An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Tomul LXII, 2016, f. 2, vol. 2

Conharmonic curvature tensor on (κ,μ)-contact metric manifold

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Received: 19.V.2014 / Revised: 14.I.2015 / Accepted: 20.I.2015

Abstract The object of the present paper is to study conharmonic curvature tensor on a (κ,μ)-contact metric manifold. In this paper we prove that a ϕ-conharmonically flat (κ,μ)-contact metric manifold is an η-Einstein manifold and obtain the value of scalar curvature of a ξ-conharmonically flat (κ,μ)-contact metric manifold. The conditions $R(X,Y)C=0$ and $C(X,Y)S=0$ are also studied on the manifold where C,R and S are conharmonic, Riemannian and Ricci curvatures respectively.

Keywords (κ,μ)-contact metric manifold · η-Einstein manifold · Quasi-conharmonically flat manifold · ϕ-conharmonically flat manifold · ξ-conharmonically flat manifold · Conharmonically semi-symmetric manifold

Mathematics Subject Classification (2010) 53C25 · 53D15

1 Introduction

Let M and $\overline{M}$ be two Riemannian manifolds with the metric tensors g and $\overline{g}$ respectively. If g and $\overline{g}$ are related by the equation

$$\overline{g}(X,Y)=e^{2\sigma}g(X,Y), \tag{1.1}$$

where σ is a real function, then the manifolds M and $\overline{M}$ are called conformally related manifolds and the correspondence between M and $\overline{M}$ is known as conformal transformation. A harmonic function is defined as a function whose Laplacian vanishes. In general a harmonic function is not transformed in a harmonic function. The transformation under which a harmonic function remains invariant have been studied by

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Ishii [7] who introduced a conharmonic transformation as a special type of conformal transformation satisfying the condition

$$\sigma^i_{,i} + \sigma_{,i}\sigma^i = 0, \tag{1.2}$$

where 'comma' denotes covariant differentiation with respect to the metric g.

A tensor C of rank four that remains invariant under conharmonic transformation for a $(2n+1)$-dimensional Riemannian manifold M^{2n+1}, is given by [8,10]

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{(2n-1)}\{&g(Y,Z)QX - g(X,Z)QY \\ &+ S(Y,Z)X - S(X,Z)Y\}. \end{aligned} \tag{1.3}$$

Or equivalently

$$\begin{aligned} C(X,Y,Z,W) &= R(X,Y,Z,W) \\ &- \frac{1}{(2n-1)}\{g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &+ S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\}, \end{aligned} \tag{1.4}$$

where $R(X,Y)Z$ denotes the Riemannian curvature tensor, $R(X,Y,Z,W) = g(R(X,Y)Z,W)$, S denotes the Ricci tensor and Q denotes the Ricci operator.

The curvature tensor given by (1.3) or equivalently (1.4) is called conharmonic curvature tensor.

2 Preliminaries

A differentiable manifold M^{2n+1} is said to be a contact manifold if it admits a global 1-form η, there exists a unique vector field ξ, called the characteristic vector field such that,

$$\eta(\xi) = 1 \text{ and } d\eta(\xi, X) = 0. \tag{2.1}$$

A Riemannian metric g on M^{2n+1} is said to be an associated metric if there exists a $(1,1)$ tensor field ϕ such that

$$d\eta(X,Y) = g(X,\phi Y), \eta(X) = g(X,\xi), \phi^2 = -I + \eta \otimes \xi. \tag{2.2}$$

From these equations we have

$$\phi\xi = 0, \eta \circ \phi = 0, g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.3}$$

A differentiable manifold M^{2n+1} equipped with the structure tensors (ϕ,ξ,η,g) satisfying (2.2) is said to be a contact metric manifold (see [2]) and is denoted by $M = (M^{2n+1},\phi,\xi,\eta,g)$.

A contact matric manifold is called an η-Einstein manifold if the Ricci tensor S is of the form $S = ag + b\eta \otimes \eta$, where a and b are smooth functions on M^{2n+1} and if $b = 0$, then the manifold is called an Einstein manifold.

It is called conharmonically semi-symmetric if $R(X,Y)C = 0$ holds good on the manifold. Given a contact metric manifold M, we can define a $(1,1)$ tensor field h

by $h = L_\xi \phi$, where L denotes the Lie differentiation. Then we may observe that h is symmetric and satisfies $h\xi = 0$, $\eta \circ h = 0$, and $h\phi = -\phi h$. We also have $Tr.h = Tr.\phi h = 0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds:

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.4}$$
$$(\nabla_X \eta)Y = g(X, \phi Y) - g(\phi h X, Y). \tag{2.5}$$

A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.6}$$

$X,\ Y \in TM$, where ∇ is the Levi-Civita connection of Riemannian metric g. A contact metric manifold for which ξ is a Killing vector field is said to be a K-contact manifold. Every Sasakian manifold is a K-contact manifold but the converse is not true. However, a three dimensional K-contact manifold is Sasakian. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$.

