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On a fixed point theorem

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Abstract In this paper we study the existence of fixed points for the product of nonlinear operators. This kind of fixed points theorems is necessary in consideration of quadratic differential and integral problems. We concentrate on applicability of obtained theorems and we prove the existence of fixed points for operators acting on some function spaces being not necessarily Banach algebras. This result essentially extend earlier ones and can be successfully applied for all considered quadratic problems which will be also presented.

Keywords Fixed point · Pointwise mulitiplication operator · Ideal space · Quadratic integral equation

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1 Introduction

In this paper we will discuss a fixed point theorem for the product (pointwise multiplication) of two, possibly nonlinear, operators $A \cdot B$. In contrast to the case of Krasnoselskii's fixed point theorem (the sum of two operators) this case is not sufficiently investigated. However, such a kind of theorems is widely discussed and they are related in some "quadratic" problems. Let us mention quadratic integral equations

$$x(t) \;=\; g(t) + \lambda u(t, x(t)) \cdot \int_a^b K(t,s) f(s, x(s))\, ds \tag{1.1}$$

considered in [6, 7, 10].

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Let us stress on one aspect of existing results: all of them are related to operators acting on Banach algebras (with respect to the pointwise multiplication). It is a significant limitation. In particular, this implies that for the above problems only continuous or even classical solutions can be found. Moreover, this approach leads to some additional assumption on the growth for considered operators. This is particularly clear when we have A (or: B) as the Nemytskii superposition operator, for which acting conditions are strictly related to the growth assumptions (cf. [10]).

We also indicate the method how to avoid this problem and we present some examples of application indicating the role of obtained theorems.

2 Pointwise multiplication in function spaces

Let $\mathbb{R}$ be the field of real numbers. In the paper we will denotes by I a compact interval $[a,b] \subset \mathbb{R}$.

Let us observe, that for the pointwise product of two operators an assumption that all of them are acting on the same space E as well as the counterdomain is in E is very restrictive. In particular, this is a restriction of the growth condition. The typical situation considered in many problems is when A is the Nemytskii superposition operator. It maps continuously $L^p(I)$ into $L^q(I)$ if and only if

$$|f(t,x)| \leq a(t) + b \cdot |x|^{\frac{p}{q}}, \tag{2.1}$$

for all $t \in I$ and $x \in \mathbb{R}$, where $a \in L_q(I)$ and $b \geq 0$. This is more natural than an assumption that $A : C(I) \to C(I)$ is continuous.

In this paper some properties of function spaces play a major role. We need to consider the triples of spaces with the following property: for a triple of function spaces E, E_1, E_2 there exists a constant k such that for arbitrary $x \in E_1$ and $y \in E_2$ a product (pointwise multiplication) $x \cdot y \in E$ and $\|x \cdot y\|_E \leq k \cdot \|x\|_{E_1} \cdot \|y\|_{E_2}$. Let us recall some special cases. Most known is the case of Banach algebras, i.e. the space of continuous functions. In this case $E = E_1 = E_2 = C(I,X)$ ($k = 1$, or so-called quasi-algebras for $k \neq 1$). Moreover, some subalgebras of this space can be interesting. Let us note, that the space of real-valued functions of finite total variation is also a Banach (quasi-) algebra with $k = 2$ (see [9]), so in this case we are not restricted to continuous functions.

However, if we try to consider "bigger" spaces we need to go outside the class of Banach (quasi-) algebras.

For discontinuous functions, let us recall the Hölder inequality for Lebesgue spaces: $\|x \cdot y\|_{L^1} \leq \|x\|_{L^p} \cdot \|y\|_{L^q}$ whenever $\frac{1}{p} + \frac{1}{q} = 1$. Thus, a triple (L^1, L^p, L^q) is good enough.

