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Incomplete generalized Tribonacci polynomials and numbers

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Abstract The main object of this paper is to present a systematic investigation of a new class of polynomials – incomplete generalized Tribonacci polynomials and a class of numbers associated with the familiar Tribonacci polynomials. The various results obtained here for these classes of polynomials and numbers include explicit representations, generating functions, recurrence relations and summation formulas.

Keywords Incomplete generalized Tribonacci polynomials · Incomplete generalized Tribonacci numbers · Generating function · Recurrence relation

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1 Introduction and definitions

Fibonacci polynomials have been studied in 1883 by the belgian mathematician E. C. Catalan and german mathematician E. Jacobsthal. Namely, the Fibonacci polynomials $F_n(x)$ studied by Catalan are defined by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \qquad n \geq 2,$$

while the Fibonacci polynomials, studied by Jacobsthal, (see [2], for $m = 2$ and $x := x/2$, [3]), are defined by

$$J_0(x) = 1,\ J_1(x) = 1,\ J_n(x) = J_{n-1}(x) + xJ_{n-2}(x), \qquad n \geq 2.$$

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Generating functions of the incomplete Fibonacci and Lucas numbers are determined in [8]. In the certain order of ideas incomplete generalized Fibonacci and Lucas numbers are defined and studied by DJORDJEVIĆ [4]. DJORDJEVIĆ and SRIVASTAVA [3], [5] defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers.

Tribonacci polynomials, defined by the recurrence relation ($n \geq 3$)

$$T_0(x)=0,\ T_1(x)=1,\ T_2(x)=x^2,\ T_n(x)=x^2T_{n-1}(x)+xT_{n-2}(x)+T_{n-3}(x),$$

are studied by HOGGATT JR. and BICKNELL [7].

For $x=1$ we get the sequence of numbers $T_n(1)=t_n$ (see [6]).

One explicit representation of the Tribonacci polynomials $T_n(x)$ is given by

$$T_n(x)=\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{i}\binom{i}{j}\binom{n-1-i-j}{i}x^{2n-2-3(i+j)},\qquad n\geq 1. \tag{1.1}$$

Hence, from (1.1), for $x=1$, we get the explicit formula of numbers t_n:

$$t_n=\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{i}\binom{i}{j}\binom{n-1-i-j}{i},\qquad n\geq 1. \tag{1.2}$$

2 Incomplete generalized Tribonacci polynomials and numbers

Here we define the generalized Tribonacci polynomials $T_{n,m}(x)$ with the following generating function:

$$F_m(x,t)=\left(1-x^2t-xt^m-t^{2m-1}\right)^{-1}=\sum_{n=1}^{\infty}T_{n,m}(x)t^{n-1},\ m\geq 2. \tag{2.1}$$

Thus, for $m=2$, from (2.1), we obtain the generating function of the Tribonacci polynomials $T_n(x)$ (see [7]):

$$F(x,t)=\left(1-x^2t-xt^2-t^3\right)^{-1}=\sum_{n=1}^{\infty}T_{n,2}(x)t^{n-1}.$$

Next, from (2.1), we find that $T_{n,m}(x)$ posses the following explicit representation

$$T_{n,m}(x)= \tag{2.2}$$

$$\sum_{i=0}^{[(n-1)/m]}\sum_{j=0}^{i}\binom{i}{j}\binom{n-1-(m-1)(i+j)}{i}x^{2n-2-(2m-1)(i+j)} \tag{2.3}$$

So, for $m=2$ in (2.3) we get the explicit formula (1.1). Also, by (2.3) for $m=3$, we get the explicit representation of $T_{n,3}(x)$:

$$T_{n,3}(x)=\sum_{i=0}^{[(n-1)/3]}\sum_{j=0}^{i}\binom{i}{j}\binom{n-1-2(i+j)}{i}x^{2n-2-3(i+j)}.$$

