{p}{q}\int_c^d\int_a^b\left[w(x,y)\left(\log\left(\frac{xy}{ac}\right)\right)^{q}H(x,y)F(x,y)^{p-1}\right]dxdy\\
&=\frac{p}{q}\int_c^d\int_a^b\left[w(x,y)^{1/p}y^{(p-1)/p}H(x,y)\left(\log\left(\frac{xy}{ac}\right)\right)^{q-(q-1)(p-1)/p}\right] \qquad (3.9)\\
&\times\left[w(x,y)^{(p-1)/p}y^{-(p-1)/p}\left(\log\left(\frac{xy}{ac}\right)\right)^{(q-1)(p-1)/p}F(x,y)^{p-1}\right]dxdy.
\end{aligned}$$

By applying Hölder's inequality, we obtain

$$\begin{aligned}
&\int_c^d\int_a^b w(x,y)\frac{1}{y}\left(\log\left(\frac{xy}{ac}\right)\right)^{q-1}F(x,y)^p dxdy\\
&\leq\frac{\beta p}{q}\left\{\int_c^d\int_a^b\left[w(x,y)y^{p-1}H(x,y)^p\left(\log\left(\frac{xy}{ac}\right)\right)^{q+p-1}\right]dxdy\right\}^{1/p} \qquad (3.10)\\
&\times\left\{\int_c^d\int_a^b\left[w(x,y)y^{-1}\left(\log\left(\frac{xy}{ac}\right)\right)^{q-1}F(x,y)^p\right]dxdy\right\}^{(p-1)/p}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_c^d\int_a^b w(x,y)\frac{1}{y}\left(\log\left(\frac{xy}{ac}\right)\right)^{q-1}F(x,y)^p dxdy\\
&\leq\left(\frac{\beta p}{q}\right)^p\int_c^d\int_a^b\left[w(x,y)y^{p-1}H(x,y)^p\left(\log\left(\frac{xy}{ac}\right)\right)^{q+p-1}\right]dxdy.
\end{aligned}$$

Similarly, if $0 < p < 1$ and (2.5) is reversed, then the inequality (2.6) is also reversed.

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