The third example (and the most important) is devoted to studying some Orlicz spaces (for definitions see [20,21], for instance). Generally speaking, the product of two functions $x, y \in L_M(I)$ is not in $L_M(I)$. However, if x and y belongs to some particular Orlicz spaces, then the product $x \cdot y$ belong to a third Orlicz space. But in general one can find two functions belonging to Orlicz spaces: $u \in L_U(I)$ and $v \in L_V(I)$ such that the product uv does not belong to any Orlicz space (this product is not integrable). Nevertheless, we have:

Lemma 2.1 ([20], Lemma 13.5, [21], Theorem 10.2) *Let φ_1, φ_2 and φ are arbitrary N-functions. The following conditions are equivalent:*

1. *For every functions $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}(I)$, $u \cdot w \in L_{\varphi}(I)$.*
2. *There exists a constant $k > 0$ such that for all measurable u, w on I we have $\|uw\|_{\varphi} \leq k\|u\|_{\varphi_1}\|w\|_{\varphi_2}$.*
3. *There exists numbers $C > 0$, $u_0 \geq 0$ such that for all $s, t \geq u_0$ $\varphi\left(\frac{st}{C}\right) \leq \varphi_1(s)+\varphi_2(t)$.*
4. $\limsup_{t\to\infty} \frac{\varphi_1^{-1}(t)\varphi_2^{-1}(t)}{\varphi(t)} < \infty$.

Let us recall the following simple sufficient condition for the above statements hold true.

Lemma 2.2 ([20], p. 223) *If there exist complementary N-functions Q_1 and Q_2 such that the inequalities $Q_1(\alpha u) < \varphi^{-1}[\varphi_1(u)]$, $Q_2(\alpha u) < \varphi^{-1}[\varphi_2(u)]$ hold, then for every functions $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}$, $u \cdot w \in L_{\varphi}(I)$. If moreover φ satisfies the Δ_2-condition, then it is sufficient that the inequalities $Q_1(\alpha u) < \varphi_1[\varphi^{-1}(u)]$, $Q_2(\alpha u) < \varphi_2[\varphi^{-1}(u)]$ hold.*

An interesting discussion about necessary and sufficient conditions for product operators can be found in [20, Section 13] and [21]. Note, that since $L^p = L_M$ for $M(t) = \frac{t^p}{p}$ the case of Lebesgue spaces L^p is included in the above consideration for Orlicz spaces.

We will use in the paper a general concept of function spaces, i.e. ideal spaces (or: Köthe function spaces). A normed space $(X, \|\cdot\|)$ of (classes of) measurable functions $x : I \to U$ (U is a normed space) is called pre-ideal if for each $x \in X$ and each measurable $y : I \to U$ the relation $|y(s)| \leq |x(s)|$ (for almost all $s \in I$) implies $y \in X$ and $\|y\| \leq \|x\|$. If X is also complete, it is called an ideal space (see [25]). An ideal normed space X is called regular if all singeltons in X have equicontinuous norm, i.e. $\lim_{\delta\to 0} \sup_{\{D: meas D \leq \delta\}} \|x \cdot \chi_D\| = 0$, where χ_D is the characteristic function of a measurable set D (cf. [15,23,24]).

We will say that an set T in a ideal space E is compact in measure if it is compact in the topology of convergence in measure, i.e. as a subset of the space of all measurable functions $L^0(I)$ (see [17,25]).

Finally we have a special case for $E_2 = L^\infty$ and some function spaces for which ($E = E_1$) $\|x \cdot y\|_E \leq \|x\|_E \cdot \|y\|_{L^\infty}$. The class of spaces with this property is known as preideal$*$ spaces (cf. [24, p. 66] or [25]). Although this case seems to be general it has one weakness from our point of view: the measure of noncompactness in L^∞ seems to be inapplicable and we do not discuss it in the paper. Moreover the continuous superposition operator with values in L^∞ should be constant. In particular, the function u in the equation (1.1) should be not depending on x, so the problem is reduced to the case of standard (non-quadratic) integral equations.

It is possible to check this property for a given triple of spaces. An open question is if is possible to characterize all such spaces?

3 Measures of noncompactness

If X is a subset of a Banach space E, then $\bar{X}$ and $convX$ denote the closure and convex closure of X, respectively. By B_r we denote a ball centered at θ with the radius r. The standard algebraic operations on sets will be denoted by the symbols $k \cdot X$ and $X + Y$.

Moreover, by $\mathcal{M}_E$ we denote the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets. We will use an axiomatic approach to the notion of a measure of noncompactness.