Now, from (2.1), we have the generating function of the generalized Tribonacci numbers $t_{n,m}$ ($t_{n,m} = T_{n,m}(1)$):

$$F_m(t) = \left(1 - t - t^m - t^{2m-1}\right)^{-1} = \sum_{n=1}^{\infty} t_{n,m} t^{n-1}, \; m \geq 2. \tag{2.4}$$

So, the corresponding explicit formula, from (2.3) is

$$t_{n,m} = \sum_{i=0}^{[(n-1)/m]} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-(m-1)(i+j)}{i}. \tag{2.5}$$

Hence, from (2.5) and for $m = 3$, we get the explicit formula for the numbers $t_{n,3}$:

$$t_{n,3} = \sum_{i=0}^{[(n-1)/3]} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-2(i+j)}{i}.$$

Now, using (2.3) and (2.5), we introduce the incomplete generalized Tribonacci polynomials $T_{n,m}^{(s)}(x)$ and incomplete generalized Tribonacci numbers $t_{n,m}(s)$ in the following way.

Definition 2.1 *The incomplete generalized Tribonacci polynomials* $T_{n,m}^{(s)}(x)$ *are defined by*

$$T_{n,m}^{(s)}(x) = \sum_{i=0}^{s} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-(m-1)(i+j)}{i} x^{2n-2-(2m-1)(i+j)}, \tag{2.6}$$

where $0 \leq s \leq [(n-1)/m], \; n \geq 1, \; m \geq 2.$

Definition 2.2 *The incomplete generalized Tribonacci numbers* $t_{n,m}(s)$ *are defined by the following explicit formula*

$$t_{n,m}(s) = \sum_{i=0}^{s} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-(m-1)(i+j)}{i}, \tag{2.7}$$

where $0 \leq s \leq [(n-1)/m], \; n \geq 1, \; m \geq 2.$

Some initial values of the incomplete generalized Tribonacci polynomials are given in Table 1.

Also, from Definition 2.1 and Definition 2.2, it follows

$$T_{n,m}^{[(n-1)/m]}(x) = T_{n,m}(x) \quad \textit{and} \quad t_{n,m}([(n-1)/m]) = t_{n,m}.$$

For $m = 3$, using Table 1, we get:

Table 1 Polynomials $T_{n,m}^{(s)}(x)$

n	$s=0$	$s=1$	$s=2$
1	1		
2	x^2		
3	x^4		
...	...	...	...
m	x^{2m-2}		
$m+1$	x^{2m}	$x^{2m}+x$	
$m+2$	x^{2m+2}	$x^{2m+2}+2x^3$	
...	...	...	
$2m+1$	x^{4m}	$x^{4m}+(m+1)x^{2m+1}+2x^2$	$x^{4m}+(m+1)x^{2m+1}+3x^2$

Table 2 Polynomials $T_{n,3}^{(s)}(x)$

n	$s=0$	$s=1$	$s=2$
1	1		
2	x^2		
3	x^4		
4	x^6	x^6+x	
5	x^8	x^8+2x^3	
6	x^{10}	$x^{10}+3x^5+1$	
7	x^{12}	$x^{12}+4x^7+2x^2$	$x^{12}+4x^7+3x^2$

3 Some properties of the polynomials $T_{n,m}^{(s)}(x)$ and numbers $t_{n,m}(s)$

Theorem 3.1 *The incomplete generalized Tribonacci polynomials $T_{n,m}^{(s)}(x)$ satisfy the following recurrence relation*

$$T_{n,m}^{(s+1)}(x)=x^2T_{n-1,m}^{(s+1)}(x)+xT_{n-m,m}^{(s)}(x)+T_{n+1-2m,m}^{(s)}(x),\ n\geq 2m-1, \tag{3.1}$$

where $0\leq s\leq[(n-1)/m]$ *and* $m\geq 2$.