A generalization of both $R(X, Y)\xi = 0$ and the Sasakian case (see Blair et al. [3]) considered the (κ, μ)-nullity condition on a contact metric manifold. The (κ, μ)-nullity distribution $N(\kappa, \mu)$ (see [3]) of a contact metric manifold is defined by $N(\kappa, \mu)$: $p \rightarrow N_p(\kappa, \mu) = \{W \in T_pM \mid R(X, Y)W = (\kappa I + \mu h)(g(Y, W)X - g(X, W)Y)\} \forall X, Y \in TM$, where $(\kappa, \mu) \in \mathbb{R}^2$. A contact metric manifold with $\xi \in N(\kappa, \mu)$ is called a (κ, μ)-contact metric manifold. Thus we have

$$R(X, Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.7}$$

on a (κ, μ)-contact metric manifold and $\kappa \leq 1$. For $\kappa = 1$, the structure becomes Sasakian($h = 0$) and if $\kappa < 1$, the (κ, μ)-nullity condition completely determines the curvature of M(Blair *et al.* 1995). For a (κ, μ)-contact manifold, the condition of being Sasakian manifold, a K-contact manifold, $\kappa = 1$ and $h = 0$ are all equivalent. A (κ, μ)-contact manifold reduces to an $N(\kappa)$-contact manifold if and only if $\mu = 0$. In a (κ, μ)-contact metric manifold, the following relations hold:

$$h^2 = (\kappa - 1)\phi^2,\ \kappa \leq 1, \tag{2.8}$$
$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.9}$$
$$R(\xi, X)Y = \kappa[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX], \tag{2.10}$$
$$S(X, \xi) = 2n\kappa\eta(X), \tag{2.11}$$
$$\begin{aligned} S(X, Y) = {} & [2(n-1) - n\mu]g(X, Y) + [2(n-1) + \mu]g(hX, Y) \\ & + [2(1-n) + n(2\kappa + \mu)]\eta(X)\eta(Y), \end{aligned} \tag{2.12}$$
$$r = 2n(2n - 2 + \kappa - n\mu) \tag{2.13}$$

and

$$S(\phi X, \phi Y) = S(X, Y) - 2n\kappa\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y), \tag{2.14}$$

where S is the Ricci tensor and r is the scalar curvature of the manifold. The curvature tensor of a $(2n+1)$-dimensional non-Sasakian (κ,μ)-contact metric manifold is given by Boeckx [4]

$$
\begin{aligned}
R(X,Y,Z,W) &= (1-\frac{\mu}{2})\,\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}\\
&-\frac{\mu}{2}\big\{2g(X,\phi Y)g(\phi Z,W)+g(X,\phi Z)g(\phi Y,W)\\
&-g(Y,\phi Z)g(\phi X,W)\big\}+\big\{g(Y,Z)g(hX,W)\\
&-g(X,Z)g(hY,W)-g(Y,W)g(hX,Z)+g(X,W)g(hY,W)\big\}\\
&+\frac{1-\frac{\mu}{2}}{1-\kappa}\,\{g(hY,Z)g(hX,W)-g(hX,Z)g(hY,W)\}\\
&+\frac{\kappa-\frac{\mu}{2}}{1-\kappa}\,\{g(\phi hY,Z)g(\phi hX,W)-g(\phi hX,Z)g(\phi hY,W)\}\\
&+\eta(X)\eta(W)\left\{\left(\kappa-1+\frac{\mu}{2}\right)g(Y,Z)+(\mu-1)g(hY,Z)\right\}\\
&-\eta(Y)\eta(W)\left\{\left(\kappa-1+\frac{\mu}{2}\right)g(X,Z)+(\mu-1)g(hX,Z)\right\}\\
&+\eta(Y)\eta(Z)\left\{\left(\kappa-1+\frac{\mu}{2}\right)g(X,W)+(\mu-1)g(hX,W)\right\}\\
&-\eta(X)\eta(Z)\left\{\left(\kappa-1+\frac{\mu}{2}\right)g(Y,W)+(\mu-1)g(hY,W)\right\}
\end{aligned}
\tag{2.15}
$$