Definition 3.1 ([4]) *A mapping $\mu : \mathcal{M}_E \rightarrow [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*
(i) $\mu(X) = 0 \Rightarrow X \in \mathcal{N}_E$.
(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(iii)$\mu(\bar{X}) = \mu(conv X) = \mu(X)$.
(iv) $\mu(\lambda X) = |\lambda|\, \mu(X)$, for $\lambda \in \mathbb{R}$.
(v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
(vi) $\mu(X \bigcup Y) = \max\{\mu(X), \mu(Y)\}$.
(vii) If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \cdots$, and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family of sets A with $\mu(A) = 0$ we will call a kernel of the measure and we will denote by $ker\ \mu$. Let us recall a classical example: the Hausdorff measure of noncompactness $\beta_H(X)$ (cf. [4]) is defined as follows $\beta_H(X) = inf\{r > 0$: there exists a finite subset Y of E such that $X \subset Y + B_r$ $\}$, where X is an arbitrary nonempty and bounded subset of E. For more examples, we refer the reader to [4].

To distinguish between measures of noncompactness μ in different spaces (if necessary) we will indicate an appropriate space as an index i.e. μ_E, μ_{E_1} etc.

Let c be a measure of uniform integrability of the set X in an ideal function space E on the compact interval I (introduced in [1], cf. also [24, Definition 3.9] or [15,16]):

$$c(X) = \limsup_{\varepsilon\to 0} \sup_{mesD\leq\varepsilon} \sup_{x\in X} \|x \cdot \chi_D\|_E,$$

where χ_D denotes the characteristic function of a measurable subset $D \subset I$.

The following theorem clarifies the connections between the two coefficients in E.

Proposition 3.2 ([16], Theorem 1) *Let X be a nonempty, bounded and compact in measure subset of an ideal regular space E. Then $\beta_H(X) = c(X)$.*

As a consequence, we obtain that bounded sets which are additionally compact in measure are compact in E iff they are equiintegrable in this space (i.e. have equiabsolutely continuous norms, in particular when X is a subset for a regular part of E). This useful property will help us to replace weak (sequential) continuity conditions for considered operators by a strong one, which seems to be interesting in many cases.

4 Main results

Since we are interested on a fixed point of some product operators we will assume, that our operators have values in some intermediate spaces and then the product will again turn the values into the original space.

First, let us apply our approach to the most applicable theorem of this type. Consider an arbitrary (in the sense of Definition 3.1) measure of noncompactness μ in $C(I, E)$. An interesting fixed point theorem in Banach algebras was proved by Banaś and Lecko (cf. [5]). Let us consider different spaces of continuous functions with a suitable choice of measures of noncompactness μ_E on $E = C(I)$, μ_{E_1} on E_1 and μ_{E_2} on E_2. By using our approach we are able to present the following extension of the mentioned theorem:

Theorem 4.1 *Let E, E_1, E_2 be regular ideal function spaces. Assume that T is nonempty, bounded, closed, and convex subset of the Banach space E, and the operators $A : E \to E_1$ and $B : E \to E_2$. Moreover, assume:*

1. *(A1) A transform continuously the set T into $T_1 \subset E_1$ and $A(T)$ is bounded in E_1,*
2. *(A2) there exists a constant $k_1 > 0$ such that A satisfies an inequality: $\mu_{E_1}(A(U)) \leq k_1 \cdot \mu_E(U)$ for arbitrary bounded subset U of E,*
3. *(B1) B transform continuously the set T into $T_2 \subset E_2$ and $B(T)$ is bounded in E_2,*
4. *(B2) there exists a constant $k_2 > 0$ such that B satisfies an inequality: $\mu_{E_2}(B(U)) \leq k_2 \cdot \mu_E(U)$ for arbitrary bounded subset U of E,*
5. *(E1) for the triple of spaces E, E_1, E_2 there exists a constant k such that for arbitrary $x \in E_1$, $y \in E_2$ and $t \in I$ a product $x \cdot y \in E$ and $\|x \cdot y\|_E \leq k \cdot \|x\|_{E_1} \cdot \|y\|_{E_2}$,*
6. *(E2) for every $x \in T_1$ and $y \in T_2$ one has $x \cdot y \in T$,*
7. *(C) $\|A(T)\|_{E_1} \cdot k_2 + \|B(T)\|_{E_2} \cdot k_1 < 1$.*

Then there exists at least one fixed point for the operator $K = A \cdot B$ in the set T and that the set of all fixed points of K belongs to the kernel $\ker \mu_E$.