Proof. Since

$$\begin{aligned}
&x^2\binom{i}{j}\binom{n-1-(m-1)(i+j)}{i}x^{2n-2-(2m-1)(i+j)}\\
&\quad+x\binom{i}{j}\binom{n-m-(m-1)(i+j)}{i}x^{2n-2m-(2m-1)(i+j)}\\
&\quad+\binom{i}{j}\binom{n+1-2m-(m-1)(i+j)}{i}x^{2n+2-4m-(2m-1)(i+j)}\\
&\quad=\binom{i}{j}\binom{n-1-(m-1)(i+j)}{i}x^{2n-(2m-1)(i+j)}\\
&\quad+\binom{i}{j}\binom{n-m-(m-1)(i+j)}{i}x^{2n+1-2m-(2m-1)(i+j)}
\end{aligned}$$

$$\begin{aligned}
&\binom{i}{j-1}\binom{n-m-(m-1)(i+j)}{i}x^{2n+1-2m-(2m-1)(i+j)}\\
&=\left(\binom{i}{j}+\binom{i}{j-1}\right)\binom{n-m-(m-1)(i+j)}{i}x^{2n+1-2m-(2m-1)(i+j)}\\
&+\binom{i}{j}\binom{n-1-(m-1)(i+j)}{i}x^{2n-(2m-1)(i+j)}\\
&=\binom{i+1}{j}\binom{n-m-(m-1)(i+j)}{i}x^{2n+1-2m-(2m-1)(i+j)}\\
&+\binom{i}{j}\binom{n-1-(m-1)(i+j)}{i}x^{2n-(2m-1)(i+j)}\\
&=\binom{i}{j}\binom{n-m-(m-1)(i+j)}{i-1}x^{2n-(2m-1)(i+j)}\\
&+\binom{i}{j}\binom{n-1-(m-1)(i+j)}{i}x^{2n-(2m-1)(i+j)}\\
&=\binom{i}{j}\binom{n-(m-1)(i+j)}{i}x^{2n-(2m-1)(i+j)},
\end{aligned}$$

the relation (3.1) holds true. □

Corollary 3.2 *The incomplete generalized numbers* $t_{n,m}(s)$ *satisfy the following relation*

$$t_{n,m}(s+1)=t_{n-1,m}(s+1)+t_{n-m,m}(s)+t_{n+1-2m,m}(s).$$

Corollary 3.3 *For the incomplete generalized numbers* $t_{n,m}(s)$ *it holds*

$$t_{n,m}(s+1)-t_{n-1-k,m}(s+1)=\sum_{i=0}^{k}\left(t_{n-i-m,m}(s)+t_{n+1-i-2m,m}(s)\right).$$

Theorem 3.4 *The incomplete generalized polynomials* $T^{(s)}_{n,m}(x)$ *satisfy*

$$\begin{aligned}
\sum_{s=0}^{l}T^{(s)}_{n,m}(x)&=(l+1)T_{n,m}(x)\\
&-\sum_{i=0}^{l}\sum_{j=0}^{i}i\binom{i}{j}\binom{n-1-(m-1)(i+j)}{i}x^{2n-2-(2m-1)(i+j)}, \qquad (3.3)
\end{aligned}$$

where $l=[(n-1)/m]$.

Proof.

$$\begin{aligned}
&T^{(0)}_{n,m}(x)+T^{(1)}_{n,m}(x)+T^{(2)}_{n,m}(x)+\cdots+T^{(l)}_{n,m}(x)\\
&\quad = x^{2n-2}+x^{2n-2}+\binom{1}{0}\binom{n-1-(m-1)}{1}x^{2n-2-(2m-1)}\\
&\quad + \binom{1}{1}\binom{n-1-2(m-1)}{1}x^{2n-2-2(2m-1)}\\
&\quad + x^{2n-2}+\binom{1}{0}\binom{n-1-(m-1)}{1}x^{2n-2-(2m-1)}\\
&\quad + \binom{1}{1}\binom{n-1-2(m-1)}{1}x^{2n-2-2(2m-1)}\\
&\quad + \binom{2}{0}\binom{n-1-2(m-1)}{2}x^{2n-2-2(2m-1)}\\
&\quad + \binom{2}{1}\binom{n-1-3(m-1)}{2}x^{2n-2-3(2m-1)}\\
&\quad + \binom{2}{2}\binom{n-1-4(m-1)}{2}x^{2n-2-4(2m-1)}+\dots\\
&\quad + x^{2n-2}+\binom{l}{0}\binom{n-1-l(m-1)}{l}x^{2n-2-l(2m-1)}\\
&\quad + \binom{l}{1}\binom{n-1-(l+1)(m-1)}{l}x^{2n-2-(l+1)(2m-1)}\\
&\quad + \cdots+\binom{l}{l}\binom{n-1-2l(m-1)}{l}x^{2n-2-2l(2m-1)}\\
&\quad = (l+1)T_{n,m}(x)\\
&\quad - \sum_{i=0}^{l}\sum_{j=0}^{i} i\binom{i}{j}\binom{n-1-(m-1)(i+j)}{i}x^{2n-2-(2m-1)(i+j)}.
\end{aligned}$$