If $\{e_1,e_2,\dots,e_{2n},e_{2n+1}=\xi\}$ be a local orthonormal basis of vector fields in M^{2n+1}, then $\{\phi e_1,\phi e_2,\dots,\phi e_{2n},\xi\}$ is also a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i,e_i)=\sum_{i=1}^{2n} g(\phi e_i,\phi e_i)=2n, \tag{2.16}$$

$$
\begin{aligned}
\sum_{i=1}^{2n} g(e_i,Y)S(X,e_i) &= \sum_{i=1}^{2n} g(\phi e_i,Y)S(X,\phi e_i)\\
&= S(X,Y)-S(X,\xi)\eta(Y)
\end{aligned}
\tag{2.17}
$$

$$R(\xi,X)\xi=\kappa(\eta(X)\xi-X)-\mu hX, \tag{2.18}$$

$$\sum_{i=1}^{2n} S(e_i,e_i)=\sum_{i=1}^{2n} S(\phi e_i,\phi e_i)=r-2n\kappa, \tag{2.19}$$

$$R(X,\xi,\xi,Y)=\kappa\,\{g(X,Y)-\eta(X)\eta(Y)\}+\mu g(hX,Y), \tag{2.20}$$

$$
\begin{aligned}
\sum_{i=1}^{2n} R(e_i,X,Y,e_i) &= \sum_{i=1}^{2n} R(\phi e_i,X,Y,\phi e_i)\\
&= S(X,Y)-\kappa\,\{g(X,Y)-\eta(X)\eta(Y)\}-\mu g(hX,Y).
\end{aligned}
\tag{2.21}
$$

3 Some structure theorem

Let W be the Weyl conformal curvature tensor of a $(2n+1)$-dimensional manifold M. At each point $p \in M^{2n+1}$, we can decompose the tangent space $T_p(M)$ into the direct sum $T_p(M) = \phi\,(T_p(M)) \oplus L(\xi_p)$, where $L(\xi_p)$ is 1-dimensional linear subspace generated by ξ_p. Then we have a map $W : T_p(M) \times T_p(M) \times T_p(M) \to T_p(M) = \phi(T_p(M)) \oplus L(\xi_p)$.

It may be natural to consider the following particular cases:

1. $W : T_p(M) \times T_p(M) \times T_p(M) \to L(\xi_p)$, i.e. the projection of the image of W in $\phi\,(T_p(M))$ is zero, then M^{2n+1} is called quasi-conformally flat.
2. $W : T_p(M) \times T_p(M) \times T_p(M) \to \phi\,(T_p(M))$, i.e. the projection of the image of W in $L(\xi_p)$ is zero, then M^{2n+1} is called ξ-conformally flat.
3. $W : \phi\,(T_p(M)) \times \phi\,(T_p(M)) \times \phi\,(T_p(M)) \to L(\xi_p)$, i.e. when W is restricted to $\phi\,(T_p(M)) \times \phi\,(T_p(M)) \times \phi\,(T_p(M))$, the projection of the image of W in $\phi\,(T_p(M))$ is zero. This condition is equivalent to $g(W(\phi X, \phi Y)\phi Z, \phi W) = 0$. Then M^{2n+1} is called ϕ-conformally flat.

These cases are studied by many geometers (see [1,11,5,6,9]). Analogous to these cases, we define the following:

Definition 3.1 *A (κ,μ)-contact metric manifold is said to be quasi-conharmonically flat if*

$$g(C(X,Y)Z, \phi W) = 0. \tag{3.1}$$

Definition 3.2 *A (κ,μ)-contact metric manifold is said to be ξ-conharmonically flat if*

$$C(X,Y)\xi = 0. \tag{3.2}$$

Definition 3.3 *A (κ,μ)-contact metric manifold is said to be ϕ-conharmonically flat if*

$$g(C(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{3.3}$$

for all $X, Y, Z, W \in T(M)$.

First we consider a conharmonically flat (κ,μ)-contact metric manifold.

A (κ,μ)-contact metric manifold is said to be conharmonically flat if its conharmonic curvature tensor vanishes, i.e.,

$$C(X,Y,Z,W) = 0, \tag{3.4}$$

or

$$R(X,Y,Z,W) = \frac{1}{2n-1}\big\{g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\big\}. \tag{3.5}$$

Putting $X = W = e_i$ in the last equation and summing over $i = 1, 2, \ldots, 2n$, we get

$$\begin{aligned}\sum_{i=1}^{2n} R(e_i, Y, Z, e_i) = \frac{1}{2n-1}\Big\{&(r - 2n\kappa)g(Y, Z) - S(Y, Z)\\ &- S(Y, \xi)\eta(Z)\sum_{i=1}^{2n} g(e_i, Z)S(Y, e_i) + 2nS(Y, Z)\\ &- \sum_{i=1}^{2n} S(e_i, Z)g(Y, e_i)\Big\}.\end{aligned} \tag{3.6}$$

Using (2.11)and (2.17), we obtain

$$S(Y, Z) = (r - \kappa)g(Y, Z) + (2n - 1)\mu g(hY, Z) + (2n + 1)\kappa\eta(Y)\eta(Z). \tag{3.7}$$

Substituting the value of $g(hY, Z)$ from equation (2.12) in the equation (3.7), we get

$$\begin{aligned}S(Y, Z) = &\left\{\frac{\left((2(n-1) + \mu\right)(r - \kappa) + n(2n - 1)\mu^2)}{2(n - 1)(1 - \mu)}\right\} g(Y, Z)\\ &+ \left\{(2n + 1)\kappa + \frac{(2n - 1)\mu[2(n - 1) + \kappa - n\mu]}{2(n - 1)(1 - \mu)}\right\} \eta(Y)\eta(Z).\end{aligned} \tag{3.8}$$

Now, we are able to state the following theorem.