This theorem was proved by Banaś in a special case of Banach algebras $E = E_1 = E_2 = C(I, \mathbb{R})$ ($k = 1$) (cf. also [11,14], for instance). Since the proof runs like in previous cases (except some changes resulting from our considerations), we omit the details.

We do not require, that the values of all operators are from the same space, but by using a property of considered spaces we are able to repeat the proof. We will present a proof for a more general case. Note, that when the above result is used for proving the existence of continuous solutions (i.e. E=$C(I, \mathbb{R})$), then the Ascoli criterion of compactness simplify the proof, because the convergence of sequences is directly related with pointwise convergence. Our result allow to use different criterion of compactness (or even weak compactness!) in ideal spaces (cf. [10]).

Now, we will consider the case of functions spaces without such a nice property. We will consider some subspaces of a space of $L^0(I)$ of measurable functions, bigger than $C(I)$. It allows us to apply the fixed point theorem for the problems with discontinuous solutions. This proof will be based on different compactness criterion (the Dunford-Pettis theorem and the Erzakova theorem).

Theorem 4.2 *Assume that T is nonempty, bounded, closed, convex and compact in measure subset of a regular ideal function space E, and the operators $A : E \to E_1$ and $B : E \to E_2$. Moreover, assume:*

1. *(A1) A transform continuously the set T into $T_1 \subset E_1$ and $A(T)$ is bounded in E_1,*
2. *(A2) there exists a constant $k_1 > 0$ such that A satisfies an inequality: $c_{E_1}(A(U)) \leq k_1 \cdot c_E(U)$ for arbitrary bounded subset U of E,*
3. *(B1) B transform continuously the set T into $T_2 \subset E_2$ and $B(T)$ is bounded in E_2,*
4. *(B2) there exists a constant $k_2 > 0$ such that B satisfies an inequality: $c_{E_2}(B(U)) \leq k_2 \cdot c_E(U)$ for arbitrary bounded subset U of E,*

5. (*E*1) *for the triple of regular ideal spaces* E, E_1, E_2 *there exists a constant* k *such that for arbitrary* $x \in E_1$ *and* $y \in E_2$ *a product* $x \cdot y \in E$ *and* $\|x \cdot y\|_E \leq k \cdot \|x\|_{E_1} \cdot \|y\|_{E_2}$,
6. (*E*2) *for every* $x \in T_1$ *and* $y \in T_2$ *one has* $x \cdot y \in T$,
7. (*C*) $k \cdot k_1 \cdot k_2 < 1$.

Then there exists at least one fixed point for the operator $K = A \cdot B$ *in the set* T *and the set of all fixed points of* K*, i.e.* $FixK$ *is relatively compact in* E.

Proof. It is obvious that the operator K is well-defined on T and by (E2) it acts between T into itself.

Denote $M_1 = \sup_{t \in T} \|A(t)\|_{E_1}$ and $M_2 = \sup_{t \in T} \|B(t)\|_{E_2}$. Let (x_n) be an arbitrary sequence in T convergent to $x \in T$. Then

$$\begin{aligned}
&\|K(x_n) - K(x)\|_E \\
&= \|A(x_n) \cdot B(x_n) - A(x) \cdot B(x)\|_E \\
&\leq \|A(x_n) \cdot B(x_n) - A(x) \cdot B(x) - A(x) \cdot B(x_n) + A(x) \cdot B(x_n)\|_E \\
&\leq \|(A(x_n) - A(x)) \cdot B(x_n)\|_E + \|(B(x_n) - B(x)) \cdot A(x)\|_E \\
&\leq k \cdot \|A(x_n) - A(x)\|_{E_1} \cdot M_2 + k \cdot \|B(x_n) - B(x)\|_{E_2} \cdot M_1.
\end{aligned}$$

From our assumptions it follows that K is continuous from T into E.