□

Corollary 3.5 *The incomplete generalized numbers $t_{n,m}(s)$ satisfy the following relation*

$$\sum_{s=0}^{l} t_{n,m}(s)=(l+1)t_{n,m}-\sum_{i=0}^{l}\sum_{j=0}^{i} i\binom{i}{j}\binom{n-1-(m-1)(i+j)}{i},$$

where $l=[(n-1)/m]$.

4 Generating function of the numbers $t_{n,m}(s)$

The following known result (due essentially to Pinter and Srivastava [8]) will be required in our investigation of the generating function of same incomplete numbers which are similar to the incomplete generalized Tribonacci numbers $t_{n,m}^{(s)}$ defined by (2.7). For the theory and applications of the various methods and techniques for deriving generating functions of special functions and polynomials, we may refer the interested reader to the recent result on generating functions by Srivastava and Manocha [9].

Lemma 4.1 *Let* $\{s_{n,m}\}_{n\in\mathbb{N},\, m\geq 2}$ *be a complex sequence satisfying the following recurrence relation*

$$s_{n,m} = s_{n-1,m} + s_{n-m,m} + s_{n+1-2m,m} + r_{n,m} \quad n \geq 2m-1, \tag{4.1}$$

where $\{r_{n,m}\}_{n\in\mathbb{N},\, m\geq 2}$ *is a complex sequence. Let* $G_m(t)$ *be the generating function of the sequence* $\{r_{n,m}\}_{n\in\mathbb{N},\, m\geq 2}$*. Then the generating function* $U_m(t)$ *of the sequence* $\{s_{n,m}\}_{n\in\mathbb{N},\, m\geq 2}$ *is*

$$\begin{aligned} U_m(t)\cdot\left(1-t-t^m-t^{2m-1}\right) &= G_m(t) + s_{0,m} - r_{0,m} \\ &+ t(s_{1,m}-s_{0,m}-r_{1,m}) + \cdots + t^m(s_{m,m}-s_{m-1,m}-r_{m,m}) \\ &+ \cdots + t^{2m-2}(s_{2m-2,m}-s_{2m-3,m}-s_{m-2,m}-r_{2m-2,m}). \end{aligned} \tag{4.2}$$

Proof. Since $U_m(t)$ and $G_m(t)$ are the generating functions of the numbers $s_{n,m}$ and numbers $r_{n,m}$, respectively, then their power series representations are:

$$\begin{aligned} U_m(t) &= s_{0,m} + s_{1,m}t + s_{2,m}t^2 + \cdots + s_{k,m}t^k + \ldots, \\ G_m(t) &= r_{0,m} + r_{1,m}t + r_{2,m}t^2 + \cdots + r_{k,m}t^k + \ldots. \end{aligned}$$

Note that

$$\begin{aligned} tU_m(t) &= s_{0,m}t + s_{1,m}t^2 + s_{2,m}t^3 + \cdots + s_{k,m}t^{k+1} + \ldots, \\ t^mU_m(t) &= s_{0,m}t^m + s_{1,m}t^{m+1} + s_{2,m}t^{m+2} + \cdots + s_{k,m}t^{m+k} + \ldots, \\ t^{2m-1}U_m(t) &= s_{0,m}t^{2m-1} + s_{1,m}t^{2m} + s_{2,m}t^{2m+1} + \cdots + s_{k,m}t^{2m-1+k} + \ldots. \end{aligned}$$