Theorem 3.4 *A conharmonically flat (κ, μ)-contact metric manifold is an η-Einstein manifold.*

Proof. The proof follows from the equation (3.8). □

Next we consider a quasi-conharmonically flat (κ, μ)-contact metric manifold, then equation (3.1) is satisfied for all $X, Y, Z, W \in T(M)$. Taking ϕX in place of X, we get

$$g(C(\phi X, Y)Z, \phi W) = 0, \tag{3.9}$$

using (1.4) in the last equation, we get

$$\begin{aligned}R(\phi X, Y, Z, \phi W) = \frac{1}{2n-1}\big\{&g(Y, Z)S(\phi X, \phi W) - g(\phi X, Z)S(Y, \phi W)\\ &+ S(Y, Z)g(\phi X, \phi W) - S(\phi X, Z)g(Y, \phi W)\big\},\end{aligned} \tag{3.10}$$

putting $X = W = e_i$ in the last equation and summing over $i = 1, 2, \ldots, 2n$, we get

$$\begin{aligned}\sum_{i=1}^{2n} R(\phi e_i, Y, Z, \phi e_i) = \frac{1}{2n-1}\big\{&g(Y, Z)\sum_{i=1}^{2n} S(\phi e_i, \phi e_i)\\ &- \sum_{i=1}^{2n} g(\phi e_i, Z)S(Y, \phi e_i) + 2nS(Y, Z) - \sum_{i=1}^{2n} S(\phi e_i, Z)g(Y, \phi e_i)\big\},\end{aligned} \tag{3.11}$$

using equations (2.17), (2.19) and (2.21) in the equation (3.11), we obtain

$$\begin{aligned} &S(Y,Z) - \kappa\{g(Y,Z) - \eta(Y)\eta(Z)\} - \mu g(hY,Z) \\ &\qquad = \frac{1}{2n-1}\big\{(r-2n\kappa)g(Y,Z) + 4n\kappa\eta(Y)\eta(Z) + 2(n-1)S(Y,Z)\big\}, \end{aligned} \tag{3.12}$$

substituting the value of $g(hY,Z)$ from equation (2.12) in the equation (3.12), we obtain

$$\begin{aligned} S(Y,Z) = &\left\{\frac{(2(n-1)+\mu)(r-\kappa) + n(2n-1)\mu^2}{2(n-1)(1-\mu)}\right\} g(Y,Z) \\ &+ \left\{\frac{(2n+1)\kappa + (2n-1)\mu[2(n-1)+\kappa-n\mu]}{2(n-1)(1-\mu)}\right\}\eta(Y)\eta(Z). \end{aligned} \tag{3.13}$$

Thus it is an η-Einstein manifold. We state this result in the form of the theorem.

Theorem 3.5 *A quasi-conharmonically flat (κ,μ)-contact metric manifold is an η-Einstein manifold.*

Now, let M be a ξ-conharmonically flat (κ,μ)-contact metric manifold, then we have from equation (3.2)

$$g(C(X,Y)\xi, W) = 0, \tag{3.14}$$

in the view of equation (1.4), the last equation implies that

$$\begin{aligned} R(X,Y,\xi,W) = &\frac{1}{2n-1}\big\{\eta(Y)S(X,W) - \eta(X)S(Y,W) \\ &+ S(Y,\xi)g(X,W) - S(X,\xi)g(Y,W)\big\}. \end{aligned} \tag{3.15}$$

Putting $X = W = e_i$ in the last equation and summing over $i = 1, 2, \ldots, 2n$, we get

$$\begin{aligned} \sum_{i=1}^{2n} R(e_i,Y,\xi,e_i) = &\frac{1}{2n-1}\Big\{\eta(Y)\sum_{i=1}^{2n} S(e_i,e_i) - \sum_{i=1}^{2n}\eta(e_i)S(Y,e_i) \\ &+ S(Y,\xi)\sum_{i=1}^{2n} g(e_i,e_i) - \sum_{i=1}^{2n} S(e_i,\xi)g(Y,e_i)\Big\}, \end{aligned} \tag{3.16}$$

using equations (2.16), (2.17), (2.19) and (2.21), we obtain

$$S(Y,\xi) = \frac{1}{2n-1}\{(r-2n\kappa)\eta(Y) + 4n\kappa\eta(Y) + 2(n-1)S(Y,\xi)\}$$

this implies that

$$S(Y,\xi) = (r-\kappa)\eta(Y) + (2n+1)\kappa\eta(Y). \tag{3.17}$$

In the view of equation (2.11), the last equation implies

$$r = 0. \tag{3.18}$$

Thus scalar curvature vanishes. This leads the following theorem.