Now, we will investigate the contraction property for a measure $c(X)$.

Assume that X is a nonempty subset of T and let $\varepsilon > 0$ be arbitrary. Then for an arbitrary $x \in X$ and for a set $D \subset I$, meas$D \leq \varepsilon$ we obtain $\|K(x) \cdot \chi_D\|_E \leq k\|A(x) \cdot \chi_D\|_E \cdot \|B(x) \cdot \chi_D\|_{E_2}$. Since for any non-negative real-valued functions f and g we have $\sup_I (f \cdot g) \leq \sup_I f \cdot \sup_I g$, by definition of $c(x)$ and by taking the supremum over all $x \in X$ and all measurable subsets D with $measD \leq \varepsilon$ we get $c(K(X)) \leq k \cdot k_1 \cdot k_2 \cdot c(X)$. Because $X \subset T$ is a nonempty, bounded and compact in measure subset of an ideal regular space E, we can use Proposition 3.2 and then $\beta_H(K(X)) \leq k \cdot k_1 \cdot k_2 \cdot \beta_H(X)$. The inequality obtained above together with the properties of the operator K and the set T established before, allow us to apply the classical Darbo fixed point theorem for β_H. If we suppose, that $\beta_H(FixK) \neq 0$, then $K = FixK$ implies $\beta_H(FixK) = \beta_H(K) < \beta_H(K)$, a contradiction. This completes the proof. □

The above theorem is presented in a form related to earlier results. Nevertheless, it is easy to check, that by the same method we are able to obtain an improved version:

Corollary 4.3 *The above theorem remains true if one of the operators is a* k_1*- contraction with respect to the measure c and another operator is bounded by* k_2 *whenever* $k \cdot k_1 \cdot k_2 < 1$.

It was observed, that a compactness criterion for Lebesgue spaces requires some uniform integrability condition, as well as, a condition of compactness in measure (see [15] or [18], for the discussion). Thus, the next goal of the paper is to indicate a big class of sets being compact in measure.

Remark 4.1 There are many natural sets which are compact in measure. Let us observe, that at least a specific class of sets should be interesting for integral (and differential) equations, namely sets consisting a.e. monotonic functions. This observation is well-known (cf. [3] or [19]). Surprisingly there is no published proof, even for sets of continuous functions.

We present this proof for a general regular ideal space. It can help us to construct more sets compact in measure. Note that this is a weak version of Helly's selection theorem (cf. [3,9]).

Lemma 4.4 *Let U be a bounded subset of subset of regular ideal space E of real-valued functions over a bounded interval I such that all the functions from U are a.e. monotonic. Then this set is compact in measure in the space E.*

Proof. Let us recall, that a topology of convergence in measure is a metric topology (cf. [17,18]), so we need only to check sequential compactness. Without loss of generality, let us restrict to the case of a.e. increasing functions. Take an arbitrary sequence $(x_n) \subset U$. Denote by W the set s.t. x_n are increasing on $J = I \setminus W$. Let Z be a dense subset of J. Since $\{x_n(t)\}$ is bounded on Z (except perhaps the point $t = b$), by a diagonal procedure we can subtract a pointwisely convergent subsequence (x_{n_k}) of (x_n) (on Z). Then a limit $x = \lim_{k\to\infty} x_{n_k}$ is an increasing function on Z. It is known, that this function can be extended to y defined on J in such a way y is a limit of (x_{n_k}) on Z.

Let t_0 be an arbitrary point in J. Since Z is dense in J we are able to find two sequences (s_n) and (τ_n) of points in Z convergent to t_0 such that $s_n < t_0 < \tau_n$. For any fixed $k \in \mathbb{N}$, (x_{n_k}) is increasing on J and we have $x_{n_k}(s_n) < x_{n_k}(t_0) < x_{n_k}(\tau_n)$ so by passing to the limit with $k \to \infty$ we obtain

$$y(s_n) \leq \liminf_{k\to\infty} x_{n_k}(t_0) \leq \limsup_{k\to\infty} x_{n_k}(t_0) \leq y(\tau_n).$$

By passing to the limit with $n \to \infty$ we get

$$y(t_0-) \leq \liminf_{k\to\infty} x_{n_k}(t_0) \leq \limsup_{k\to\infty} x_{n_k}(t_0) \leq y(t_0+).$$

For any point of continuity of y we have $y(t) = \lim_{k\to\infty} x_{n_k}(t)$. The set D of all points of discontinuity of this function is at most countable. Then y is a.e. increasing.