Hence

$$\begin{aligned} U_m(t)\cdot\left(1-t-t^m-t^{2m-1}\right) &= s_{0,m} - r_{0,m} + t(s_{1,m}-s_{0,m}-r_{1,m}) \\ &+ t^2(s_{2,m}-s_{1,m}-r_{2,m}) + t^3(s_{3,m}-s_{2,m}-r_{3,m}) + \ldots \\ &+ t^m(s_{m,m}-s_{m-1,m}-s_{0,m}-r_{m,m}) + \ldots \\ &+ t^{2m-2}(s_{2m-2,m}-s_{2m-3,m}-s_{m-2,m}-r_{2m-2,m}) + G_m(t). \end{aligned}$$

From the last relation we can conclude that relation (4.2) holds true. □

Theorem 4.2 *The generating function* $Q_m(z)$ *of the incomplete generalized Tribonacci numbers* $t_{n,m}(s)$ *is given by*

$$\begin{aligned} Q_m(z)\cdot\left(1-z-z^m-z^{2m-1}\right) &= t_{ms+1,m} \\ &+ z(t_{ms+2,m}-t_{ms+1,m}) + z^2(t_{ms+3,m}-t_{ms+2,m}) + \ldots \\ &+ z^m(t_{ms+m+1,m}-t_{ms+m,m}-r_{m,m}) + \cdots + \\ &z^{2m-2}(t_{ms+2m-1,m}-t_{ms+2m-2,m}-t_{ms+m-1,m}-r_{2m-2,m}) \\ &-(z^m+z^{2m-1})\frac{(1+z)^s}{(1-z)^{s+1}}. \end{aligned}$$

Proof. Let s be a fixed positive integer and let $t_{n,m}(s)$ be the n-th incomplete generalized Tribonacci numbers. If $Q_s(z)$ is the generating function of $t_{n,m}(s)$, then $Q_s(z) = \sum_{i=0}^{\infty} t_{i,m}(s) z^i$.

We can easily conclude that: $t_{n,m}(s) = 0$ for $0 \leq n < ms+1$, and

$$s_{0,m} = t_{ms+1,m}(s), \quad s_{1,m} = t_{ms+2,m}(s),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$s_{m,m} = t_{ms+m+1,m}(s), \quad s_{n,m} = t_{n+ms+1,m}(s).$$

Also, $r_{0,m} = r_{1,m} = \ldots r_{m-1} = 0$, $r_{m,m} = 1$, and

$$r_{n,m} = \sum_{j=0}^{s} \binom{s}{j} \binom{n+s-m-j}{s} + \sum_{j=0}^{s} \binom{s}{j} \binom{n+s+1-2m-j}{s}.$$

We can prove that

$$\left(z^m + z^{2m-1}\right) \frac{(1+z)^s}{(1-z)^{s+1}}$$

is the generating function of the sequence $\{r_{n,m}\}_{n \in \mathbb{N},\, m \geq 2}$ (see [1]).

Thus, from Lemma 4.1, we find

$$\begin{aligned} Q_s(z) \cdot \left(1 - z - z^m - z^{2m-1}\right) &= t_{ms+1,m} + z(t_{ms+2,m} - t_{ms+1,m}) \\ &+ z^2(t_{ms+3,m} - t_{ms+2,m}) + \cdots + z^m(t_{ms+m+1,m} - t_{ms+m,m} - 1) \\ &+ \cdots + z^{2m-2}(t_{ms+2m-1,m} - t_{ms+2m-2,m} - t_{ms+m+1} - r_{2m-2,m}) \\ &- (z^m + z^{2m-1}) \frac{(1+z)^s}{(1-z)^{s+1}}. \end{aligned}$$

□

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