Theorem 3.6 *If a (κ,μ)-contact metric manifold is ξ-conharmonically flat, then its scalar curvature vanishes.*

Finally we consider a ϕ-conharmonically flat (κ,μ)-contact metric manifold. Then we have from equation (3.3), $g(C(\phi X,\phi Y)\phi Z,\phi W)=0$, this implies that

$$\begin{aligned} R(\phi X,\phi Y,\phi Z,\phi W) = \frac{1}{2n-1}\big\{&g(\phi Y,\phi Z)S(\phi X,\phi W) \\ &- g(\phi X,\phi Z)S(\phi Y,\phi W)+S(\phi Y,\phi Z)g(\phi X,\phi W) \\ &- S(\phi X,\phi Z)g(\phi Y,\phi W)\big\}. \end{aligned} \tag{3.19}$$

Putting $X=W=e_i$ in the equation (3.19) and summing over $i=1,2,\ldots,2n$, we get

$$\begin{aligned} \sum_{i=1}^{2n} R(\phi e_i,\phi Y,\phi Z,\phi e_i) = \frac{1}{2n-1}\big\{&g(\phi Y,\phi Z)\sum_{i=1}^{2n} S(\phi e_i,\phi e_i) \\ &- \sum_{i=1}^{2n} g(\phi e_i,\phi Z)S(\phi Y,\phi e_i)+2nS(\phi Y,\phi Z) \\ &- \sum_{i=1}^{2n} S(\phi e_i,\phi Z)g(\phi Y,\phi e_i)\big\}, \end{aligned} \tag{3.20}$$

using equations (2.17), (2.19) and (2.21) in the equation (3.20), we obtain $S(\phi Y,\phi Z)-\kappa g(\phi Y,\phi Z)-\mu g(h\phi Y,\phi Z)=\frac{1}{2n-1}\big\{(r-2n\kappa)g(\phi Y,\phi Z)+2(n-1)S(\phi Y,\phi Z)\big\}$, or

$$S(\phi Y,\phi Z)=(r-\kappa)g(\phi Y,\phi Z)+(2n-1)\mu g(h\phi Y,\phi Z). \tag{3.21}$$

Replacing Y by ϕY and Z by ϕZ in the equation (3.21), we get $S(\phi^2 Y,\phi^2 Z)=(r-\kappa)g(\phi Y,\phi Z)+(2n-1)\mu g(h\phi^2 Y,\phi^2 Z)$, this implies that

$$S(Y,Z)=(r-\kappa)g(Y,Z)+(2n-1)g(hY,Z)-(r-(2n+1)\kappa)\eta(Y)\eta(Z), \tag{3.22}$$

substituting the value of $g(hY,Z)$ from equation (2.12) in the equation (3.22), we obtain $S(Y,Z)=\alpha g(Y,Z)+(2nk-\alpha)\eta(Y)\eta(Z)$, where

$$\alpha=\frac{(r-\kappa)(2(n-1)+\mu)+(2n-1)\mu(2(1-n)+n\mu)}{2(n-1)(1-\mu)}.$$

This shows that M is an η-Einstein manifold with $a+b=2n\kappa$. Thus we have the following theorem.

Theorem 3.7 *A ϕ-conharmonically flat (κ,μ)-contact metric manifold is an η-Einstein manifold with $a+b=2n\kappa$.*

4 (κ, μ)-contact metric manifold satisfying $R(X,Y)C = 0$

In this section we consider a conharmonically semi-symmetric (κ, μ)-contact metric manifold i.e. (κ, μ)-contact metric manifold satisfying $R(X,Y)C = 0$. We begin with the lemma:

Lemma 4.1 *Let M be a $(2n+1)$-dimensional (κ, μ)-contact metric manifold then*

$$\begin{aligned}\sum_{i=1}^{2n+1} C(e_i, Y, Z, he_i) = & \left\{\frac{2(n-1)(1-\kappa)(1-\mu)}{2n-1}\right\} g(Y,Z) \\ & + \left\{2\mu + \kappa + \frac{(2n+1)(2(n-1)-n\mu)}{2n-1}\right\} g(hY,Z) \\ & + (1-\kappa)\left\{\frac{n(\mu-2)+3+(\mu-1)}{2n-1}\right\} \eta(Y)\eta(Z).\end{aligned} \tag{4.1}$$