Since the measure of I is finite, the sequence (x_n) contains an a.e. convergent subsequence. By the Lebesgue theorem the last is also convergent in measure. Finally: arbitrary sequence in U contains a subsequence convergent in measure to some $y \in U$ and then this set is compact in measure. □

It is clear that the subfamily of U consisting of nonnegative functions is compact in measure too (it is an important property for positive monotonic solutions).

Remark 4.2 We need to remark, that one of our assumptions can be easily relaxed. We assume, that the space is regular. Denote by E_0 a regular part of E. It is sufficient to assume, that $K : T \cap E_0 \to T \cap E_0$. It seems to be important for the case of so-called improving operators (taking bounded subsets of E into the sets with equiabsolutely continuous norms, i.e. into E_0). A detailed theory of compactness in regular ideal spaces can be found in [23].

Remark 4.3 Finally, let us note, that the proof can be easily adapted to the case of a multivalued version of the main results (see [12,13] for a few basic results with multivalued operators A and B). The basic result was obtained by Petruşel [22]) and the current status of research can be found in: for lsc case see [13, Theorem 3.3] or [12, Theorem 2.2] and for usc case [12, Theorem 2.4] or [13, Theorem 3.5].

Nevertheless, in such a kind of results there is a serious problem how to ensure, that the product of two multivalued operators has still closed, convex values. To the best of our knowledge, there is no such a kind of results till now. In general, this property is assumed to be satisfied. Thus, the multivalued case seems to be interesting only in a case when either A or B is single-valued, in which this property holds true. This is exactly the case indicated in some applications. Our theorems can be easily extended to the multivalued context in both cases.

Let us recall some problems mentioned in the introductory part of the paper. Our approach allows to find discontinuous solutions for these problems.

In the first case let us restrict ourself to the case of the Chandrasekhar equation

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \psi(s) x(s)\, ds,$$

which is one of our prototypes for considered problem. In order to apply earlier results we have to impose an additional condition that the so-called "characteristic" function ψ is a polynomial (as in the book of Chandrasekhar [8, Chapter 5]) or at least continuous (cf. [7, Theorem 3.2], [2]). This function is immediately related to the angular pattern for single scattering and then our results allow to consider some peculiar states of the atmosphere. In astrophysical applications of the Chandrasekhar equation the only restriction, that $\int_0^1 \psi(s)\, ds \leq 1/2$ is treated as necessary (cf. [7, Chapter VIII; Corollary 2 p. 187]). The continuity assumption for ψ implies the continuity of solutions for the considered equation (cf. [7]) and then it seems to be too restrictive even from the theoretical point of view.

Consider $A(x)(t) = x(t)$ and $B(x)(t) = \int_0^1 \frac{t}{t+s} \psi(s) x(s)\, ds$. We are able check under some very general conditions the operators A and B satisfy all the conditions of Theorem 4.2 or Corollary 4.3 and then to find a solution belonging to some Lebesgue or Orlicz spaces (cf. [10]).

Finally, let us note, that our results can be easily applied to a class of quadratic operator equations of the form $x(t) = U(x)(t) \cdot H(x)(t)$. In our equation (1.1) we have the Nemytskii superposition operator U generated by $u(t, x(t))$ and the Hammerstein integral operator, but we are not restricted only to this case.

Another typical example is a Cauchy problem ($f_1 : I \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, f_2 and g satisfy some regularity conditions, cf. [14])

$$\left(\frac{x(t) - g(t)}{f_1(t,x)} \right)' = f_2(t, x(t)) \quad x(0) = 0. \tag{4.1}$$

This problem is rewritten in a form of quadratic integral equations, so our theorems apply. When we are looking for continuous solutions for the quadratic integral equations, the solutions for the above differential problems are classical. Our approach allows to investigate weaker types of solutions (in Orlicz-Sobolev spaces, for instance).

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