Proof. Let $\{e_1, e_2, \ldots, e_2n, e_{2n+1} = \xi\}$ be an orthonormal basis of the tangent space at each point of the manifold M then the equation (1.4) implies that

$$\begin{aligned}\sum_{i=1}^{2n+1} C(e_i, Y, Z, he_i) = & \sum_{i=1}^{2n+1} R(e_i, Y, Z, he_i) - \frac{1}{2n-1}\Big\{g(Y,Z)\sum_{i=1}^{2n+1} S(e_i, he_i) \\ & - \sum_{i=1}^{2n+1} g(e_i, Z)S(Y, he_i) + S(Y,Z)\sum_{i=1}^{2n+1} g(e_i, he_i) \\ & - \sum_{i=1}^{2n+1} S(e_i, Z)g(Y, he_i)\Big\}.\end{aligned} \tag{4.2}$$

From equation (2.12) we have

$$\sum_{i=1}^{2n+1} S(e_i, he_i) = 2n(2(n-1)+\mu)(1-\kappa). \tag{4.3}$$

Also from equation (2.15) we have

$$\begin{aligned}\sum_{i=1}^{2n+1} R(e_i, Y, Z, he_i) = & \, 2(n-1)(1-\kappa)g(Y,Z) \\ & + 2(1-n+n\mu)(1-\kappa)\eta(Y)\eta(Z) + (2(n-1)-(n-2)\mu+\kappa)g(hY,Z).\end{aligned} \tag{4.4}$$

The equation (4.1) is obtained from equations (4.2), (4.3) and (4.7). Hence the lemma. □

The condition $R(X,Y)C = 0$ implies that

$$\begin{aligned}R(X,Y)C(U,V)W - C(R(X,Y)U,V)W \\ - C(U,R(X,Y)V)W - C(U,V)R(X,Y)W = 0,\end{aligned} \tag{4.5}$$

for $U, V \in T(M)$. Now

$$\begin{aligned}\eta(C(X,Y)Z) = &\frac{\kappa}{2n-1}\left\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\right\} \\ &+ \mu\left\{g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)\right\} \\ &+ \frac{1}{2n-1}\left\{S(X,Z)\eta(Y) - S(Y,Z)\eta(X)\right\},\end{aligned} \tag{4.6}$$

putting $Z = \xi$, we get

$$\eta(C(X,Y)\xi) = 0, \tag{4.7}$$

also we have

$$\eta(C(X,X)Z) = 0. \tag{4.8}$$

From the equation (4.5), (4.6), (4.7), (4.8), (2.7) and (2.10), we get

$$\begin{aligned}&\kappa\{C(U,V,W,Y)\xi - \eta(C(U,V)W)Y - g(Y,U)C(\xi,V)W \\ &\quad + \eta(U)C(Y,V)W - g(Y,V)C(U,\xi)W + \eta(V)C(U,Y)W \\ &\quad - g(Y,W)C(U,V)\xi + \eta(W)C(U,V)Y\} \\ &\quad + \mu\{C(U,V,W,hY)\xi - \eta(C(U,V)W)hY - g(hY,U)C(\xi,V)W \\ &\quad + \eta(U)C(hY,V)W - g(hY,V)C(U,\xi)W + \eta(V)C(U,hY)W \\ &\quad - g(hY,W)C(U,V)\xi + \eta(W)C(U,V)hY\} = 0.\end{aligned}$$

Putting $Y = U$ in the last equation and taking inner product with ξ, we get

$$\begin{aligned}&\kappa\{C(U,V,W,U) - g(U,U)\eta(C(\xi,V)W) - g(U,V)\eta(C(U,\xi)W) \\ &+ \eta(W)\eta(C(U,V)U)\} + \mu\{C(U,V,W,hU) + \eta(U)\eta(C(hU,V)W) \\ &- g(hU,V)\eta(C(U,\xi)W) + \eta(V)\eta(C(U,hU)W) + \eta(W)\eta(C(U,V)hU) = 0.\end{aligned}$$

Putting $U = e_i$ and taking summation over $i = 1, 2, \ldots, 2n+1$, we obtain

$$\begin{aligned}&\kappa\left\{\sum_{i=1}^{2n+1} C(e_i,V,W,e_i) - (2n+1)\eta(C(\xi,V)W) - \eta(C(V,\xi)W)\right. \\ &\left. + \eta(W)\sum_{i=1}^{2n+1}\eta(C(e_i,V)e_i)\right\} + \mu\left\{\sum_{i=1}^{2n+1} C(e_i,V,W,he_i)\right. \\ &+ \sum_{i=1}^{2n+1}\eta(e_i)\eta(C(he_i,V)W) - \sum_{i=1}^{2n+1} g(he_i,V)\eta(C(e_i,\xi)W) \\ &\left. + \eta(V)\sum_{i=1}^{2n+1}\eta(C(e_i,he_i)W) + \eta(W)\sum_{i=1}^{2n+1}\eta(C(e_i,V)he_i)\right\} = 0.\end{aligned} \tag{4.9}$$

But in a (κ,μ)-contact metric manifold we have the following relations

$$\sum_{i=1}^{2n+1} C(e_i,V,W,e_i) = \frac{-2n(2n-2+\kappa-n\mu)}{2n-1}g(V,W), \tag{4.10}$$

$$\begin{aligned}\eta(C(\xi,V)W) = &\frac{-\kappa}{2n-1}g(V,W)+\mu g(hV,W)\\ &-\frac{1}{2n-1}S(V,W)+\frac{(2n+1)\kappa}{2n-1}\eta(V)\eta(W),\end{aligned} \tag{4.11}$$

$$\sum_{i=1}^{2n+1} \eta(C(e_i,V)e_i) = \frac{r}{2n-1}\eta(V) = \frac{2n(2n-2+\kappa-n\mu)}{2n-1}\eta(V), \tag{4.12}$$

$$\begin{aligned}\sum_{i=1}^{2n+1} g(he_i,V)\eta(C(e_i,\xi)W) = &\left\{\mu(\kappa-1)+\frac{2(n-1)-n\mu}{2n-1}\right\}g(V,W)\\ &+\left\{\mu(1-\kappa)+\frac{2(1-n)+n(2\kappa+\mu)}{2n-1}\right\}\eta(V)\eta(W)\\ &+\left\{\frac{2(n-1)+\mu+\kappa}{2n-1}\right\}g(hV,W)\end{aligned} \tag{4.13}$$

$$\sum_{i=1}^{2n+1} \eta(C(e_i,V)he_i) = \frac{4n(1-\kappa)(n-1+n\mu)}{2n-1}\eta(V), \tag{4.14}$$

$$\sum_{i=1}^{2n+1} \eta(C(e_i,he_i)W) = 0. \tag{4.15}$$

Using equations (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15) in the equation (4.9), we obtain

$$\begin{aligned}&2n\kappa S(V,W)+[\mu^2(1+n-\kappa)+2\kappa\mu(n^2-n+1)-4n(n-1)\kappa]g(V,W)\\ &+[-\mu^2(2n^2-3n+3)+4n(n-1)\mu-2\kappa\mu(2n^2-2n+1)]g(hV,W)\\ &+[\mu^2\left(2(3n^2-2n+1)-\kappa(6n^2-3n+2)\right)\\ &-\mu\left(2\kappa(n^2+3n-2)-6(n-1)\right)+4n\kappa(n-1-n\kappa)]\eta(V)\eta(W)=0,\end{aligned} \tag{4.16}$$

substituting the value of $g(hY,Z)$ from equation (2.12) in the equation (4.16), we obtain,

$$S(V,W) = \alpha_1 g(V,W) + \beta_1\eta(V)\eta(W), \tag{4.17}$$

where

$$\alpha_1 = \frac{\begin{array}{c}[n\mu^3\left\{\kappa+2(n-1)^2\right\}+2(n-1)\mu^2\left\{(n^2+1)\kappa-2(2n^2-n+2)\right\}\\ -4(n-1)\mu\left\{\kappa(2n^2-3n+2)-2n(n-1)\right\}+8n(n-1)^2]\end{array}}{4n(n-1)\kappa+2(n-1)(2n-(2n-1)\kappa)\mu-(2n^2-3n+3)\mu^2}$$

and

$$\beta_1 = \frac{\begin{array}{c}[n\mu^3\left\{\kappa(6n^2-3n+2)-(8n^2-7n+5)\right\} \\ +2\mu^2\left\{\kappa(7n^3-5n^2+n-2)-(n-1)(6n^2+2n-1)\right\} \\ -4(n-1)\mu\left\{(n-1)\kappa^2-(5n-2)\kappa+(n-1)(2n-3)\right\} \\ +8n(n-1)\kappa(n\kappa-n+1)]\end{array}}{4n(n-1)\kappa+2(n-1)(2n-(2n-1)\kappa)\mu-(2n^2-3n+3)\mu^2}.$$

This leads the following theorem.

Theorem 4.2 *A conharmonically semi-symmetric (κ,μ)-contact metric manifold is an η-Einstein manifold.*

Proof. The proof follows from the equation (4.17). □

5 (κ,μ)-contact metric manifold satisfying $C(X,Y)S=0$

In this section we consider a $(2n+1)$-dimensional (κ,μ)-contact metric manifold satisfying $C(X,Y)S=0$. This equation implies that

$$S(C(X,Y)Z,W)+S(Z,C(X,Y)W)=0, \tag{5.1}$$

taking $X=W=\xi$, we get

$$S(C(\xi,Y)Z,\xi)+S(Z,C(\xi,Y)\xi)=0, \tag{5.2}$$

from the equations (2.10), (2.11) and (1.3) we get

$$\begin{aligned} C(\xi,Y)Z=&\frac{\kappa}{2n-1}\left[-g(Y,Z)\xi+\eta(Z)Y\right]+\mu\left[g(hY,Z)\xi-\eta(Z)hY\right] \\ &+\frac{1}{2n-1}\left[\eta(Z)QY-S(Y,Z)\xi\right], \end{aligned} \tag{5.3}$$

putting $Z=\xi$ in the equation (5.3), we obtain

$$C(\xi,Y)\xi=\frac{\kappa}{2n-1}Y+\kappa\eta(Y)\xi-\mu hY+\frac{1}{2n-1}QY, \tag{5.4}$$

using equations (5.3) and (5.4) in the equation (2.11), we obtain

$$\begin{aligned} S(C(\xi,Y)Z,\xi)=&\frac{2n\kappa}{2n-1}\{-\kappa g(Y,Z)+(2n-1)\mu g(hY,Z) \\ &+2n\kappa\eta(Y)\eta(Z)-S(Y,Z)\}, \end{aligned} \tag{5.5}$$

substituting the value of $g(hY,Z)$ from equation (2.12) in the equation (5.5), we obtain,

$$\begin{aligned} S(C(\xi,Y)Z,\xi)=&\frac{2n\kappa}{(2n-1)(2(n-1)+\mu)}[\mu S(Y,Z)-\{-n\mu^2(2n-1) \\ &+\mu(\kappa+2(n-1)(2n-1))+2(n-1)\kappa\}g(Y,Z) \\ &+\{-n\mu^2(2n-1)+\mu\left(\kappa(1+4n-4n^2)+2(n-1)(2n-1)\right) \\ &+2(n-1)(2n+1)\kappa\}\eta(Y)\eta(Z)]. \end{aligned} \tag{5.6}$$

In similar way from the equations (5.4), (2.12) and (2.12), we obtain

$$\begin{aligned} S(Z, C(\xi, Y)\xi) &= \left[\frac{-n\mu^2 + \mu(\kappa - 2n^2 + 3n - 2) + 2(n-1)(\kappa - 1)}{(2n-1)(2(n-1)+\mu)}\right] S(Y,Z) \\ &+ \left[\frac{\mu^2(n^2+\kappa-1) - \mu\left(4n(n-1) - 4(n-1)(1-\kappa)\right) + 4(n-1)^2\kappa}{2(n-1)+\mu}\right] g(Y,Z) \\ &+ \frac{1}{(2n-1)(2(n-1)+\mu)}[\mu^2\left((2n-1)(1-n^2) + \kappa(2n^2-2n+1)\right) \\ &+\mu(4(2n-1)(n^2-1) - 2\kappa(7n^2-6n+1) - 2n\kappa^2) \\ &-4(n-1)\kappa(6n^2-6n+1-n\kappa)]\eta(Y)\eta(Z), \end{aligned} \tag{5.7}$$

using equations (5.6) and (5.7) in the equation (5.2), we obtain

$$S(Y, Z) = \alpha_2 g(Y, Z) + \beta_2 \eta(Y)\eta(Z), \tag{5.8}$$

where

$$\alpha_2 = \frac{\begin{matrix}[(2n-1)\mu^2\left\{(2n^2+1)\kappa + n^2 - 1\right\} \\ -2\mu\left\{n\kappa^2 + 2(n-1)(2n-1)((n+1)\kappa + n - 1)\right\} \\ +4(n-1)\kappa(n\kappa + (n-1)(2n-1)]\end{matrix}}{n\mu^2 - \{(2n+1)\kappa - 2n^2 + 3n - 2\}\mu + 2(n-1)(1-\kappa)}$$

and

$$\beta_2 = \frac{\begin{matrix}[\mu^2\left\{\kappa(1-2n+4n^2(1-n)) + (2n-1)(1-n^2)\right\} \\ +2\mu\left\{4n^2(1-n)\kappa^2 + \kappa(4n^3-13n^2+8n-1) + 2(2n-1)(n^2-1)\right\} \\ +4(n-1)\kappa\left\{2n\kappa(n+1) - 6n^2 + 6n - 1\right\}]\end{matrix}}{(n\mu^2 - \{(2n+1)\kappa - 2n^2 + 3n - 2\}\mu + 2(n-1)(1-\kappa)}.$$

Hence it is an η-Einstein manifold. Now, we can state the following theorem.

Theorem 5.1 *A (κ, μ)-contact metric manifold satisfying $C(X, Y)S = 0$ is an η-Einstein manifold.*

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